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Implementation of summation theorems of Andrews and Gessel-Stanton

Generalized hypergeometric functions and their natural generalizations in one and several variables appear in many mathematical problems and their applications. Solving partial differential equations encountered in many applied problems of mathematics physics is expressed in terms of such generalized hypergeometric functions. In particular, the Srivastava-Daoust double hypergeometric function (S-D function) has proved its practical utility in representing solutions to a wide range of problems in pure and applied mathematics. In this paper, we introduce two general double-series identities involving bounded sequences of arbitrary complex numbers employing the finite summation theorems of Gessel-Stanton and Andrews for terminating $_3F_2$ hypergeometric series with arguments $3/4$ and $4/3$, respectively. Using these double-series identities, we establish two reduction formulas for the (S-D function) with arguments $z, 3z/4$ and $z, -4z/3$ expressed in terms of two generalized hypergeometric function of arguments proportional to z^3 and $-z^3$ respectively. All the results mentioned in the paper are verified numerically using Mathematica Program.

Keywords: Generalized hypergeometric function; Srivastava-Daoust double hypergeometric function; Reduction formulas; Mathematica Program.

1 Introduction and preliminaries

The $_pF_q$ $(p, q \in \mathbb{N}_0)$ is the generalized hypergeometric series defined by (see, e.g., [1; Section 1.5]):

$$_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,; \end{array} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \tag{1}$$
$$= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),$$

being a natural generalization of the Gaussian hypergeometric series $_2F_1$, where $(\lambda)_\nu$ denotes the Pochhammer symbol (for $\lambda, \nu \in \mathbb{C}$) defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} \qquad\qquad (\lambda, \nu + \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-)$$

$$= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases}$$

Here Γ is the familiar Gamma function (see, e.g., [1; Section 1.1]) and it is assumed that $(0)_0 := 1$, an empty product as 1, and that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zero appear in the denominator of (1), that is, that

$$(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ j = 1, \ldots, q).$$

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Here and elsewhere, let $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be respectively the sets of integers, real numbers, and complex numbers, and let

$$\mathbb{N} := \{1,\,2,\,3\ldots\};\ \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,;\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0,-1,-2,-3,\cdots\}\,.$$

For more details of $_pF_q$ including its convergence, its various special and limiting cases, and its further diverse generalizations, one may referred, for example [2,3].

Whenever the generalized hypergeometric function $_pF_q$, including $_2F_1$, can be expressed in terms of Gamma functions through summation of its specified argument, which may include unit or $\frac{1}{2}$ argument, the outcome holds significant value from both theoretical and practical perspectives.

The generalized hypergeometric series has classical summation theorems, including those of Gauss, Gauss second, Kummer, and Bailey for the $_2F_1$ series, as well as Watson's, Dixon's, Whipple's, and Saalschütz's summation theorems for the $_3F_2$ series and others. These theorems have significant importance in both theory and application.

From 1992 to 1996, Lavoie et al. [4–6] published a series of works that generalized the aforementioned classical summation theorems for the $_3F_2$ series of Watson, Dixon, and Whipple. They also presented many special and limiting cases of their results, which have been further extended and generalized by Rakha-Rathie [7], Kim et al. [8], and more recently by Qureshi et al. [9]. These results have also been verified, using computer programs such as Mathematica.

Srivastava and Daoust [10; 199] introduced a generalization of the Kampé de Fériet function [11; 150] by means of the double hypergeometric series (see also [12,13]):

$$F_{C:\ D;\ D'}^{A:\ B;\ B'} \left(\begin{array}{c} [(a_A):\vartheta,\ \varphi]: \quad [(b_B):\psi]; \quad [(b'_{B'}):\psi']; \\ [(c_C):\delta,\ \varepsilon]: \quad [(d_D):\eta]; \quad [(d'_{D'}):\eta']; \end{array} x,\ y \right)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\displaystyle\prod_{j=1}^{A} (a_j)_{m\vartheta_j+n\varphi_j} \prod_{j=1}^{B} (b_j)_{m\psi_j} \prod_{j=1}^{B'} (b'_j)_{n\psi'_j}}{\displaystyle\prod_{j=1}^{C} (c_j)_{m\delta_j+n\varepsilon_j} \prod_{j=1}^{D} (d_j)_{m\eta_j} \prod_{j=1}^{D'} (d'_j)_{n\eta'_j}} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{2}$$

