Kelvin-Voigt equations with memory: existence, uniqueness and regularity of solutions

In general, the study of inverse problems is realizable only in the case when the corresponding direct problems have the unique solution with some necessary properties such as continuity and regularity. In this paper, we study initial-boundary value problems for the system of 2D-3D nonlinear Kelvin-Voigt equations with memory, which describes a motion of an incompressible homogeneous non-Newtonian fluids with viscoelastic and relaxation properties. The investigation of these direct problems is related to the study of inverse problems for this system, which requires the continuity and regularity of solutions to these direct problems and their derivatives. The system, in addition to the initial condition, is supplemented with one of the boundary conditions: stick and slip boundary conditions. In both cases of these boundary conditions, the global in time existence and uniqueness of strong solutions to these initial-boundary value problems were proved. Moreover, under suitable assumptions on the data, the regularity of solutions and their derivatives were established.

Keywords: Kelvin-Voigt system, slip and stick boundary conditions, strong solutions, global existence and uniqueness, smoothness.

Introduction

Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with a smooth boundary $\partial \Omega$, and $Q_T = \Omega \times (0, T)$ be a cylinder with a lateral $\Gamma_T = \partial \Omega \times [0, T]$. Let us consider the following initial-boundary value problem for the system of nonlinear Kelvin-Voigt (Navier-Stokes-Voigt) equations with memory

$$\begin{align*}
 v_t + (v \cdot \nabla)v - \kappa \Delta v_t - \nu \Delta v - \int_0^t K(t - \tau) \Delta v(x, \tau) d\tau + \nabla p &= f, \quad (x, t) \in Q_T, \\
 \text{div} v(x, t) &= 0, \quad (x, t) \in Q_T,
\end{align*}$$

supplemented with the initial condition

$$v(x, 0) = v_0(x), \quad x \in \Omega \tag{3}$$

and one of the following boundary conditions: stick boundary condition

$$v(x, t) = 0, \quad (x, t) \in \Gamma_T \tag{4}$$

or slip boundary condition

$$v_n(x, t) = v \cdot n = 0, \quad \text{rot} v \times n = 0, \quad (x, t) \in \Gamma_T. \tag{5}$$

System (1)-(2) is called a Kelvin-Voigt (also called Navier-Stokes-Voigt) system with memory or an integro-differential Kelvin-Voigt system, and models a motion of viscoelastic incompressible
non-Newtonian fluids [1–5]. Most of hydrodynamics problems were considered with stick-boundary condition (4), however, in recent years works have appeared on initial-boundary value problems with a slip-boundary condition like (5), see for instance [6–8] et al. Because this is related to the fact that these boundary conditions have an important meaning for non-Newtonian fluids [9,10]. In the case of slip boundary condition (5), we assume that Ω is a simply connected bounded domain [11]. System (1)-(2), in some particular cases, can be considered as a nonlinear pseudoparabolic equation due to the term ∆vt, therefore all established below results will be hold true also for initial-boundary value problems for such type PDEs.

The issue of study of problems (1)–(4) and (1)–(3), (5) is aroused due to the investigation of inverse problems for system (1)-(2) that is supplemented with some additional conditions on solutions of the corresponding direct problem. In generally, the study of inverse problems are realizable only there is information such as unique solvability of the corresponding direct problems and smoothness of their solutions [12–14]. The direct problems for (1)-(2) with various statements have been studied before in some works as [5,7,15,16], where the existence and uniqueness of weak solutions were established. The existence, uniqueness, and the regularities of smooth solutions of the initial-boundary value problems for system (1)-(2) without the memory term have been investigated in [17] for homogeneous fluids, and in [18], in the case for non-homogeneous fluids. However, by our knowledge, there is not work for problems (1)–(4) and (1)–(3),(5). By this purpose, in this paper, we investigate the existence and uniqueness of strong solutions of problems (1)–(4) and (1)–(3),(5), and their regularities. First we work on problem (1)–(4) and the study problem (1)–(3),(5) is similar to the first one, therefore, we omit some details of proofs.

1 Preliminaries

In this section, we introduce the main functional spaces and some useful inequalities related to boundary conditions (4) and (5) from [8]. We distinguish vectors from scalars by using boldface letters. The symbol C will denote a generic constant – generally a positive one, value of which will not be specified; it can change from one inequality to another. We denote by L2(Ω) the usual Lebesgue space of square integrable vector-valued functions on Ω, and by Wm,2(Ω) the Sobolev space of functions in L2(Ω) whose weak derivatives of an order not greater than m are in L2(Ω). The norm and inner product in L2(Ω) denoted by ∥ · ∥2,Ω and ⟨ · , · ⟩2,Ω, respectively.

