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Coefficients of multiple Fourier-Haar series and variational modulus of continuity

In this paper, we introduce the concept of a variational modulus of continuity for functions of several variables, give an estimate for the sum of the coefficients of a multiple Fourier-Haar series in terms of the variational modulus of continuity, and prove theorems of absolute convergence of series composed of the coefficients of multiple Fourier-Haar series. In this paper, we study the issue of the absolute convergence for multiple series composed of the Fourier-Haar coefficients of functions of several variables of bounded p-variation. We estimate the coefficients of a multiple Fourier-Haar series in terms of the variational modulus of continuity and prove the sufficiency theorem for the condition for the absolute convergence of series composed of the Fourier-Haar coefficients of the considered function class. This paper researches the question: under what conditions, imposed on the variational modulus of continuity of the fractional order of several variables functions, there is the absolute convergence for series composed of the coefficients of multiple Fourier-Haar series.

Keywords: Fourier-Haar series, variational modulus of continuity, coefficients of multiple Fourier-Haar series.

Introduction

It is known that the definition of p-variation functions for one variable was introduced by Wiener [1], for functions of two variables this definition was given by Clarkson and Adams [2]. Similar questions for trigonometric and multiplicative systems were considered in the works [3, 4]. Let us give the necessary definitions.

Let $f(x_1, ...x_n)$ be defined on the set $[0,1]^N$ and $\rho = \rho_1 \times \rho_2 \times ... \times \rho_N$, here $\rho_j = \{0 = x_j^0 < x_j^1 < ... < x_j^s = 1\}$, $s_j \geq 1$, $j = 1, ..., N$ is an arbitrary partition of a set $[0,1]^N$. Variational sum of order p of the function $f(x_1, ...x_n)$ with respect to the partitions ρ is called the quantity ($1 \leq\ p \leq \infty$)

$$\aleph_\rho^p(f) = \left(\sum_{r_1=1}^{s_1} ... \sum_{r_N=1}^{s_N} |\Delta_1(f; x_1^{r_1-1}, ..., x_N^{r_N-1}; h_1^{r_1}, ..., h_N^{r_N})|^p \right)^{1/p},$$

here

$$\Delta_1(f; x_1, ..., x_N; h_1, ..., h_N) := \textstyle\sum_{\eta_1=0}^1 ... \sum_{\eta_n=0}^1 (-1)^{\eta_1+...+\eta_N} f(x_1 + \eta_1 h_1, ..., x_N + \eta_N h_N),$$

$$(x_1, ..., x_N) \in [0,1]^N, h_j > 0, h_j^{r_j} := x_j^{r_j} - x_j^{r_j-1}, r_j = 1, 2, ...s_j, j = 1, 2, ..., n.$$

Variational modulus of continity $\omega_{1-1/p}(f, \delta_1, ..., \delta_N)$ of an order $1 - \frac{1}{p}$ of the function $f(x_1, ...x_n)$ is called the value

$$\omega_{1-1/p}(f, \delta_1, ..., \delta_N) = \sup_{|\rho_j| \leq \delta_j} \aleph_\rho^p(f), \tag{1}$$

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here $|\rho_j| = \max\limits_{1\le\rho_j\le s_j}(x_j^{r_j} - x_{j-1}^{r_j})$.

We say that $f \in V_p[0,1]^N$, $1 \le\ p \le \infty$, if $V_p\left(f,[0,1]^N\right) \equiv \omega_{1-1/p}(f,1,...,1) < \infty$, and if $f \in C_p[0,1]^N$, $1 \le\ p < \infty$, if $\lim\limits_{\delta_i\to 0}\omega_{1-1/p}(f,\delta_1,...,\delta_N) = 0$.. The properties of a variational modulus of continuity was studied by A.P. Terekhin (see [5,6]).

