DOI 10.31489/2023M4/41-55

UDC 517.956.32

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# Boundary value problems with displacement for one mixed hyperbolic equation of the second order

The paper studies two nonlocal problems with a displacement for the conjugation of two equations of second-order hyperbolic type, with a wave equation in one part of the domain and a degenerate hyperbolic equation of the first kind in the other part. As a non-local boundary condition in the considered problems, a linear system of FDEs is specified with variable coefficients involving the first-order derivative and derivatives of fractional (in the sense of Riemann-Liouville) orders of the desired function on one of the characteristics and on the line of type changing. Using the integral equation method, the first problem is equivalently reduced to a question of the solvability for the Volterra integral equation of the second kind with a weak singularity; and a question of the solvability for the second problem is equivalently reduced to a question of the solvability for the Fredholm integral equation of the second kind with a weak singularity. For the first problem, we prove the uniform convergence of the resolvent kernel for the resulting Volterra integral equation of the second kind and we prove that its solution belongs to the required class. As to the second problem, sufficient conditions are found for the given functions that ensure the existence of a unique solution to the Fredholm integral equation of the second kind with a weak singularity of the required class. In some particular cases, the solutions are written out explicitly.

*Keywords:* wave equation, degenerate hyperbolic equation of the first kind, Volterra integral equation, Fredholm integral equation, Tricomi method, method of integral equations, methods of fractional calculus theory.

### *Introduction. Notation. Formulation of the problem*

In the Euclidean plane with independent variables $x$ and $y$ we consider the equation

$$0 = \begin{cases} (-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x, & y < 0, \\ u_{xx} - u_{yy} + f, & y > 0, \end{cases} \tag{1}$$

where $\lambda$, $m$ are given numbers, and $m > 0$, $|\lambda| \leq \frac{m}{2}$; $f = f(x, y)$ is the given function; $u = u(x, y)$ is the desired function.

Equation (1) for $y < 0$ coincides with the equation form

$$(-y)^m u_{xx} - u_{yy} + \lambda (-y)^{\frac{m-2}{2}} u_x = 0, \tag{2}$$

and for $y > 0$ equation (1) is a inhomogeneous wave equation

$$u_{xx} - u_{yy} + f(x, y) = 0. \tag{3}$$

Equation (2) belongs to the class of degenerate hyperbolic equations of the first kind [1; 21]. An important property in equation (2) is that at $|\lambda| \leq \frac{m}{2}$ the Cauchy problem is valid in its ordinary formulation with type degeneration along the line $y = 0$, even though it violates Protter condition [2].

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At $m = 2$ equation (2) turns into the Bitsadze-Lykov equation [3; 37], [4], [5; 234], while when $\lambda = 0$ equation (2) turns into the Gellerstedt equation [6], applicable to determine the shape of a slot in a dam [7; 234]. A special case of equation (2) is also the Tricomi equation, which plays an important role in the theory of aerodynamics and gas dynamics [8; 38], [9; 280], [10; 373] .

Equation (1) is considered in the domain $\Omega = \Omega_1 \cup \Omega_2 \cup I$, where $\Omega_1$ is the domain bounded by characteristics $\sigma_1 = AC : x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$, $\sigma_2 = CB : x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = r$ of equation (2) at $y < 0$, emanating from the point $C = (r/2, y_c)$, $y_c = -\left[\frac{(m+2)r}{4}\right]^{\frac{2}{m+2}}$, passing through the points $A = (0, 0)$ and $B = (r, 0)$, and the segment $I = AB$ of the line $y = 0$; $\Omega_2$ is the domain bounded by characteristics $\sigma_3 = AD : x - y = 0$, $\sigma_4 = BD : x + y = r$ of equation (3), emanating from the points $A$ and $B$, intersecting at the point $D = \left(\frac{r}{2}, \frac{r}{2}\right)$ and the line segment $I = AB$.

By a *regular* solution to Eq. (1) in the domain $\Omega$ we mean the function $u = u(x, y)$ which belongs to the class $C(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$, $u_x, u_y \in L_1(I)$, substitution of which turns Eq. (1) into an identity.

*Problem 1.* Find a regular solution of equation (1) in the domain $\Omega$ that satisfies the conditions

$$u[\theta_{r1}(x)] = \psi_1(x), \qquad 0 \le x \le r, \tag{4}$$

$$\alpha_1(x)(r-x)^{\beta_2} D_{rx}^{1-\beta_1}\{u[\theta_{r0}(t)]\} + \alpha_2(x) D_{rx}^{1-\beta} u(t, 0) + \alpha_3(x) u_y(x, 0) = \psi_2(x), \;\; 0 < x < r, \tag{5}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ are defined functions on the segment $0 \le x \le r$ and $\alpha_1^2(x) + \alpha_2^2(x) + \alpha_3^2(x) \ne 0 \; \forall x \in [0, r]$.