Let us introduce the function spaces regarding to the slip and stick boundary conditions (5) and (4), respectively (see [3,6]):

\[ H_n(Ω) \equiv \{ v ∈ L^2(Ω) : \text{div } v = 0, v_n|_{∂Ω} = 0 \}; \]
\[ H^1_n(Ω) \equiv \{ v ∈ W^1_2(Ω) : \text{div } v = 0, v_n|_{∂Ω} = 0 \}; \]
\[ H^2_n(Ω) \equiv \{ v ∈ H^1_n(Ω) ∩ W^{2,2}(Ω) : (\text{rot } v × n)|_{∂Ω} = 0 \}; \]
\[ H^1(Ω) \equiv \{ v ∈ H^1(Ω) ∩ W^{2,2}(Ω) \} \]

and for the simplicity, we use the following common notation for both cases

\[ V := \begin{cases} H(Ω), \text{ in the case (4)}; \\ H_n(Ω), \text{ in the case (5)}; \end{cases} \]
\[ V^i := \begin{cases} H^i(Ω), \text{ in the case (4)}; \\ H^i_n(Ω), \text{ in the case (5)}; \end{cases} \]

where i = 1, 2.

The scalar product and the norm in V^1_n(Ω) we define by \[ ⟨ \text{rot } v, \text{rot } u ⟩_{2,Ω} \] and \[ ∥ v ∥_{V^1_n(Ω)} := ∥ \text{rot } v ∥_{2,Ω}, \]
respectively. According to [3,6,8,11] and the references cited in them (see for example [9,19]), the following inequalities are hold:

Poincaré’s inequality
\[ ∥ v ∥_{2,Ω} ≤ C_1(Ω) ∥ \nabla v ∥_{2,Ω}, \quad v ∈ V^1(Ω); \] (6)
\[ N_1(Ω) ∥ v ∥_{W^{1,2}(Ω)} ≤ ∥ \text{rot } v ∥_{2,Ω} ≤ N_2(Ω) ∥ v ∥_{W^{1,2}(Ω)}, \quad ∀ v ∈ V^1(Ω); \]
\[ N_3(\Omega) \|v\|_{W^{2,2}(\Omega)} \leq \|\Delta v\|_{2,\Omega} = \|\text{rot rot } v\|_{2,\Omega} \leq N_4(\Omega) \|v\|_{W^{2,2}(\Omega)}, \quad \forall v \in V^2(\Omega); \quad (7) \]

and Ladyzhenskaya inequalities [6].

Let us introduce a bilinear and continuous form \( a \) on \( V^1 \), associated with the operator \( \Delta \):

\[ a(v, u) = \langle \nabla v, \nabla u \rangle_{2,\Omega}, \quad \forall v, u \in V^1(\Omega) \quad (8) \]

in case (4), and

\[ a(v, u) = \langle \text{rot } v, \text{rot } u \rangle_{2,\Omega}, \quad \forall v, u \in V^1(\Omega) \quad (9) \]

in case (5). It is clear that \( a(v, v) \) is a norm on \( V^1(\Omega) \), which is equivalent to \( W^{1,2}(\Omega) \)-norm. In particular, due to (6), in \( V^1 \) the norm \( \|\text{rot } v\|_{2,\Omega} \) is equivalent to the norm \( \|v\|_{W^{1,2}(\Omega)} \), and therefore equivalent to the norm \( \|\nabla v\|_{2,\Omega} \).

Thus, \( a \) defines an isomorphism \( A \) from \( V^1(\Omega) \) to \( V^{-1}(\Omega) \),

\[ \langle Av, u \rangle \equiv a(v, u), \quad \forall v, u \in V^1(\Omega), \]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing of \( V^1 \) and \( V^{-1} \). There hold the following continuous inclusions

\[ V^1(\Omega) \rightarrow L^2(\Omega) \rightarrow V^{-1}(\Omega), \]

where each of the first two spaces is dense in the next one.

It follows from (7) also that in \( V^2 \) the norm \( \|\Delta v\|_{2,\Omega} \) is equivalent to the norm \( \|\text{rot rot } v\|_{2,\Omega} \).