Modulus of continuity $\omega(f,\delta_1,...,\delta_N)$ for function $f(x_1,...x_n)$ is called the value

$$\omega(f,\delta_1,...,\delta_N) = \sup_{0<h_i\le\delta_i}|f(x_1+h_1,...,x_N+h_N)... - f(x_1,...,x_i+h_i,...,x_N) - ... + f(x_1,...,x_N)|.$$

The functions of the Haar system on the semi-open interval $[0,1)$ is defined by $h_0(x) = 1$ if $x \in [0,1)$; if $n = 2^k + j$, $k \in P = N \cup \{0\}$, $0 \le j < 2^k$ and $\Delta_j^{(k)} = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right)$, then

$$h_n(x) = \begin{cases} 2^{k/2}, x \in \Delta_{2j}^{(k+1)} \\ -2^{k/2}, x \in \Delta_{2j+1}^{(k+1)} \\ 0, x \in [0,1) \setminus \Delta_j^{(k)} \end{cases},$$

(see [7]).
Then the multiplicative Haar system is defined as follows:

$$h_{k_1,...,k_n}(x_1,...,x_n) = h_{k_1}(x_1)...h_{k_n}(x_n),$$

$$(x_1,...,x_n) \in [0,1)^N.$$

The Fourier-Haar coefficients for functions of several variables are determined by the equality: $a_{n_1,...,n_N}(f) = \int_0^1 ... \int_0^1 f(x_1,...,x_N)h_{n_1}(x_1)...h_{n_N}(x_N)dx_1...dx_N$, $n_1,...,n_N \in N$.

This paper researches the question: under what conditions, imposed on the variational modulus of continuity of the fractional order of several variables functions, does the series converge?

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} |a_{n_1,...,n_N}(f)|^{\beta}, \beta > 0,$$

where $a_{n_1,...,n_N}(f)$ are the Fourier-Haar coefficients of the function f. For the case of functions of one variable, such questions were considered by S.S. Volosivets [8].

1 *Formulas and theorems*

Theorem 1. Let $f \in C_p[0,1]$, $1 < p < \infty$ and $a_n(f) = \int_0^1 f(x)h_n(x)dx$, $n \in N$. The following inequality is valid

$$\left(\sum_{i=2^k}^{2^{k+1}-1}|a_i(f)|^p\right)^{\frac{1}{p}} \le \omega_{1-1/p}\left(f,\frac{1}{2^k}\right)2^{-\frac{k}{2}-1}.$$

You can see the proof of Theorem 1 in [8].

Further, we are considering the functions of several variables. We need the following auxiliary statements.

Lemma 1. Let $f \in V_p[0,1]^N$, $1 \le\ p < \infty$ и $0 < \delta_1, \delta_2 < 1$. The following inequality is valid

$$\omega(f,\delta_1,...,\delta_N)_{L_p} \le \omega_{1-1/p}(f,\delta_1,...,\delta_N)\delta_1^{\frac{1}{p}}...\delta_N^{\frac{1}{p}}.$$

This lemma is an analogue of the corresponding lemma from the work [5], it is proved for the case of functions of one variable, for functions of several variables, the proof is proved similarly to the one variable case.

The following theorem gives an estimate for the Fourier-Haar coefficients of two variables functions in terms of the variational modulus of continuity of the order $(1-1/p)$.

Theorem 2. Let $f \in C_p[0,1]^N$, $1<p<\infty$ and

$$a_{n_1,...,n_N}(f) = \int_0^1 ... \int_0^1 f(x_1,...,x_N) h_{n_1}(x_1)...h_{n_N}(x_N)dx_1...dx_N, \quad n_1,...,n_N \in N,$$

the following inequality is valid

$$\left(\sum_{i=2^{k_1}}^{2^{k_1+1}-1} ... \sum_{i_N=2^{k_N}}^{2^{k_N+1}-1} |a_{i_1...i_N}(f)|^p\right)^{\frac{1}{p}} \leq \omega_{1-1/p}\left(f, \frac{1}{2^{k_1}}, ..., \frac{1}{2^{k_N}}\right) 2^{-\frac{k_1+...+k_N}{2}-2}. \tag{2}$$