where the coefficients

$$\vartheta_1,...,\vartheta_A;\ \varphi_1,...,\varphi_A;\ \psi_1,...,\psi_B;\ \psi'_1,...,\psi'_{B'};\ \delta_1,...,\delta_C;$$
$$\varepsilon_1,...,\varepsilon_C;\ \eta_1,...,\eta_D;\ \eta'_1,...,\eta'_{D'}$$

are real and positive. Let

$$\Delta_1 := 1 + \left(\sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \eta_j \right) - \left(\sum_{j=1}^{A} \vartheta_j + \sum_{j=1}^{B} \psi_j \right)$$

and

$$\Delta_2 := 1 + \left(\sum_{j=1}^{C} \varepsilon_j + \sum_{j=1}^{D'} \eta'_j \right) - \left(\sum_{j=1}^{A} \varphi_j + \sum_{j=1}^{B'} \psi'_j \right).$$

Then

(i) The double power series in (2) converges for all complex values of x and y when $\Delta_1 > 0$ and $\Delta_2 > 0$.

(ii) The double power series in (2) is convergent for suitably constrained values of $|x|$ and $|y|$ when $\Delta_1 = 0$ and $\Delta_2 = 0$.

(iii) The double power series in (2) would diverge except when, trivially, $x = y = 0$ when $\Delta_1 < 0$ and $\Delta_2 < 0$.

Qureshi et al. [14] provided insightful remarks on previous studies, specifically [15–17]. They employed a double-series manipulation technique, utilizing Whipple's transformation (see [18; 266, Eq.(6.6)]):

$$
{}_5F_4 \left[\begin{array}{c} -\frac{m}{2}, \frac{-m+1}{2}, E, 1-m-B-C, 1-m-D \, ; \\ 1-m-B, 1-m-C, \frac{1+E-D-m}{2}, \frac{2+E-D-m}{2} \, ; \end{array} 1 \right]
$$

$$
= \frac{(D)_m}{(D-E)_m} \, {}_4F_3 \left[\begin{array}{c} -m, B, C, E \, ; \\ 1-m-B, 1-m-C, D \, ; \end{array} 1 \right]
$$

$$
\left(m \in \mathbb{N}_0; \; B, \, C \, D, \, \frac{1+E-D-m}{2}, \, \frac{2+E-D-m}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^- \right),
$$

and (see [19; p. 537, Eq.(10.11)]; see also [17; Eq. (2.5)])

$$
{}_4F_3 \left[\begin{array}{c} -m, X, Y, Z \, ; \\ U, W, X+Y+Z+1-U-W-m \, ; \end{array} 1 \right]
$$

$$
= \frac{(U-X)_m (Y+Z+1-U-W-m)_m}{(U)_m (X+Y+Z+1-U-W-m)_m}
$$

$$
\times \, {}_4F_3 \left[\begin{array}{c} -m, W-Y, W-Z, X \, ; \\ 1-m+X-U, U+W-Y-Z, W \, ; \end{array} 1 \right]
$$

$$
\left(\begin{array}{c} m \in \mathbb{N}_0; \; U, W, X+Y+Z+1-U-W-m, \\ 1-m+X-U, U+W-Y-Z, W \in \mathbb{C} \setminus \mathbb{Z}_0^- \end{array} \right).
$$

Through this approach, they introduced three double-series identities, which incorporated a bounded sequence of complex numbers. In addition, they [14] demonstrated that the application of double-series identities enables the provision of numerous reduction formulas, whether they are already known or newly discovered. Subsequently and concurrently, a number of papers have utilized series manipulation techniques along with, among several others, transformation formulas for ${}_2F_1$ in Chan et al. [20], the reduction and transformation formulas of Kampé de Fériet and Srivastava-Daoust functions [21], implications of Bailey transformations in double-series and their consequences [22], the reduction formula for ${}_2F_1$ in Karlsson [23], terminating ${}_3F_2 \left(\frac{4}{3} \right)$ [24; Eq.(1.3)] (see also Gessel-Stanton summation theorem [25; Eq.(5.21)] and terminating ${}_3F_2 \left(\frac{3}{4} \right)$ [24; Eq.(1.4)] (see also [26; Eq.(1.12)])) in Qureshi et al. [24]. These papers have presented multiple or double series identities, which have been employed to derive a range of reduction formulas for the Kampé de Fériet, Srivastava-Daoust function and other intriguing identities for the ${}_pF_q$ functions.