Regarding to sliding condition (5), we have the Green formulas (see [6] and [8, 9]):

\[ (-\Delta v, u)_{2,\Omega} = -\langle \nabla \text{div } v, u \rangle_{2,\Omega} + \langle \text{rot } v, \text{rot } u \rangle_{2,\Omega} = -\int_{\partial \Omega} \text{div } v \cdot u_n dS + \int_{\partial \Omega} u \cdot (\text{rot } v \times n) dS + \langle \text{rot } v, \text{rot } u \rangle_{2,\Omega} = (\text{rot } v, \text{rot } u)_{2,\Omega} \quad (10) \]

in case \( d = 3 \), and

\[ (-\Delta v, u)_{2,\Omega} = \langle \text{div } v, \text{div } u \rangle_{2,\Omega} + \langle \overline{\text{rot }} (\text{rot } v), u \rangle_{2,\Omega} = \int_{\partial \Omega} (\text{rot } v \times n) u dS + \langle \text{rot } v, \text{rot } u \rangle_{2,\Omega} = (\text{rot } v, \text{rot } u)_{2,\Omega}, \quad (11) \]

in case \( d = 2 \), where \( \overline{\text{rot }} \varphi \) is the vector \( (\varphi_{x_2}, -\varphi_{x_1})_{2,\Omega} \) for the scalar function \( \varphi \).

The regularity properties of solutions will be proved under the following lemma, which the proof is given in [20].

**Lemma 1.** If \( f \in L^p(0, T; X) \) and \( \frac{\partial f}{\partial t} \in L^p(0, T; X) \) \((1 \leq p \leq \infty)\), then \( f \), after, which can be changing on a set of measure zero (from segment \((0, T)\)) be a continuous mapping \([0, T] \rightarrow X\).

**Definition 1.** A vector function \( v(x, t) \) is a strong solution to problem (1)–(4) ((1)–(3), (5)) if:

1. \( v(x, t) \in C((0, T); V^1(\Omega) \cap V^2(\Omega)) \cap W^1_t((0, T); V^1(\Omega) \cap V^2(\Omega)); \)
2. Each equation in (1)–(4) ((1)–(3), (5)) holds in the distribution sense in the their corresponding domain.
2 Main results

Throughout the work, we assume that
\[ K(t) \in L^2([0,T]) \text{ and } \|K\|_{L^2([0,T])} \equiv K_0 < \infty. \] (12)

For the problems (1)–(4) and (1)–(3),(5) the following results are hold.

**Theorem 1.** Suppose that \( f \in L^2(0,T;L^2(\Omega)) \), \( v_0 \in V_1(\Omega) \cap V_2(\Omega) \), and (12) are hold. Then problems (1)–(4) and (1)–(3),(5) have a unique strong solution and the following estimate is valid
\[
\|v\|_{L^\infty(0,T;V_1(\Omega) \cap V_2(\Omega))} + \|vt\|_{L^2(0,T;V_1(\Omega) \cap V_2(\Omega))} + \|v\|_{L^2(0,T;V_1(\Omega))} \leq C < \infty,
\]
where \( C \) is a positive constant depending on data of the problem.

**Proof.** The proof consists of following steps: by Gelerkin’s method constructing a sequence of approximated solutions; obtaining a priori estimates and passage to limit.

### 2.1 Galerkin’s approximations

To prove the existence of a strong solution to problem (1)–(5), we use the Faedo–Galerkin method with a special basis of eigenfunctions of the spectral problem
\[
-\Delta \varphi_j + \nabla q = \lambda_j \varphi_j, \quad \varphi_j \in V_2(\Omega)
\]
is closely connected with problems (1)–(4) and (1)–(3),(5). In case (5), it is equivalent to the problem [6,8]
\[
\mathcal{A} \varphi_j \equiv -\Delta \varphi_j = \lambda_j \varphi_j, \quad \varphi_j \in V_2(\Omega)
\]
since \( \nabla q \equiv 0 \) due to the fact
\[
(\Delta \varphi, \nabla p) = 0, \text{ for any } \varphi \in V_2(\Omega) \text{ and any } p \in W_2^1(\Omega).
\]

For the problem (1)–(3), (4), \( \mathcal{A} \varphi_j \equiv \Delta \varphi_j \) [21]. Given \( m \in \mathbb{N} \), let us consider the \( m \)-dimensional spaces \( X^m \) spanned by the first \( m \) eigenfunctions \( \varphi_1,\ldots,\varphi_m \). For each \( m \in \mathbb{N} \), we search for approximate solutions in the form
\[
v^m(x,t) = \sum_{j=1}^{m} c^m_j(t) \varphi_j(x), \quad \varphi_j \in X^m,
\]
where unknown coefficients \( c^m_j(t) \), \( j = 1,\ldots,m \) are defined as solutions of the following system of ordinary differential equations derived from
\[
\frac{d}{dt} \left( (v^m, \varphi_k)_{2,\Omega} - \kappa (\Delta v^m, \varphi_k)_{2,\Omega} \right) + ((v^m \cdot \nabla) v^m, \varphi_k)_{2,\Omega} - \nu (\Delta v^m, \varphi_k)_{2,\Omega} -
\]
\[
- \int_0^t K(t-\tau) (\Delta v^m, \varphi_k)_{2,\Omega} d\tau = (f, \varphi_k)_{2,\Omega},
\] (13)
for $k = 1, 2, \ldots, m$. System (13) is supplemented with the Cauchy data