*Problem 2.* Find a regular solution to equation (1) in the domain $\Omega$ that satisfies the nonlocal condition (5) and the boundary condition

$$u[\theta_{01}(x)] = \psi_1(x), \qquad 0 \le x \le r, \tag{6}$$

where $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ defined functions on the segment $0 \le x \le r$ while $\alpha_1^2(x) + \alpha_2^2(x) + \alpha_3^2(x) \ne 0 \; \forall x \in [0, r]$.

Hence $\theta_{00}(x) = \left(\frac{x}{2}, -(2-2\beta)^{\beta-1} x^{1-\beta}\right)$; $\theta_{01}(x) = \left(\frac{x}{2}, \frac{x}{2}\right)$; $\theta_{r0}(x) = \left(\frac{r+x}{2}, -(2-2\beta)^{\beta-1}(r-x)^{1-\beta}\right)$; $\theta_{r1}(x) = \left(\frac{r+x}{2}, \frac{r-x}{2}\right)$ are affixes of intersection of characteristics emanating from the point $(x, 0)$ with characteristics of $AC$, $AD$, $BC$, $BD$ correspondingly; affixes of points $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$, $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$; $D_{cx}^{\alpha} g(t)$ is a fractional integro-differential operator (in the sense of Riemann-Liouville) of an order $|\alpha|$ with origin at the point $c$ [5], [7], [11].

The Goursat problem for a hyperbolic equation degenerating inside a domain was previously studied in [12, 13]. In [12], the criterion for the continuity of the solution to the Goursat problem for an equation of form (2) is studied and in [13], the solution to the Goursat problem for a model equation that degenerates inside the domain is written explicitly. Paper [14] considers the first boundary value problem for a hyperbolic equation degenerating inside a domain. Papers [15–17] study boundary value problems for degenerate hyperbolic equations in a characteristic quadrangle with data on opposite characteristics.

*Problems 1 and 2* formulated above and studied in this paper belong to the class of boundary problems with the Zhegalov-Nakhushev displacement [18–20]. Problems with a displacement for hyperbolic equations degenerate inside the domain were previously studied in [21–25]. Previously, various problems with a displacement for parabolic-hyperbolic type equations of the second and third orders were studied in the works [26, 27]. A more complete scientific literature review on boundary value problems with a displacement one can find in monographs [28–34]. As part of this work, we established sufficient conditions for the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x, y)$, for a unique regular solution to *problems 1 and 2* in the considered domain. In some special cases, the solutions are written out explicitly.

*Study of Problem 1*

The study of problem 1. The following Theorem holds.

*Theorem 1.* Assume the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f\,(x,\,y)$ are such that

$$\alpha_1(x), \alpha_2(x), \alpha_3(x), \psi_2(x) \in C[0, r] \cap C^2(0, r), \tag{7}$$

$$\psi_1(x) \in C^1[0, r] \cap C^2(0, r), \tag{8}$$

$$f\,(x,\,y) \in C\left(\bar{\Omega}_2\right), \tag{9}$$

and one of the below conditions is met: either

$$\alpha_3(x) - \gamma_2 \alpha_1(x) \neq 0 \quad \forall\; x \in [0, r] \tag{10}$$

*or*

$$\alpha_3(x) - \gamma_2 \alpha_1(x) \equiv 0,\;\; \alpha_2(x) + \gamma_1 \alpha_1(x) \neq 0 \quad \forall\; x \in [0, r]. \tag{11}$$

Then there exists a unique regular solution to Problem 1 in the domain $\Omega$.

*Proof.* Let there is a solution to problem (1), (4), (5) and let

$$u\,(x,\,0) = \tau(x), \quad 0 \le x \le r, \tag{12}$$

$$u_y\,(x,\,0) = \nu(x), \quad 0 < x < r. \tag{13}$$

At $|\lambda| \le \frac{m}{2}$ the solution to Cauchy problem (12)-(13) for equation (2) is written out according to one of the formulas [35; 14]

$$u\,(x,\,y) = \frac{1}{B\,(\beta_1,\,\beta_2)} \int\limits_0^1 \tau\left[x + (1-\beta)\,(-y)^{1/(1-\beta)}\,(2t-1)\right] t^{\beta_2 - 1}\;(1-t)^{\beta_1 - 1}\,dt +$$

$$+ \frac{y}{B\,(1-\beta_1,\,1-\beta_2)} \int\limits_0^1 \nu\left[x + (1-\beta)\,(-y)^{1/(1-\beta)}\,(2t-1)\right] t^{-\beta_1}\;(1-t)^{-\beta_2}\,dt, \quad |\lambda| < \frac{m}{2}, \tag{14}$$

$$u\,(x,\,y) = \tau\left[x + (1-\beta)\,(-y)^{1/(1-\beta)}\right] +$$

$$+ (1-\beta)\;y \int\limits_0^1 \nu\left[x + (1-\beta)\,(-y)^{1/(1-\beta)}\,(2t-1)\right]\,(1-t)^{-\beta}\,dt, \quad \lambda = \frac{m}{2}, \tag{15}$$