Inspired by the aforementioned papers, especially [14, 21], and utilizing the reversing order of the finite summation theorem of Gessel-Stanton [25; 305, Eq.(5.21)])

$$
{}_3F_2 \left[\begin{array}{c} -n, -2b-\frac{2n}{3}, -6b-n \, ; \\ -3b-n, \frac{1}{2}-3b-n \, ; \end{array} \frac{3}{4} \right] = \begin{cases} 0 & ; n = 3m+1 \ \text{and} \ 3m+2, \\ \frac{\left(\frac{1}{3}\right)_m \left(\frac{2}{3}\right)_m (6b+1)_{3m}(2b+1)_{2m}(3)^{3m}}{(1+2b)_m(3b+1)_{3m}(3b+\frac{1}{2})_{3m}(4)^{3m}} & ; n = 3m, \end{cases} \tag{3}
$$

where $m = 0, 1, 2, 3, \ldots$

(also, reversing order of the terms in finite summation theorem of George Andrews [26; 4, Eq.(1.12); see also p.16, Eq.(4.8)])

$$
{}_3F_2 \left[\begin{array}{c} -n, \frac{1-3b-2n}{2}, \frac{2-3b-2n}{2} \, ; \\ 1-b-n, 1-3b-2n \, ; \end{array} \frac{4}{3} \right] = \begin{cases} 0 & ; n = 3m+1 \ \text{and} \ 3m+2, \\ \frac{(3m)!(-1)^m (b)_m}{m!(b)_{3m}(3)^{3m}} & ; n = 3m, \end{cases} \tag{4}
$$

where $m = 0, 1, 2, 3, ...$

Our objective is to introduce two double-series identities. These identities incorporating bounded sequences of complex numbers are derived using series rearrangement techniques and Pochhammer symbol identities. These issues are further discussed in Section 2. In Section 3, we employ these general double-series identities to establish two reduction formulas for Srivastava-Daoust double hypergeometric function in terms of generalized hypergeometric functions with arrangements proportional to z^3 and $-z^3$. We achieve this by using Cauchy's double series identity (see, e.g., [27; 56])

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Theta(n, r) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\Theta(n-r, r), \tag{5}$$

provided that the associated double series are absolutely convergent. We also have the following identities involving the Pochhammer symbol:

$$(-n)_r = \frac{n!(-1)^r}{(n-r)!}; \qquad 0 \le r \le n, \tag{6}$$

$$\prod_{i=1}^{D}(d_i)_{3n} = 3^{3nD}\prod_{i=1}^{D}\left(\frac{d_i}{3}\right)_n \prod_{i=1}^{D}\left(\frac{1+d_i}{3}\right)_n \prod_{i=1}^{D}\left(\frac{2+d_i}{3}\right)_n, \tag{7}$$

$$\prod_{j=1}^{E}(e_j)_{3n} = 3^{3nE}\prod_{j=1}^{E}\left(\frac{e_j}{3}\right)_n \prod_{j=1}^{E}\left(\frac{1+e_j}{3}\right)_n \prod_{j=1}^{E}\left(\frac{2+e_j}{3}\right)_n. \tag{8}$$

Remark 1.1 Wolfram's MATHEMATICA has implemented the $_pF_q$ function as Hypergeometric PFQ, which is appropriate for performing both symbolic and numerical computations.

Throughout this article, we assume that any values of parameters and arguments, which would render the results in Sections 2 to 3 invalid or undefined, are tacitly excluded.

2 Two general double-series identities

This section demonstrates two double-series identities that involve bounded sequences by primarily utilizing Gessel-Stanton and George Andrews (3) and (4). The first identity takes the following form:

Theorem 1. Let $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0) \ne 0$. Then, the following general double-series identity holds true:

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n(4)^r r!n!}$$

$$= \sum_{n=0}^{\infty}\Psi(3n)\frac{\left(\frac{6b+1}{3}\right)_n\left(\frac{6b+2}{3}\right)_n z^{3n}}{\left(\frac{3b+1}{3}\right)_n\left(\frac{3b+2}{3}\right)_n\left(\frac{6b+1}{6}\right)_n\left(\frac{6b+5}{6}\right)_n(432)^n n!} \tag{9}$$

provided $\left(-2b, \frac{1}{2}+3b, 1+3b, 1+6b, \frac{3b+1}{3}, \frac{3b+2}{3}, \frac{6b+1}{6}, \frac{6b+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right)$, and the infinite series occurring on both sides of equation (9), are absolutely convergent.