$$v^m(0) = v^m_0,$$

where

$$v^m_0 = \sum_{j=1}^{m} (v_0, \varphi_j)_{2, \Omega} \varphi_j$$

is a sequence in $L^2(\Omega) \cap V^1(\Omega)$ such that

$$v^m_0 \to v_0(x) \text{ as } m \to \infty \text{ in } V^1(\Omega) \cap V^2(\Omega).$$

According to a general theory of ordinary differential equations, Cauchy problem (13)--(14) has a solution $c^m(t)$ in $[0, T_s]$. By a priori estimates which we shall establish below, $[0, T_s]$ can be extended to $[0, T]$.

### 2.2 A priori estimates

**Lemma 2.** Assume that

$$f \in L^2(0, T; L^2(\Omega)), \quad v_0(x) \in V^1(\Omega),$$

and the conditions (12) and (15) are fulfilled. Then, for all $t \in [0, T]$, the following a priori estimate is valid

$$\|v^m\|_{L^\infty(0, T; V^1(\Omega))}^2 + \|v^m\|_{L^2(0, T; V^1(\Omega))}^2 \leq M_0 < \infty,$$

where $M_0$ is a positive constant depending only on data of the problem.

**Proof.** Multiply $k$-th equation of (13) by $c^m_k(t)$ and summing up from 1 to $m$, then using Green’s formulas (10)-(11), we obtain

$$\frac{d}{dt} \left( \|v^m\|^2_{2, \Omega} + \varkappa \|v^m\|^2_{V^1(\Omega)} \right) + \nu \|v^m\|^2_{V^1(\Omega)} =$$

$$= \int_0^t K(t - \tau) a(v^m(t), v^m(\tau)) d\tau + \langle f, v^m \rangle \equiv I_1,$$

where $a$ is defined by (8) and (9), regarding to the boundary conditions. Next, we estimate the terms on the right-hand side of (17) by Hölder’s and Young’s inequalities

$$I_1 \leq \int_0^t |K(t - \tau)| \|v^m(\tau)\|_{V^1(\Omega)} \|v^m(t)\|_{V^1(\Omega)} d\tau + \|v^m\|^2_{2, \Omega} \|f\|^2_{2, \Omega} \leq$$

$$\leq \frac{\nu}{2} \|v^m\|^2_{V^1(\Omega)} + \frac{K_0^2}{2\nu} \int_0^t \|v^m(\tau)\|^2_{V^1(\Omega)} d\tau + \frac{1}{2} \|v^m\|^2_{2, \Omega} + \frac{1}{2} \|f\|^2_{2, \Omega}.$$

Substituting last inequality into (17), and integrating by $s$ from 0 to $t$, we obtain

$$\|v^m\|^2_{2, \Omega} + \varkappa \|v^m\|^2_{V^1(\Omega)} + \nu \int_0^t \|v^m\|^2_{V^1(\Omega)} ds \leq \|v_0(x)\|^2_{2, \Omega} + \varkappa \|v_0(x)\|^2_{V^1(\Omega)} + \int_0^t \|v^m\|^2_{2, \Omega} ds +$$

$$+ \frac{K_0^2}{\nu} \int_0^t \|v^m(s)\|^2_{V^1(\Omega)} d\tau ds + \|f\|^2_{2, \Omega} \leq C_1 \int_0^t \left( \|v^m\|^2_{2, \Omega} + \varkappa \|v^m\|^2_{V^1(\Omega)} \right) ds + C_2,$$

where $C_1$ and $C_2$ are positive constants depending only on data of the problem.
where
\[ C_1 = \max\{1, \frac{K_0^2 T}{\nu \kappa}\}, \quad C_2 = \|v_0(x)\|_{2, \Omega}^2 + \kappa \|v_0(x)\|_{V^1(\Omega)}^2 + \|f\|_{2, Q_T}^2. \]

Omitting the third term on the left hand side of (18) and applying Grönwall’s lemma and taking supremum, we get
\[
\sup_{t \in [0, T]} \left( \|v^m\|_{2, \Omega}^2 + \|v^m\|_{V^1(\Omega)}^2 \right) \leq C_3 < \infty, \tag{19}
\]
where \( C_3 = C_3(\nu, \kappa, T, C_1, C_2) \). Plugging (19) with (18), we obtain the first energy estimate (16).