$$u\,(x,\,y) = \tau\left[x - (1-\beta)\,(-y)^{1/(1-\beta)}\right] +$$

$$+ (1-\beta)\;y \int\limits_0^1 \nu\left[x + (1-\beta)\,(-y)^{1/(1-\beta)}\,(1-2t)\right]\,(1-t)^{-\beta}\,dt, \quad \lambda = -\frac{m}{2}, \tag{16}$$

where $\tau(x) \in C[0, r] \cap C^2(0, r)$, $\nu(x) \in C^1(0, r) \cap L_1(0, r)$; $\beta_1 = \frac{m-2\lambda}{2(m+2)}$, $\beta_2 = \frac{m+2\lambda}{2(m+2)}$, $\beta = \beta_1 + \beta_2 = \frac{m}{m+2}$; $\Gamma\,(p) = \int\limits_0^\infty \exp\,(-t)\;t^{p-1}dt$, $B\,(p,\,q) = \int\limits_0^1 t^{p-1}\,(1-t)^{q-1}\,dt$ are Euler integrals of the first and second kind, $B\,(p,\,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$.

Consider first the case for $|\lambda| < \frac{m}{2}$. By (14) and taking into account (8), we have

$$u\left[\theta_{r0}(x)\right] = u\left(\frac{r+x}{2},\, -\left(2-2\beta\right)^{\beta-1}\left(r-x\right)^{1-\beta}\right) =$$

$$= \frac{1}{B\left(\beta_1,\, \beta_2\right)}\int\limits_0^1 \tau\left[x+\left(r-x\right)t\right]\, t^{\beta_2-1}\,\left(1-t\right)^{\beta_1-1} dt -$$

$$-\frac{1}{B\left(1-\beta_1,\, 1-\beta_2\right)}\left(2-2\beta\right)^{\beta-1}\left(r-x\right)^{1-\beta}\int\limits_0^1 \nu\left[x+\left(r-x\right)t\right]\, t^{-\beta_1}\,\left(1-t\right)^{-\beta_2} dt.$$

Introducing a new variable $z = x + (r-x)\, t$, we can rewrite the last equality as follows

$$u\left[\theta_{r0}(x)\right] = \frac{\left(r-x\right)^{1-\beta}}{B\left(\beta_1,\, \beta_2\right)}\int\limits_x^r \frac{\tau\left(z\right)\,\left(r-z\right)^{\beta_1-1}}{\left(z-x\right)^{1-\beta_2}}dz - \frac{\left(2-2\beta\right)^{\varepsilon-1}}{B\left(1-\beta_1,\, 1-\beta_2\right)}\int\limits_x^r \frac{\nu\left(z\right)\,\left(r-z\right)^{-\beta_2}}{\left(z-x\right)^{\beta_1}}dz.$$

In terms of fractional differentiation operators (in the sense of Riemann-Liouville) defined above, we rewrite the last equality

$$u\left[\theta_{r0}(x)\right] = \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta_1\right)}\left(r-x\right)^{1-\beta} D_{rx}^{-\beta_2}\left[\tau\left(t\right)\,\left(r-t\right)^{\beta_1-1}\right] -$$

$$-\frac{\Gamma\left(2-\beta\right)}{\Gamma\left(1-\beta_2\right)}\left(2-2\beta\right)^{\beta-1}\, D_{rx}^{\beta_1-1}\left[\nu\left(t\right)\,\left(r-t\right)^{-\beta_2}\right]. \qquad (17)$$

Next, let us use the laws of weighted composition operators of fractional differentiation and integration with the same origins [5], [7; 18], [36; 20]

$$D_{cx}^{-\gamma}\, D_{ct}^{\gamma}\varphi\left(s\right) = \varphi(x), \qquad (18)$$

$$D_{cx}^{\alpha}\left|t-c\right|^{\alpha+\gamma}\, D_{ct}^{\gamma}\varphi\left(s\right) = \left|x-c\right|^{\gamma}\, D_{cx}^{\alpha+\gamma}\left|t-c\right|^{\alpha}\varphi\left(t\right), \qquad (19)$$

where $0 < \alpha \leq 1$, $\gamma < 0$, $\alpha + \gamma > -1$; $\varphi(x) \in L\left[a,b\right]$, and for $\alpha + \gamma > 0$ the function $\varphi(x)$ has a fractional derivative $D_{cx}^{\alpha+\gamma}\varphi\left(t\right)$.

Applying the operator $D_{rx}^{1-\beta_1}$ to both parts of equality (17) and using composition laws (18), and (19) we obtain

$$\left(r-x\right)^{\beta_2} D_{rx}^{1-\beta_1} u\left[\theta_{r0}\left(t\right)\right] = \gamma_1 D_{rx}^{1-\beta}\tau\left(t\right) - \gamma_2\nu(x), \qquad (20)$$

where $\gamma_1 = \frac{\Gamma(\beta)}{\Gamma(\beta_1)}$, $\gamma_2 = \frac{\Gamma(2-\beta)\,(2-2\beta)^{\beta-1}}{\Gamma(1-\beta_2)}$ .