Proof.

$$\text{Let} \qquad \Xi_1(z) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\Psi(n+r)\frac{(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n(4)^r r!n!}. \tag{10}$$

Replacing n by $(n-r)$ in equation (10) and using Cauchy's double-series identity (5), we have

$$\Xi_1(z) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \Psi(n) \frac{\left(-2b - \frac{2n}{3}\right)_r (-6b-n)_r (-3)^r z^n}{(-3b-n)_r \left(\frac{1}{2} - 3b - n\right)_r (4)^r (n-r)! r!}. \tag{11}$$

Multiplying numerator and denominator by $n!$ and using Pochhammer symbol identity (6) to the right hand side of equation (11), we obtain

$$\Xi_1(z) = \sum_{n=0}^{\infty} \Psi(n) \frac{z^n}{n!} \sum_{r=0}^{n} \frac{(-n)_r \left(-2b - \frac{2n}{3}\right)_r (-6b-n)_r (3)^r}{(-3b-n)_r \left(\frac{1}{2} - 3b - n\right)_r (4)^r \, r!}$$

$$= \sum_{n=0}^{\infty} \Psi(n) \frac{z^n}{n!} \, {}_3F_2 \left[\begin{array}{c} -n, -2b - \frac{2n}{3}, -6b - n; \\ \\ \frac{1}{2} - 3b - n, -3b - n; \end{array} \, \frac{3}{4} \right]. \tag{12}$$

We now apply the decomposition identity

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(3n) + \sum_{n=0}^{\infty} \Phi(3n+1) + \sum_{n=0}^{\infty} \Phi(3n+2),$$

provided that each of the sums is absolutely convergent, to the right-hand side of (12). This produces

$$\Xi_1(z) = \sum_{n=0}^{\infty} \Psi(3n) \frac{z^{3n}}{(3n)!} \, {}_3F_2 \left[\begin{array}{c} -(3n), -2b - 2n, -6b - 3n; \\ \\ -3b - 3n, \frac{1}{2} - 3b - 3n; \end{array} \, \frac{3}{4} \right] + \sum_{n=0}^{\infty} \Psi(3n+1) \frac{z^{3n+1}}{(3n+1)!} \times$$

$$\times {}_3F_2 \left[\begin{array}{c} -(3n+1), -2b - 2n - \frac{2}{3}, -6b - 3n - 1; \\ \\ -3b - 3n - 1, -3b - 3n - \frac{1}{2}; \end{array} \, \frac{3}{4} \right] + \sum_{n=0}^{\infty} \Psi(3n+2) \frac{z^{3n+2}}{(3n+2)!} \times$$

$$\times {}_3F_2 \left[\begin{array}{c} -(3n+2), -6b - 2n - \frac{4}{3}, -6b - 3n - 2; \\ \\ -3b - 3n - 2, -3b - 3n - \frac{3}{2}; \end{array} \, \frac{3}{4} \right]. \tag{13}$$

Finally, using the summation theorem (3) to the right hand side of equation (13), we get

$$\Xi_1(z) = \sum_{n=0}^{\infty} \Psi(3n) \frac{z^{3n}}{(3n)!} \left[\frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n (6b+1)_{3n} (2b+1)_{2n} (3)^{3n}}{(2b+1)_n (3b+1)_{3n} \left(\frac{6b+1}{2}\right)_{3n} (4)^{3n}} \right].$$

After further simplification, we get the required result (9).

The second identity is given by the following theorem:

Theorem 2. Let $\{\Psi(\mu)\}_{\mu=1}^{\infty}$ be a bounded sequence of essentially arbitrary complex numbers or real numbers such that $\Psi(0) \neq 0$. Then, the following general double-series identity holds true:

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \Psi(n+r) \frac{(3b)_{2n+r} \left(\frac{1+3b}{2}\right)_{n+r} \left(\frac{3b}{2}\right)_{n+r} (b)_n (-4)^r z^{n+r}}{(3b)_{2n+2r} (b)_{n+r} \left(\frac{1+3b}{2}\right)_n \left(\frac{3b}{2}\right)_n (3)^r \, r! \, n!}$$

$$= \sum_{n=0}^{\infty} \Psi(3n) \frac{(b)_n (-z^3)^n}{\left(\frac{b}{3}\right)_n \left(\frac{b+1}{3}\right)_n \left(\frac{b+2}{3}\right)_n (729)^n \, n!} \tag{14}$$

provided $\left(3b, \frac{1+3b}{2}, b, \frac{3b}{2}, \frac{b}{3}, \frac{b+1}{3}, \frac{b+2}{3} \in \mathbb{C} \backslash \mathbb{Z}_0^- \right)$, and the infinite series occurring on both sides of equation (14) are absolutely convergent.