**Lemma 3.** Assume that all conditions of Lemma 2 are fulfilled. Then the following estimate is valid
\[
\|v^m\|_{L^\infty(0, T; V(\Omega) \cap V^1(\Omega))}^2 + \|v^m\|_{L^2(0, T; V(\Omega) \cap V^1(\Omega))}^2 \leq M_1 < \infty, \quad \forall t \in [0, T], \tag{20}
\]
where \( M_1 \) is a positive constant depending on data of the problem.

**Proof.** Multiplying both sides of \( k \)th equation of (13) by \( \frac{dv^m}{dt} \) and summing up from \( k = 1 \) to \( k = m \), we obtain
\[
\|v^m\|_{2, \Omega}^2 + \nu \|v^m\|_{V^1(\Omega)}^2 + \nu \frac{d}{dt} \|v^m\|_{V^1(\Omega)}^2 = (\nu (\nu, \nu) v^m, v^m)_{2, \Omega} - \int K(t - \tau) a(v^m(\tau), v^m(t)) d\tau + (f, v^m)_{2, \Omega} \equiv I_{21} + I_{22}. \tag{21}
\]

Using Hölder and Ladyzhenskaya together with Young inequalities, we have
\[
I_{21} \leq \|((\nu \nu, \nu) v^m, v^m)\| \leq \|v^m\|_{4, \Omega} \|v^m\|_{4, \Omega}^2 \leq \frac{\varepsilon_1}{2} \|v^m\|_{V^1(\Omega)}^2 + \frac{C^4(\Omega)}{2\varepsilon_1} \|v^m\|_{V^1(\Omega)}^4, \tag{22}
\]
\[
I_{22} \leq \frac{\varepsilon_2}{2} \|v^m(t)\|_{V^1(\Omega)}^2 + \frac{K_2^2}{2\varepsilon_2} \int_0^t \|v^m(\tau)\|_{V^1(\Omega)}^2 d\tau + \frac{\varepsilon_3}{2} \|v^m(t)\|_{2, \Omega}^2 + \frac{1}{2\varepsilon_3} \|f\|_{2, \Omega}^2. \tag{23}
\]

Plugging (22)-(23) with \( \varepsilon_i = \frac{\varepsilon}{3}, i = 1, 2, 3 \) into (21), and integrating the result by \( s \) in \([0, t], \ t \leq T\), we have
\[
\nu \|v^m\|_{V^1(\Omega)}^2 + 2 \|v^m(t)\|_{2, Q_T}^2 + \kappa \|v^m(t)\|_{L^2(0, T; V^1(\Omega))}^2 \leq \nu \|v_0\|_{V^1(\Omega)}^2 + \frac{3C^4(\Omega)}{\kappa} M_0^2 + \frac{3K_2^2}{\kappa} M_0 T + \frac{3}{\kappa} \|f\|_{2, Q_T}^2 := C_4 < \infty
\]
which follows that (20).

**Lemma 4.** Assume that in addition to the conditions of Lemma 2 holds
\[
v_0 \in V(\Omega) \cap V^2(\Omega).
\]
Then for all \( t \in [0, T] \), the estimate is valid
\[
\sup_{t \in [0, T]} \|A v^m\|_{2, \Omega}^2 + \|A v^m\|_{2, Q_T}^2 \leq M_2 < \infty, \tag{24}
\]
where \( A = \tilde{\Delta} \) for the problem with (4), and \( A = \Delta \) for the problem with (5).
By applying Granwall’s lemma and the standard techniques, we get from (27) estimate (24).

\[
\nu \| \mathbf{v}^m \|^2_{\mathbf{V}^1(\Omega)} + \kappa \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} + \frac{\nu}{2} \frac{d}{dt} \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} = (\nabla \cdot \mathbf{v}^m, \mathbf{A} \mathbf{v}^m)_{2,\Omega} - \\
\int_0^t K(t - \tau) (\Delta \mathbf{v}^m(\tau), \mathbf{A} \mathbf{v}^m(\tau))_{2,\Omega} d\tau + (\mathbf{f}(t), \mathbf{A} \mathbf{v}^m)_{2,\Omega} = I_{31} + I_{32}.
\]