Substituting $\left(r-x\right)^{\beta_2} D_{rx}^{1-\beta_1} u\left[\theta_{r0}\left(t\right)\right]$ by (20) we can specify condition (5) as follows

$$\left[\alpha_2(x) + \gamma_1\alpha_1(x)\right] D_{rx}^{1-\beta}\tau\left(t\right) + \left[\alpha_3(x) - \gamma_2\alpha_1(x)\right]\nu(x) = \psi_2(x). \qquad (21)$$

Relation (21) is fundamental between the desired functions $\tau(x)$ and $\nu(x)$, the domain $\Omega_1$ to the line $y = 0$ for $|\lambda| < \frac{m}{2}$. At $\lambda = \frac{m}{2}$ by (15) under condition (5) we arrive again at (21) but in this case for $\beta_1 = 0$, $\beta_2 = \beta = \frac{m}{m+2}$, $\gamma_1 = 0$, $\gamma_2 = 2^{\beta-1}\left(1-\beta\right)^{\beta}$, while for $\lambda = -\frac{m}{2}$ by (16) and (5) we get (21) for $\beta_1 = \beta = \frac{m}{m+2}$, $\beta_2 = 0$, $\gamma_1 = 1$, $\gamma_2 = \Gamma\left(2-\beta\right)\left(2-2\beta\right)^{\beta-1}$.

Next, we should obtain the fundamental relation between $\tau(x)$ and $\nu(x)$ transferred to the line$y = 0$ from $\Omega_2$.

For this purpose, we study a representation of the regular solution to problem (12), (13) in $\Omega_2$ for equation (3), which is written out by the d'Alembert formula [37; 59]:

$$u\left(x,y\right)=\frac{\tau\left(x+y\right)+\tau\left(x-y\right)}{2}+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu\left(t\right)dt+\frac{1}{2}\int\limits_{0}^{y}\int\limits_{x-y+t}^{x+y-t}f\left(s,t\right)ds\,dt, \tag{22}$$

where $\tau(x)\in C[0,r]\cap C^2(0,r)$, $\nu(x)\in C^1(0,r)\cap L_1(0,r)$, $f\left(x,\,y\right)\in C\left(\bar{\Omega}_2\right)$.

Satisfying condition (4) in (22) obtain

$$u\left[\theta_{r1}(x)\right]=u\left(\frac{r+x}{2},\,\frac{r-x}{2}\right)=\frac{\tau\left(r\right)+\tau(x)}{2}+\frac{1}{2}\int\limits_{x}^{r}\nu\left(t\right)dt+\frac{1}{2}\int\limits_{0}^{\frac{r-x}{2}}\int\limits_{x+t}^{r-t}f\left(s,\,t\right)ds\,dt=\psi_1(x),$$

whence, using the differentiation, we get the relation form

$$\tau'(x)-\nu(x)-\int\limits_{0}^{\frac{r-x}{2}}f\left(x+t,\,t\right)dt=2\psi_1'(x). \tag{23}$$

Relation (23) is the fundamental relation between $\tau(x)$ and $\nu(x)$, transferred from $\Omega_2$ to the segment $I$ of the straight line $y=0$.

Eliminating the desired function $\nu(x)$ from the above relations (21) and (23) and taking into account the matching condition $\tau\left(r\right)=\psi_1\left(r\right)$ and condition (10) of *Theorem 1*, with respect to $\tau(x)$ we arrive at the first-order ordinary differential equation with a fractional-order derivative in lower terms

$$\tau'(x)+a(x)D_{rx}^{1-\beta}\tau\left(t\right)=2\psi_1'(x)+\frac{\psi_2(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)}+\int\limits_{0}^{\frac{r-x}{2}}f\left(x+t,\,t\right)dt,\ \ 0<x<r, \tag{24}$$

$$\tau\left(r\right)=\psi_1\left(r\right), \tag{25}$$

where $a(x)=\frac{\alpha_2(x)+\gamma_1\alpha_1(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)}$.

We integrate equation (24) from $x$ to $r$, in view of the initial condition (25), and get the integral equation corresponding to problem (24)-(25)

$$\tau(x)-\frac{1}{\Gamma\left(\beta\right)}\int\limits_{x}^{r}K\left(x,\,t\right)\,\tau\left(t\right)dt=F_1(x), \tag{26}$$

where $K\left(x,\,t\right)=\frac{a(x)}{\left(t-x\right)^{1-\beta}}+\int\limits_{x}^{t}\frac{a'(s)dt}{\left(t-s\right)^{1-\beta}}$,

$F_1(x)=2\psi_1(x)-\psi_1\left(r\right)-\int\limits_{x}^{r}\frac{\psi_2(t)}{\alpha_3(t)-\gamma_2\alpha_1(t)}dt-\int\limits_{x}^{r}\int\limits_{0}^{(r-t)/2}f\left(t+s,s\right)ds\,dt.$