Proof of Theorem 2. We present the proof for the two variables case [9,10]. In many variables it is proved in a similar way. Using the definition of the Haar function $h_{n_1,n_2}(x,y) = h_{n_1}(x)\, h_{n_2}(x)$ if $n_1 = 2^{k_1}+m_1$,$n_2 = 2^{k_2}+m_2$, we have

$$a_{n_1,n_2}(f) = \int_0^1 \int_0^1 f(x,y) h_{n_1}(x) h_{n_2}(y) dxdy =$$

$$= \int_{\frac{m_1}{2^{k_1}}}^{\frac{m_1+1}{2^{k_1}}} \int_{\frac{m_2}{2^{k_2}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y) h_{n_1}(x) h_{n_2}(y) dxdy =$$

$$= 2^{\frac{k_1+k_2}{2}} \left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}} \int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy - \int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}} \int_{\frac{2m_2+1}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy - \right.$$

$$\left. - \int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}} \int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y)dxdy + \int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}} \int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y)dxdy \right),$$

then, replacing the variables taking the shift of the arguments, we get

$$a_{n_1,n_2}(f) = 2^{\frac{k_1+k_2}{2}} \left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}} \int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x,y)dxdy - \int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}} \int_{\frac{2m_2+1}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} f(x+2^{-k_1-1},y)dxdy - \right.$$

$$- \int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}} \int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x,y+2^{-k_2-1})dxdy +$$

$$\left. + \int_{\frac{2m_1+1}{2^{k_1+1}}}^{\frac{m_1+1}{2^{k_1}}} \int_{\frac{2m_2+1}{2^{k_2+1}}}^{\frac{m_2+1}{2^{k_2}}} f(x+2^{-k_1-1},y+2^{-k_2-1})dxdy \right) =$$

$$= 2^{\frac{k_1+k_2}{2}} \left(\int_{\frac{m_1}{2^{k_1}}}^{\frac{2m_1+1}{2^{k_1+1}}} \int_{\frac{m_2}{2^{k_2}}}^{\frac{2m_2+1}{2^{k_2+1}}} (f(x,y) - f(x+2^{-k_1-1},y) - \right.$$

$$\left. - f(x,y+2^{-k_2-1}) + f(x+2^{-k_1-1},y+2^{-k_2-1}))dxdy \right).$$

Now, based on Holder and Lemma 1, we have (here $\frac{1}{p}+\frac{1}{q}=1$)

$$a_{n_1,n_2}(f) \leq 2^{\frac{k_1+k_2}{2}} \left(\int_{\Delta_{2m_1}^{(k_1+1)}} \int_{\Delta_{2m_2}^{(k_2+1)}} |(f(x,y) - f(x+2^{-k_1-1},y) - \right.$$

$$\left. -f(x,y+2^{-k_2-1}) + f(x+2^{-k_1-1},y+2^{-k_2-1}))|^p dxdy \right)^{\frac{1}{p}} \left(\frac{1}{2^{k_1+k_2+2}}\right)^{\frac{1}{q}} \leq$$

$$\leq 2^{\frac{k_1+k_2}{2}} \left(\sup_{\substack{h_1 \leq \frac{1}{2^{k_1+1}} \\ h_2 \leq \frac{1}{2^{k_2+1}}}} \int_{\Delta_{2m_1}^{(k_1+1)}} \int_{\Delta_{2m_2}^{(k_2+1)}} |\Delta_2(f,x,y,h_1,h_2)|^p dxdy \right)^{\frac{1}{p}} \left(\frac{1}{2^{k_1+k_2+2}}\right)^{\frac{1}{q}} \leq$$

$$\leq 2^{\frac{k_1+k_2}{2}} V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right) \times$$

$$\times \left(\frac{1}{2^{k_1+k_2+2}}\right)^{\frac{1}{q}} \left(\frac{1}{2^{k_1+k_2+2}}\right)^{\frac{1}{p}} =$$

$$= 2^{\frac{k_1+k_2}{2}} V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right) \left(\frac{1}{2^{k_1+1}}\right)^{\frac{1}{q}+\frac{1}{p}} \left(\frac{1}{2^{k_2+1}}\right)^{\frac{1}{q}+\frac{1}{p}} =$$

$$= 2^{\frac{k_1+k_2}{2}} V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right) \left(\frac{1}{2^{k_1+1}}\right) \left(\frac{1}{2^{k_2+1}}\right) =$$

$$= 2^{\frac{k_1+k_2}{2}-k_1-k_2-2} V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right) =$$

$$= 2^{-\frac{k_1+k_2}{2}-2} V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right).$$