Estimating the terms on right hand side (25) by using Hölder, Ladyzhenskaya, Sobolev and Young inequalities, we get the following inequalities

\[
I_{31} \leq \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} \| \mathbf{v}^m \|^4_{4,\Omega} \| \nabla \mathbf{v}^m \|^4_{4,\Omega} \leq \frac{\varepsilon_1}{2} \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} + \frac{C^2(\Omega)}{2\varepsilon_1} \| \mathbf{v}^m \|^2_{\mathbf{V}^1(\Omega)} \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega},
\]

\[
I_{32} \leq \frac{\varepsilon_2}{2} \| \mathbf{A} \mathbf{v}^m(t) \|^2_{2,\Omega} + \frac{K^2_0}{2\varepsilon_2} \int_0^t \| \mathbf{A} \mathbf{v}^m(\tau) \|^2_{2,\Omega} d\tau + \frac{\varepsilon_3}{2} \| \mathbf{A} \mathbf{v}^m(t) \|^2_{2,\Omega} + \frac{1}{2\varepsilon_3} \| \mathbf{f} \|^2_{2,\Omega},
\]

where \(C(\Omega)\) is a constant from embedding inequalities.

Substituting (26) with \(\varepsilon_i = \frac{\varepsilon_i}{2}, i = 1, 2, 3\) into (25) and integrating the result by \(\tau \in (0, t)\) and using estimates (16), (20), we have

\[
\nu \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} + \int_0^t \| \mathbf{v}^m(t) \|^2_{\mathbf{V}^1(\Omega)} d\tau + \kappa \int_0^t \| \mathbf{A} \mathbf{v}^m(\tau) \|^2_{2,\Omega} d\tau \leq C_5 \int_0^t \| \mathbf{A} \mathbf{v}^m(\tau) \|^2_{2,\Omega} d\tau + C_6,
\]

where

\[
C_5 = \frac{3K_0}{\kappa} (C^2(\Omega) + K_0 T), \quad C_6 = \nu \| \mathbf{v}_0 \|^2_{\mathbf{V}^2(\Omega)} + \frac{3}{\kappa} \| \mathbf{f} \|^2_{2,\Omega}.
\]

By applying Granwall’s lemma and the standard techniques, we get from (27) estimate (24).

Along with the above estimates, one can establish the following more regular estimate assuming an additional smoothness for data.

**Lemma 5.** Assume that in addition to the conditions of Lemma 5 holds

\[
\mathbf{f} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)).
\]

Then for all \(t \in [0, T]\) the following estimate is valid

\[
\sup_{t \in [0, T]} \| \mathbf{v}^m(t) \|^2_{\mathbf{V}^1(\Omega)} + \sup_{t \in [0, T]} \| \mathbf{A} \mathbf{v}^m(t) \|^2_{2,\Omega} \leq M_3 < \infty.
\]

**Proof.** Let us multiply the \(k^{th}\) equation of (13) by \(-\mu_k \frac{d \mathbf{v}^m(t)}{dt}\), and sum with respect to \(k\), from 1 to \(m\). Then we have

\[
\| \mathbf{v}^m(t) \|^2_{\mathbf{V}^1(\Omega)} + \kappa \| \mathbf{A} \mathbf{v}^m \|^2_{2,\Omega} = (\nabla \cdot \mathbf{v}^m, \mathbf{A} \mathbf{v}^m)_{2,\Omega} - \\
\int_0^t K(t - \tau) (\Delta \mathbf{v}^m(\tau), \mathbf{A} \mathbf{v}^m(\tau))_{2,\Omega} d\tau + (\mathbf{f}(t), \mathbf{A} \mathbf{v}^m)_{2,\Omega} = I_{41} + I_{42},
\]

where

\[
I_{41} = ((\nabla \cdot \mathbf{v}^m) \mathbf{v}^m, \mathbf{A} \mathbf{v}^m)_{2,\Omega}.
\]
Choosing $\varepsilon_t$ Now, taking the supremum by inequality $I$, Estimating compactness lemma, it follows that

On the other hand, due to the compact embedding

Let $\zeta_t \in \Omega$ be an arbitrary function. Multiplying (13) by $\zeta(t)$ and integrating the result by $t$ from 0 to $T$, we obtain

Choosing $\varepsilon_i$, $i = 1, 2, 3, 4$ in (30), (31), and substituting into (29), we get

Now, taking the supremum by $t \in [0, T]$ on both sides of (32), and using (20) and (24), we obtain

2.3 Passage to the limit as $m \to \infty$

By means of reflexivity and up to some subsequences, estimates (16), (20), (28) imply that

On the other hand, due to the compact embedding $W^{1,2}_0(\Omega) \hookrightarrow L^2(\Omega)$ and the Aubin-Lions compactness lemma, it follows that