The properties of the given functions (7), (8), (9) suggest that equation (26) is a Volterra integral equation of the second kind with the kernel $K\left(x,\,t\right)\in L_1\left([0,r]\times[0,r]\right)$ having a weak singularity for $x=t$ and the right side $F_1(x)\in C[0,r]\cap C^2(0,r)$. According to the general theory of Volterra integral equations, a solution to Eq. (26), is the unique solution, and can be written out by the formula

$$\tau(x)=F_1(x)+\int\limits_{x}^{r}K\left(x,\,t\right)\,F_1\left(t\right)dt, \tag{27}$$

where $R(x, t) = \sum\limits_{n=0}^{\infty} \frac{K_n(x,t)}{\Gamma^{n+1}(\beta)}$ is the resolvent kernel $K(x, t)$; $K_0(x, t) = K(x, t)$, $K_{n+1}(x, t) = \int\limits_t^x K(x, s)\, K_n(s, t)\, ds$ are the iterated kernels.

Let us show that the resolvent $R(x, t)$, like the kernel $K(x, t)$ of Eq. (26), belongs to the class $R(x, t) \in L_1([0, r] \times [0, r])$ and has a weak singularity at $x = t$, and the solution to Eq. (27), and its right-hand side $F_1(x)$, belongs to $\tau(x) \in C[0, r] \cap C^2(0, r)$.

Indeed, considering $\alpha(x) \in C^1[0, r] \cap C^2(0, r)$ we get estimates for iterated kernels $\frac{K_n(x,t)}{\Gamma^{n+1}(\beta)}$. Let $|\alpha(x)| \leq M_1$ and $|\alpha'(x)| \leq M_2 \quad \forall\, x \in [0, r]$ . Then for the first iterated kernel we have the estimate

$$\left|\frac{K_0(x, t)}{\Gamma(\beta)}\right| = \left|\frac{K(x, t)}{\Gamma(\beta)}\right| = \frac{1}{\Gamma(\beta)}\left|\frac{\alpha(x)}{(t-x)^{1-\beta}} + \int\limits_x^t \frac{\alpha'(s)\, dt}{(t-s)^{1-\beta}}\right| \leq \frac{M_1 (t-x)^{\beta-1}}{\Gamma(\beta)} + \frac{M_2 (t-x)^{\beta}}{\Gamma(\beta+1)}.$$

Next

$$\left|\frac{K_1(x, t)}{\Gamma^2(\beta)}\right| = \left|\int\limits_x^t \frac{K(x, s)}{\Gamma(\beta)} \frac{K_0(s, t)}{\Gamma(\beta)}\right| \leq \int\limits_x^t \left[\frac{M_1 (s-x)^{\beta-1}}{\Gamma(\beta)} + \frac{M_2 (s-x)^{\beta}}{\Gamma(\beta+1)}\right] \times$$

$$\times \left[\frac{M_1 (t-s)^{\beta-1}}{\Gamma(\beta)} + \frac{M_2 (t-s)^{\beta}}{\Gamma(\beta+1)}\right] dt = \frac{M_1^2}{\Gamma^2(\beta)} \int\limits_x^t (s-x)^{\beta-1} (t-s)^{\beta-1}\, dt +$$

$$+ \frac{M_1 M_2}{\Gamma(\beta)\Gamma(\beta+1)} \int\limits_x^t (s-x)^{\beta-1} (t-s)^{\beta}\, dt + \frac{M_1 M_2}{\Gamma(\beta)\Gamma(\beta+1)} \int\limits_x^t (s-x)^{\beta} (t-s)^{\beta-1}\, dt +$$

$$+ \frac{M_2^2}{\Gamma^2(\beta+1)} \int\limits_x^t (s-x)^{\beta} (t-s)^{\beta}\, dt = \frac{M_1^2\, (t-x)^{2\beta-1}}{\Gamma(2\beta)} + \frac{2M_1 M_2\, (t-x)^{2\beta}}{\Gamma(2\beta+1)} + \frac{M_2^2\, (t-x)^{2\beta+1}}{\Gamma(2\beta+2)}.$$

Similarly,

$$\left|\frac{K_2(x, t)}{\Gamma^3(\beta)}\right| = \left|\int\limits_t^x \frac{K(x, s)}{\Gamma(\beta)} \frac{K_1(s, t)}{\Gamma^2(\beta)}\right| \leq \frac{M_1^3\, (t-x)^{3\beta-1}}{\Gamma(3\beta)} + \frac{3M_1^2 M_2\, (t-x)^{3\beta}}{\Gamma(3\beta+1)} +$$

$$+ \frac{3M_1 M_2^2\, (t-x)^{3\beta+1}}{\Gamma(3\beta+2)} + \frac{M_2^3\, (t-x)^{3\beta+2}}{\Gamma(3\beta+3)}.$$

It's clear that

$$\left|\frac{K_{n-1}(x, t)}{\Gamma^n(\beta)}\right| \leq \sum_{k=0}^{n} \frac{C_n^k\, M_1^{n-k} M_2^k\, (t-x)^{n\beta+k-1}}{\Gamma(n\beta+k)}, \tag{28}$$

where $C_n^k = \frac{n!}{k!\,(n-k)!}$ is a number of combinations of $n$ elements taken $k$.