Proof.

Let
$$\Xi_2(z) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \Psi(n+r) \frac{(3b)_{2n+r} \left(\frac{1+3b}{2}\right)_{n+r} \left(\frac{3b}{2}\right)_{n+r} (b)_n (-4)^r z^{n+r}}{(3b)_{2n+2r} (b)_{n+r} \left(\frac{1+3b}{2}\right)_n \left(\frac{3b}{2}\right)_n (3)^r \; r! \; n!}. \tag{15}$$

Replacing n by $(n-r)$ in equation (15) and using Cauchy's double-series identity (5), we have

$$\Xi_2(z) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \Psi(n) \frac{\left(\frac{1-3b-2n}{2}\right)_r \left(\frac{2-3b-2n}{2}\right)_r (-4)^r z^n}{(1-b-n)_r (1-3b-2n)_r (3)^r \; r! \; (n-r)!}. \tag{16}$$

Multiplying numerator and denominator by $n!$ and using Pochhammer symbol identity (6) to right hand side of equation (16), we obtain

$$\Xi_2(z) = \sum_{n=0}^{\infty} \Psi(n) \frac{z^n}{n!} \sum_{r=0}^{n} \frac{(-n)_r \left(\frac{1-3b-2n}{2}\right)_r \left(\frac{2-3b-2n}{2}\right)_r (4)^r}{(1-b-n)_r (1-3b-2n)_r (3)^r \; r!}$$

$$= \sum_{n=0}^{\infty} \Psi(n) \frac{z^n}{n!} \; {}_3F_2 \left[\begin{array}{cc} -n, \frac{1-3b-2n}{2}, \frac{2-3b-2n}{2}; & \\ & \frac{4}{3} \\ 1-b-n, 1-3b-2n; & \end{array} \right] \tag{17}$$

We now apply the decomposition identity

$$\sum_{n=0}^{\infty} \Phi(n) = \sum_{n=0}^{\infty} \Phi(3n) + \sum_{n=0}^{\infty} \Phi(3n+1) + \sum_{n=0}^{\infty} \Phi(3n+2),$$

provided that each of the sums is absolutely convergent, to the right-hand side of (17). This produces

$$\Xi_2(z) = \sum_{n=0}^{\infty} \Psi(3n) \frac{z^{3n}}{(3n)!} \; {}_3F_2 \left[\begin{array}{cc} -3n, \frac{1-3b-6n}{2}, \frac{2-3b-6n}{2}; & \\ & \frac{4}{3} \\ 1-b-3n, 1-3b-6n; & \end{array} \right] + \sum_{n=0}^{\infty} \Psi(3n+1) \frac{z^{3n+1}}{(3n+1)!} \times$$

$$\times {}_3F_2 \left[\begin{array}{cc} -(3n+1), \frac{-3b-6n-1}{2}, \frac{-3b-6n}{2}; & \\ & \frac{4}{3} \\ -b-3n, -3b-6n-1; & \end{array} \right] + \sum_{n=0}^{\infty} \Psi(3n+2) \frac{z^{3n+2}}{(3n+2)!} \times$$

$$\times {}_3F_2 \left[\begin{array}{cc} -(3n+2), \frac{-3b-6n-3}{2}, \frac{-3b-6n-2}{2}; & \\ & \frac{4}{3} \\ -b-3n-1, -3b-6n-3; & \end{array} \right]. \tag{18}$$

Finally, using the summation theorem (4) to the right hand side of equation (18), we get

$$\Xi_2(z) = \sum_{n=0}^{\infty} \Psi(3n) \frac{z^{3n}}{(3n)!} \left[\frac{(3n)!(-1)^n (b)_n}{n!(b)_{3n}(3)^{3n}} \right]$$

$$= \sum_{n=0}^{\infty} \Psi(3n) \frac{(b)_n (-z^3)^n}{(3)^{3n} (b)_{3n} n!}.$$

After simplification, we get the result (14).