Let $\zeta(t) \in C_0^\infty([0, T])$ be an arbitrary function. Multiplying (13) by $\zeta(t)$ and integrating the result by $t$ from 0 to $T$, we obtain

$$
I_{42} = - \int_0^t K(t - \tau) (\Delta \mathbf{v}^m(\tau), \mathbf{A}\mathbf{v}_t^m(\tau) + (f(t), \mathbf{A}\mathbf{v}_t^m) - \nu (\Delta \mathbf{v}^m, \mathbf{A}\mathbf{v}_t^m)_{2, \Omega}.
$$

Estimating $I_{41}$ and $I_{42}$ by using Hölder and Cauchy inequalities as above, we obtain the following inequality

$$I_{41} \leq \|\mathbf{A}\mathbf{v}_t^m\|_{2, \Omega} \|\mathbf{v}^m\|_{4, \Omega} \|
abla \mathbf{v}^m\|_{4, \Omega} \leq \frac{\varepsilon_1}{2} \|\mathbf{A}\mathbf{v}_t^m\|^2_{2, \Omega} + \frac{C^2(\Omega)}{2\varepsilon_1} \|\mathbf{v}^m\|^2_{V^1(\Omega)} \|\mathbf{A}\mathbf{v}_t^m\|^2_{2, \Omega}, \quad (30)
$$

$$I_{42} \leq \frac{\varepsilon_2}{2} \|\mathbf{A}\mathbf{v}_t^m(t)\|^2_{2, \Omega} + \frac{K_0}{2\varepsilon_2} \int_0^t \|\Delta \mathbf{v}^m(\tau)\|^2_{2, \Omega} d\tau + \frac{\varepsilon_3}{2} \|\mathbf{A}\mathbf{v}_t^m(t)\|^2_{2, \Omega} + \frac{1}{2\varepsilon_3} \|f\|^2_{2, \Omega} + \frac{\nu}{2\varepsilon_4} \|\Delta \mathbf{v}^m\|^2_{2, \Omega} + \frac{\varepsilon_4}{2} \|\mathbf{A}\mathbf{v}_t^m(t)\|^2_{2, \Omega}. \quad (31)
$$

Choosing $\varepsilon_i = \frac{\varepsilon_i}{2}$, $i = 1, 2, 3, 4$ in (30), (31), and substituting into (29), we get

$$\|\mathbf{v}_t^m\|^2_{V^1(\Omega)} + \frac{\varepsilon}{2} \|\mathbf{A}\mathbf{v}_t^m\|^2_{2, \Omega} \leq \frac{2C^2(\Omega)}{\varepsilon} \|\mathbf{v}^m\|^2_{V^1(\Omega)} \|\mathbf{A}\mathbf{v}_t^m\|^2_{2, \Omega} + \frac{2K_0}{\varepsilon} \int_0^t \|\Delta \mathbf{v}^m(\tau)\|^2_{2, \Omega} d\tau + \frac{2}{\varepsilon} \|f\|^2_{2, \Omega} + \frac{2\nu}{\varepsilon} \|\Delta \mathbf{v}^m\|^2_{2, \Omega}. \quad (32)
$$

By means of reflexivity and up to some subsequences, estimates (16), (20), (28) imply that

$$
\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-* in } L^\infty(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \quad (33)
$$

$$
\mathbf{v}^m \to \mathbf{v} \quad \text{weakly in } L^2(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \quad (34)
$$

$$
\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly-* in } L^2(0, T; \mathbf{V}(\Omega) \cap \mathbf{V}^1(\Omega)), \quad \text{as } m \to \infty, \quad (35)
$$

$$
\mathbf{v}^m \rightharpoonup \mathbf{v} \quad \text{weakly-* in } L^\infty(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty, \quad (36)
$$

$$
\mathbf{v}^m \to \mathbf{v} \quad \text{weakly in } L^2(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty, \quad (37)
$$

$$
\mathbf{v}_t^m \rightharpoonup \mathbf{v}_t \quad \text{weakly in } L^2(0, T; \mathbf{V}^2(\Omega)), \quad \text{as } m \to \infty. \quad (38)
$$

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for \( k \in \{1, \ldots, m\} \). Then, fixing \( k \), we can pass in equation (40) to the limit \( m \to \infty \), by using the convergence results (33)–(39). Then, we obtain
\[
\int_{Q_T} v_t \cdot \varphi_k \zeta \, dx \, dt + \int_{Q_T} (v \cdot \nabla) v \cdot \varphi_k \zeta \, dx \, dt + \nu \int_{Q_T} \Delta v \cdot \varphi_k \zeta \, dx \, dt + \kappa \int_{Q_T} \Delta v_t \cdot \varphi_k \zeta \, dx \, dt = \int_{Q_T} \int_0^T K(\tau - s) \Delta v \cdot \varphi_k \zeta \, ds \, d\tau + \int_{Q_T} f \cdot \varphi_k \zeta \, dx \, dt.
\]
for \( k \in \{1, \ldots, m\} \).