Noting that $\Gamma(n\beta + k) \geq \Gamma(n\beta)\ \forall\ k = 0, 1, 2, ...$ from (28) we obtain the estimate

$$\left|\frac{K_{n-1}(x, t)}{\Gamma^n(\beta)}\right| < \frac{(t-x)^{n\beta-1}}{\Gamma(n\beta)} \sum_{k=0}^{n} C_n^k\, M_1^{n-k} M_2^k\, (t-x)^k = \frac{(t-x)^{n\beta-1}}{\Gamma(n\beta)} \left[M_1 + M_2 (t-x)\right]^n. \tag{29}$$

For sufficiently large $n$ index $n\beta - 1$ at $(t-x)$ in (29) is positive. And in this case, the difference $(t-x)$ can be replaced by a higher numerical value $r$. Thus, for the resolvent $R(x, t)$ of the kernel $K(x, t)$ we obtain the estimate:

$$|R(x, t)| = \left|\sum_{n=1}^{\infty} \frac{K_{n-1}(x, t)}{\Gamma^n(\beta)}\right| < \sum_{n=1}^{\infty} \frac{(M_1 + M_2 r)^n\, r^{n\beta-1}}{\Gamma(n\beta)}. \tag{30}$$

Using the Stirling formula for the Gamma function:

$$\Gamma\left(n\right)=\frac{1}{\sqrt{2\pi n}}n^{n}e^{-n+\frac{\eta}{12n}},0<\eta<1.$$

Cauchy criterion for the convergence of numerical series, it is easy to see that the right side series of inequality (30) converges. Thus, the series for the resolvent $R\left(x,\,t\right)$ of the kernel $K\left(x,\,t\right)$ in Eq. (26) converges absolutely and uniformly, and we can conclude that the resolvent of the kernel is continuous for any $0<\beta<1$ and any $x\neq t\in[0,r]$, having a weak singularity for $x=t$.

Further, by representation (27) and estimates (29), (30) with a continuous right-hand side, obtain the estimate

$$\left|\tau(x)\right|=\left|F_1(x)+\int\limits_x^r K\left(x,\,t\right)\,F_1\left(t\right)dt\right|<M_3\left[1+\sum_{n=1}^{\infty}\frac{\left(M_1+M_2r\right)^n r^{n\beta}}{\Gamma\left(n\beta\right)}\right], \tag{31}$$

where $M_3=\max\limits_{0\le x\le r}\left|F_1(x)\right|$.

The convergence of the majorizing sequences (the right side of inequality (31) ) implies the absolute and uniform convergence according to the Weierstrass test. Whence we conclude the continuity of the limit function $\tau(x)\in C[0,r]$.

Now $F_1(x)\in C^2(0,\,r)$. In this case, by double integration by parts on the right side of representation (27), we can see clearly that $\tau(x)\in C^2(0,r)$ that is, the solution to Eq. (26), as well as its right side is belong to $\tau(x)\in C[0,r]\cap C^2(0,r)$.

When $a(x)=a=const$, the solution to (26) is written out explicitly using the formula:

$$\tau(x)=F_1(x)+a\int\limits_x^r R\left(x,\,t;\,a\right)\,F_1\left(t\right)dt,$$

where $R\left(x,t;\,a\right)=\left(t-x\right)^{\beta-1}\,E_{\frac{1}{\beta}}\left[a\,\left(t-x\right)^{\beta};\,\beta\right]$, and $E_\rho\left(z;\mu\right)=\sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu+n\rho^{-1})}$ is the Mittag-Leffler type function [38; 117], which coincides with the Mittag-Leffler function $E_\rho\left(z;1\right)=E_{1/\rho}\left(z\right)$ at $\mu=1$.

If condition (11) is satisfied, then using system (21), (23) we can immediately get:

$$\tau(x)=D_{rx}^{\beta-1}\left[\frac{\psi_2\left(t\right)}{\alpha_2\left(t\right)+\gamma_1\alpha_1\left(t\right)}\right],\nu(x)=D_{rx}^{\beta}\left[\frac{\psi_2\left(t\right)}{\alpha_2\left(t\right)+\gamma_1\alpha_1\left(t\right)}\right]-2\psi_1'(x)-\int\limits_0^{(r-x)/2}f\left(x+t,t\right)dt.$$

*Study of Problem 2*

Now we proceed to the study of problem 2. Satisfying condition (6) for (22) we obtain:

$$u\left[\theta_{01}(x)\right]=u\left(\frac{x}{2},\,\frac{x}{2}\right)=\frac{\tau(x)+\tau\left(0\right)}{2}+\frac{1}{2}\int\limits_0^x\nu\left(t\right)dt+\frac{1}{2}\int\limits_0^{x/2}\int\limits_t^{x-t}f\left(s,\,t\right)dsdt=\psi_1(x),$$

then by differentiation, we get

$$\nu(x)+\tau'(x)+\int\limits_0^{x/2}f\left(x-t,\,t\right)dt=2\psi_1'(x). \tag{32}$$