3 Certain consequences of general double-series identities (9) and (14)

In this section, we establish a result for reducibility of Srivastava-Daoust double hypergeometric function as in the following theorem.

Theorem 3. The following results hold true:

$$F_{E+3:1;0}^{D+2:2;0} \left(\begin{array}{c} [(d_D):1,1],[-2b:-\frac{2}{3},\frac{1}{3}],[1+6b:1,1]:[\frac{1}{2}+3b:1],[1+3b:1] \; ;- \; ; \\ \\ [(e_E):1,1],[-2b:-\frac{2}{3},-\frac{2}{3}],[\frac{1}{2}+3b:1,1],[1+3b:1,1]:[1+6b:1] \; ;- \; ; \end{array} \; z, \frac{3z}{4} \right)$$

$$= \; _{2+3D}F_{4+3E} \left[\begin{array}{c} \Delta[3;(d_D)],\frac{6b+1}{3},\frac{6b+2}{3}; \\ \\ \Delta[3;(e_E)],\frac{3b+1}{3},\frac{3b+2}{3},\frac{6b+1}{6},\frac{6b+5}{6}; \end{array} \; \frac{z^3}{16\times(27)^{(1+E-D)}} \right], \qquad (19)$$

and

$$F_{E+2:2;0}^{D+3:1;0} \left(\begin{array}{c} [(d_D):1,1],[3b:2,1],[\frac{1+3b}{2}:1,1],[\frac{3b}{2}:1,1]: \qquad [b:1] \qquad ;- \; ; \\ \\ [(e_E):1,1],[3b:2,2],[b:1,1]:[\frac{3b}{2}:1],[\frac{1+3b}{2}:1];- \; ; \end{array} \; z, -\frac{4z}{3} \right)$$

$$= \; _{1+3D}F_{3+3E} \left[\begin{array}{c} \Delta[(3;(d_D)],b; \\ \\ \Delta[3;(e_E)],\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3}; \end{array} \; \frac{-z^3}{(27)^{(2+E-D)}} \right], \qquad (20)$$

where $(e_1,e_2,...,e_E,b,-2b,3b,1+6b,\frac{3b}{2},\frac{1+3b}{2},\frac{1+6b}{2},\frac{b}{3},\frac{b+1}{3},\frac{b+2}{3},\frac{3b+1}{3},\frac{3b+2}{3},\frac{6b+1}{6},\frac{6b+5}{6} \in \mathbb{C}\backslash\mathbb{Z}_0^-)$. When $D \leqslant E$ then above transformations are always convergent for $|z| < \infty$. When $D = 1 + E$ then above transformations are convergent for suitably constrained values of $|z|$.

Proof.

$$\text{Put} \qquad \Psi(\mu) = \frac{(d_1)_\mu (d_2)_\mu ... (d_D)_\mu}{(e_1)_\mu (e_2)_\mu ... (e_E)_\mu} = \frac{\prod_{i=1}^{D}(d_i)_\mu}{\prod_{i=1}^{E}(e_i)_\mu} \; ; \qquad \mu = 0,1,2,3,...,$$

on the both sides of general double-series identity (9), we obtain

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_{n+r}(-2b)_{-\frac{2n}{3}+\frac{r}{3}}(1+6b)_{n+r}\left(\frac{1}{2}+3b\right)_n(1+3b)_n(3)^r z^{n+r}}{\prod_{i=1}^{E}(e_i)_{n+r}(-2b)_{-\frac{2n}{3}-\frac{2r}{3}}\left(\frac{1}{2}+3b\right)_{n+r}(1+3b)_{n+r}(1+6b)_n(4)^r \; r!n!}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{D}(d_i)_{3n}\left(\frac{6b+1}{3}\right)_n\left(\frac{6b+2}{3}\right)_n z^{3n}}{\prod_{i=1}^{E}(e_i)_{3n}\left(\frac{3b+1}{3}\right)_n\left(\frac{3b+2}{3}\right)_n\left(\frac{6b+1}{6}\right)_n\left(\frac{6b+5}{6}\right)_n (432)^n n!}. \qquad (21)$$

Now applying the definition of double hypergeometric function (2) of Srivastava-Daoust to the left hand side of equation (21) and definition of the generalized hypergeometric function (1), together with the Pochhammer symbol identities (7) and (8) to the right hand side of equation (21), we get the desired result (19).