By linearity, equation (41) holds for any finite linear combination of \( \{z_k = \varphi_k \cdot \zeta(t)\}_{k=1}^m \) with \( \zeta(t) \in C^\infty_0([0,T]) \), and, by a continuity argument, it is still true for any \( z \in L^2(0,T; \mathbf{V}(\Omega)) \). Hence, we can see that \( v \) satisfies to
\[
\int_{Q_T} v_t \cdot z \, dx \, dt + \int_{Q_T} (v \cdot \nabla) v \cdot z \, dx \, dt + \nu \int_{Q_T} \Delta v \cdot z \, dx \, dt + \kappa \int_{Q_T} \Delta v_t \cdot z \, dx \, dt = \int_{Q_T} \int_0^T K(\tau - s) \Delta v \cdot z \, ds \, d\tau + \int_{Q_T} f \cdot z \, dx \, dt,
\]
i.e. \( v \) is a strong solution to problem (1)–(4).

### 3 Regularity of solutions

**Theorem 2.** Let all conditions of Theorem 1 be fulfilled. Then
\[
v \in C(0,T; \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)), \ p \in C(0,T; G(\Omega)).
\]
If, in addition,
\[
f \in C(0,T; L^2(\Omega))
\]
holds, then for all \( t \in (0,T) \)
\[
v \in C^1(0,T; \mathbf{V}(\Omega) \cap \mathbf{V}^2(\Omega)), \ p \in C(0,T; G(\Omega))
\]
holds.

**Proof.** Embedding (42) follows from Lemma 1, under estimates (16), (20), (24). The second assertion follows from the embedding theorems under the estimates from [20,22].

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Жады мүшесі бар Кельвин-Фойгт тендеулері: шешімдердің бар болуы, жалғыздығы және регулярлығы

Жалпы алғанда керек есептерді зерттеу өлгөр айкес келетін турға есептердің біріндегі шешімділігі және шешімдерінің ызілісіздігі мен қорғаға регулярлықта сияқты қойғыр қаққестерге не болған жағдайлда жұқе жасауы асқыралады. Макала ұлға дәрісінің және релаксациялық қаққестерге есептерді ескерілген сығындығының біртекті ұлға сүйкіштарының қозғалысының сипаттайды жағдайда мүшесі бар 2D–3D тұрғымды сияқты өлім Кельвин-Фойгт тендеулері жүйесінің үшін қойылған бастапқыш-құрлықты есептер әртістелген. Бұл турға есептерді зерттеу өсі жұқе үшін қойылған керек есептерді зерттеумен байланысты. Себебі, олардың турға есептердің шешімдері және олардың тұындыларының ызілісіздігі мен регулярлықта сияқты қаққестері қаққесті етіледі. Карастырылып отырған есептерде тендеулерің жүйесі бастықшасы шартпен қатар жұқе және сұрынған сияқтың шекаралық шарттарының бірімен толықтыралады. Осы ең шекаралық шарттар жүйедің бастықшасы шекаралық шарттык ерекшелік екі және олардың тұындыларының ызілісіздігі мен регулярлығы тармақтағы тәуелділік.

Кілт сөздер: Кельвин-Фойгт жүйесі, жұқе және сұрынғаның шекаралық шарттары, олім, шешімдер, глобальдық бар болуы және жалғыздығы, тегістік.

Уравнения Кельвина–Фойгта с памятью: существование, единственность и регулярность решений

В общем случае изучение обратных задач осуществляется только в том случае, когда соответствующие прямые задачи имеют единственное решение, обладающее некоторыми необходимыми свойствами, такими как непрерывность и регулярность. В статье исследованы начально-краевые задачи для системы 2D–3D нелинейных уравнений Кельвина–Фойгта с памятью, описывающей движение несжимаемой однородной ньютоновской жидкости с вязкоупругими и релаксационными свойствами. Исследование таких прямых задач сопряжено с изучением соответствующих обратных задач для данной системы, которое требует свойств как непрерывности и регулярности решения и их производных этих прямых задач решений. Система уравнений, помимо начального условия, дополняется одним из граничных
условий: условием прилипания или скольжения. В обоих случаях доказано глобальное во времени существование и единственность сильных решений этих начально-краевых задач. Более того, при соответствующих предположениях на данные была установлена регулярность решений и их производных.

Ключевые слова: система Кельвина–Фойгта, граничные условия скольжения и прилипания, сильные решения, глобальное существование и единственность, гладкость.

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