Relation (32) is the fundamental relation between $\tau(x)$ and $\nu(x)$, transferred from $\Omega_2$ to the segment $I$ of the strait line $y=0$, in case with *Problem 2.*

Thus, when $|\lambda| < \frac{m}{2}$ with respect to the desired $\tau(x)$ and $\nu(x)$ one gets a system of equations expressed through (21) and (32). Eliminating from (21) and (32) the unknown $\nu(x)$ with respect to $\tau(x)$, in view of the matching condition $\tau(0)=\psi_1(0)$, the same way as with *problem 1*, we arrive at the following boundary value problem for a first-order ordinary differential equation with a fractional-order derivative in lower terms

$$\tau'(x) - a(x)\, D_{rx}^{1-\beta}\tau(t) = F_2(x), \qquad 0<x<r, \tag{33}$$

$$\tau(0) = \psi_1(0), \tag{34}$$

where $F_2(x) = 2\psi_1'(x) - \frac{\psi_2(x)}{\alpha_3(x)-\gamma_2\alpha_1(x)} - \int\limits_0^{x/2} f(x-t,\, t)\, dt$.

Integrating equation (33) with respect to the variable $x$ from $x$ to $r$, considering condition (34), we obtain the integral equation corresponding to problem (33)-(34)

$$\tau(x) + \frac{1}{\Gamma(\beta)} \int\limits_0^r L(x,\, t)\, \tau(t)\, dt = \psi_1(0) + \int\limits_0^x F_2(t)\, dt, \tag{35}$$

where $L(x,\, t) = \begin{cases} K(0,\, t) - K(x,\, t), & 0 \le x < t, \\ K(0,\, t), & t < x \le r. \end{cases}$

If the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x,y)$ have the properties (7)–(9) listed in Theorem 1, then equation (35) is a Fredholm integral equation of the second kind with the kernel $L(x,\, t) \in L_1([0,r]\times[0,r])$, with a weak singularity at $x=t$, and a right-hand side of $C[0,r]\cap C^2(0,r)$.

Let us further find sufficient conditions that ensure the unique solvability to Eq. (35). To this end, let's consider a homogeneous problem corresponding to *Problem 2*, setting $\psi_1(x)\equiv 0$, $\psi_2(x)\equiv 0$ $\forall\, x\in[0,r]$ and $f(x,\, y)\equiv 0$ $\forall\ (x,y)\in\bar{\Omega}_2$. In this case, problem (33)-(34) turns into the corresponding homogeneous problem

$$\frac{1}{a(x)}\,\tau'(x) - D_{rx}^{1-\beta}\tau(t) = 0, \qquad 0<x<r, \tag{36}$$

$$\tau(0) = 0. \tag{37}$$

Multiplying equation (36) by the function $\tau(x)$, and integrating the resulting equality with respect to the variable $x$ from 0 to $r$, with condition (37) we have

$$\int\limits_0^r a^{-1}(x)\,\tau(x)\,\tau'(x)dx - \int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx =$$

$$= \frac{\tau^2(r)}{2a(r)} + \int\limits_0^r \frac{a'(x)}{2a^2(x)}\,\tau^2(x)dx - \int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx = 0. \tag{38}$$

To estimate $\int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx$, we use Lemma 1 by [39], according to which $\tau(x)\, D_{rx}^{\alpha}\varphi(t) \ge \frac{1}{2}\, D_{rx}^{\alpha}\varphi^2(t)$, $0<\alpha\le 1$. With this inequality, we have

$$\int\limits_0^r \tau(x)\, D_{rx}^{1-\beta}\tau(t)\, dx \ge \frac{1}{2}\int\limits_0^r D_{rx}^{1-\beta}\tau^2(t)\, dx = \frac{1}{2\Gamma(\beta)}\int\limits_0^r t^{\beta-1}\,\tau^2(t)\, dt \ge 0. \tag{39}$$

If the function $a(x)$ is a nonincreasing negative, then, as follows by (39), equality (38) can take place if and only if $\tau(x) \equiv 0\, \forall x \in [0, r]$. Then by (21) and (32) at $\psi_1(x) \equiv 0$, $\psi_2(x) \equiv 0\, \forall x \in [0, r]$, $f(x, y) \equiv 0 \quad \forall (x, y) \in \bar{\Omega}_2$ and $[\alpha_3(x) - \gamma_2\alpha_1(x)]\, [\alpha_2(x) + \gamma_1\alpha_1(x)] \neq 0\, \forall x \in [0, r]$ it follows that $\nu(x) \equiv 0\, \forall x \in [0, r]$ as well. Therefore, under the above conditions Eq. (32) has a unique solution within $\tau(x) \in C[0, r] \cap C^2(0, r)$.