The proof of (20) follows exactly the same procedure and will be omitted. This completes the proof of Theorem 3.

4 Conclusions and Remarks

In our present investigation, we have obtained two general double-series identities by using the finite summation theorems of Gessel-Stanton and George Andrews for the terminating hypergeometric series $_3F_2$ with arguments $3/4$ and $4/3$ respectively. These results have been used to derive two reduction formulas for the (S-D function) with arguments $(z, 3z/4)$ and $(z, -4z/3)$ in terms of two generalized hypergeometric functions $_{2+3D}F_{4+3E}$ and $_{1+3D}F_{3+3E}$ with arguments $\frac{z^3}{16 \times 27^{(1+E-D)}}$ and $\frac{-z^3}{(27)^{(2+E-D)}}$ respectively. We believe that the results established in this paper have not appeared in the literature and represent a contribution to the theory of generalized hypergeometric functions of one and two variables. The various results, which we have presented in this article, are potentially useful in mathematical analysis and applied mathematics.

Acknowledgments

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Эндрюс пен Гессель-Стэнтонның қосындылау теоремаларын жүзеге асыру

Жалпыланған гипергеометриялық функциялар мен олардың бір және бірнеше айнымалылардағы заңды жалпылануы көптеген математикалық есептер мен олардың қосымшаларында кездеседі. Математикалық физиканың көптеген қолданбалы есептері бар дербес туындылы дифференциалдық теңдеулердің шешімі осындай жалпыланған гипергеометриялық функциялар арқылы өрнектеледі. Атап айтқанда, Шривастава-Дausттың қос гипергеометриялық функциясы (S-D функциясы) іргелі және қолданбалы математикадағы кең ауқымды есептердің шешімдерін ұсыну үшін өзінің практикалық пайдалылығын дәлелдеді. Мақалада сәйкесінше $3/4$ және $4/3$ аргументтері бар $_3F_2$ аяқталатын гипергеометриялық қатарға арналған Гессель-Стэнтон және Эндрюстің ақырлы қосындылар теоремаларын пайдалана отырып, еркін кешенді сандардың шектелген тізбегінен тұратын қос қатарлар үшін екі жалпы сәйкестендіру енгізілген. Осы қос қатар сәйкестіктерін пайдалана отырып, $z, 3z/4$ және $z, -4z/3$ аргументтері бар (S-D функциясы) екі келтіру формуласы дәлелденген, олар z^3 және $-z^3$ аргументіне пропорционал екі жалпыланған гипергеометриялық функциялар арқылы өрнектеледі. Сонымен қатар мақалада айтылған барлық нәтижелер «Mathematica» бағдарламасы арқылы сандық түрде тексерілді.

Кілт сөздер: жалпыланған гипергеометриялық функция, Шривастава-Дausттың қос гипергеометриялық функциясы, келтіру формулалары, «Mathematica» бағдарламасы.

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Реализация теорем суммирования Эндрюса и Гесселя–Стэнтона

Обобщенные гипергеометрические функции и их естественные обобщения от одной и нескольких переменных встречаются во многих математических задачах и их приложениях. Решение уравнений в частных производных, возникающих во многих прикладных задачах математической физики, выражается через такие обобщенные гипергеометрические функции. В частности, двойная гипергеометрическая функция Шриваставы–Дауста (S–D-функция) доказала свою практическую полезность для представления решений широкого круга задач фундаментальной и прикладной математики. В настоящей статье мы вводим два общих тождества двойных рядов, включающие ограниченные последовательности произвольных комплексных чисел, используя теоремы конечного суммирования Гесселя–Стэнтона и Эндрюса для завершающих гипергеометрических рядов $_3F_2$ с аргументами $3/4$ и $4/3$ соответственно. Используя данные тождества двойного ряда, устанавливаем две формулы приведения для (S–D-функции) с аргументами $z, 3z/4$ и $z, -4z/3$, выраженными через две обобщенные гипергеометрические функции с аргументами, пропорциональными z^3 и $-z^3$ соответственно. Все результаты, упомянутые в статье, проверены численно с использованием программы «Mathematica».

Ключевые слова: обобщенная гипергеометрическая функция, двойная гипергеометрическая функция Шриваставы–Дауста, формулы приведения, программа «Mathematica».