Thus, we have proved the following theorem.

*Theorem 2.* Let the given functions $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\psi_1(x)$, $\psi_2(x)$ and $f(x, y)$ be such that they have properties (7)–(9) and let

$$a(x) < 0,\, a'(x) \leq 0 \ \ \forall x \in [0, r], \tag{40}$$

$$[\alpha_3(x) - \gamma_2\alpha_1(x)]\, [\alpha_2(x) + \gamma_1\alpha_1(x)] \neq 0 \ \ \forall x \in [0, r]. \tag{41}$$

Then there exists a unique regular solution to Problem 2 in $\Omega$.

In the case when $a(x) = a = const$ the solution to problem (33)-(34) is written out explicitly according to

$$\tau(x) = \frac{E_\beta\left[-a\,(r-x)^\beta\right]}{E_\beta\left[-a\,r^\beta\right]}\psi_1(0) + \frac{E_\beta\left[-a\,(r-x)^\beta\right]}{E_\beta\left[-a\,r^\beta\right]}\int\limits_0^r E_\beta\left[-at^\beta\right]\, F_2(t)\, dt - \int\limits_x^r E_\beta\left[-a\,(r-x)^\beta\right]\, F_2(t)\, dt,$$

and

$$E_\beta\left[-ar^\beta\right] \neq 0. \tag{42}$$

As follows from conditions (40)-(41) Theorem 2, inequality (42) will be satisfied, for example, for all $a < 0$.

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# Екінші ретті аралас-гиперболалық теңдеу үшін ығысуы бар шеттік есептер

Мақалада облыстың бір бөлігінде толқындық теңдеуден және екінші бөлігінде бірінші типті гиперболалық теңдеуден тұратын екінші ретті гиперболалық типті екі теңдеудің түйіндесуінде ығысуы бар екі бейлокалды есеп зерттелген. Зерттелген есептердегі бейлокалды шеттік шарт ретінде сипаттамалардың бірінде және типті өзгерту сызығында қажет функцияның бірінші ретті туынды және бөлшек ретті туынды (Риман-Лиувилль мағынасында) мәндерінің айнымалы коэффициенттері бар сызықтық комбинациясы берілген. Интегралдық теңдеулер әдісін қолдана отырып, бірінші есептің шешімділігі екінші текті әлсіз сингулярлығы бар Вольтерра интегралдық теңдеуінің шешімділігіне, ал екінші есептің шешілетіндігі туралы мәселе әлсіз сингулярлығы бар екінші текті Фредгольм интегралдық теңдеуінің шешімділігіне көшеді. Бірінші есеп үшін екінші текті Вольтерра интегралдық теңдеуінің нәтижесінде алынған ядроның резольвентасына бірқалыпты жинақтылығын және оның шешімі қажетті класқа жататыны дәлелденген. Екінші есеп үшін талап етілетін кластан әлсіз ерекшелікпен екінші текті Фредгольм интегралдық теңдеуінің жалғыз шешімінің болуын қамтамасыз ететін берілген функциялар үшін жеткілікті шарттар табылды. Кейбір ерекше жағдайлар үшін есептердің шешімдері анық жазылған.

*Кілт сөздер:* толқындық теңдеу, бірінші текті өзгешеленген гиперболалық теңдеу, Вольтерра интегралдық теңдеуі, Фредгольм интегралдық теңдеуі, Трикоми әдісі, интегралдық теңдеулер әдісі, бөлшекті есептеу теориясының әдістері.

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## Краевые задачи со смещением для одного смешанно-гиперболического уравнения второго порядка

В статье исследованы две нелокальные задачи со смещением на сопряжение двух уравнений гиперболического типа второго порядка, состоящего из волнового уравнения в одной части области и вырождающегося гиперболического уравнения первого рода — в другой. В качестве нелокального граничного условия в исследуемых задачах задана линейная комбинация с переменными коэффициентами значений производной первого порядка и производных дробного (в смысле Римана–Лиувилля) порядка от искомой функции на одной из характеристик и на линии изменения типа. С использованием метода интегральных уравнений вопрос разрешимости первой задачи эквивалентным образом редуцирован к вопросу разрешимости интегрального уравнения Вольтерра второго рода со слабой особенностью, а вопрос разрешимости второй задачи — к вопросу разрешимости интегрального уравнения Фредгольма второго рода со слабой особенностью. По первой задаче доказаны равномерная сходимость резольвенты ядра получающегося интегрального уравнения Вольтерра второго рода и принадлежность его решения требуемому классу. По второй задаче найдены достаточные условия на заданные функции, обеспечивающие существование единственного решения интегрального уравнения Фредгольма второго рода со слабой особенностью из требуемого класса. В некоторых частных случаях решения задач выписаны в явном виде.

*Ключевые слова:* волновое уравнение, вырождающееся гиперболическое уравнение первого рода, интегральное уравнение Вольтерра, интегральное уравнение Фредгольма, метод Трикоми, метод интегральных уравнений, методы теории дробного исчисления.

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