Therefore

$$|a_{n_1,n_2}(f)|^p \leq 2^{-\frac{p}{2}(k_1+k_2)-2p} \left(V_p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right)\right)^p. \tag{3}$$

We take $\varepsilon > 0$ such that $m_1 = 0,1,...,2^{k_1}-1$ and $m_2 = 0,1,...,2^{k_2}-1$ and find partition ξ_{m_1} and η_{m_2} squares $\left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right]$ and $\left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]$ (look (1)) such that

$$\left(\aleph^p_{\xi_{m_1},\eta_{m_2}}(f)\right)^p \geq \left(V_p^p\left(f, \left[\frac{m_1}{2^{k_1}}, \frac{m_1+1}{2^{k_1}}\right], \left[\frac{m_2}{2^{k_2}}, \frac{m_2+1}{2^{k_2}}\right]\right)\right)^p - \frac{\varepsilon}{2^{k_1-k_2}}.$$

Combining all these partitions of the square $[0,1]^2$ with a diameter no more $\frac{1}{2^{k_1}}$, $\frac{1}{2^{k_2}}$ accordingly and, summing inequalities (3), we get

$$\sum_{i=2^{k_1}}^{2^{k_1+1}-1} \sum_{i=2^{k_2}}^{2^{k_2+1}-1} |a_{ij}(f)|^p \leq \omega^p_{1-1/p}\left(f, \frac{1}{2^{k_1}}, \frac{1}{2^{k_2}} - \varepsilon\right) 2^{-\frac{k_1+k_2}{2}p-2p},$$

and since ε we can be made arbitrarily small, then inequality (2) is proved. Theorem 2 is proved.
In the case of one variable functions, a similar estimate for the Fourier-Haar coefficients was obtained in [8].

The following theorem gives sufficient conditions for the convergence of double series, composed of Fourier-Haar coefficients.

Theorem 3. Let $f \in C_p[0,1]^N$, $1 < p < \infty$ and

$$a_{n_1,...,n_N}(f) = \int_0^1 ... \int_0^1 f(x_1,...,x_N)h_{n_1}(x_1)...h_{n_N}(x_N)dx_1...dx_N, \quad n_1,...,n_N \in N.$$

1) Let $\beta > 0$, $p \geq \beta$. Then, under the condition of convergence for the series

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{-\frac{\beta}{2}-\frac{\beta}{p}} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{n_1}, ..., \frac{1}{n_N}\right),$$

the following series converges

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} |a_{n_1...n_N}(f)|^{\beta}.$$

2) Let $\beta > 0$, $p \geq \beta$, $\gamma > \frac{1}{p} + \frac{1}{2}$, $\gamma \in R$. Then, under the condition of convergence for the series

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{\gamma-\frac{1}{p}-\frac{1}{2}} \omega_{1-1/p}^{p}\left(f, \frac{1}{n_1}, ..., \frac{1}{n_N}\right),$$

the following series converges

$$\sum_{n_1=1}^{\infty} ... \sum_{n_N=1}^{\infty} (n_1...n_N)^{\gamma} |a_{n_1...n_N}(f)| < \infty.$$

Proof of Theorem 3. We present the proof for the two variables case [9, 10]. In many variables it is proved in a similar way. Consider the case 1).

Using Holder's inequality and Theorem 2, we have

$$\sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta} \leq \left(\sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta\frac{p}{\beta}}\right)^{\frac{\beta}{p}} \left(\sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^l}^{2^{l+1}-1} 1\right)^{1-\frac{\beta}{p}} =$$

$$= (2^k 2^l)^{1-\frac{\beta}{p}} \left(\sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta\frac{p}{\beta}}\right)^{\frac{\beta}{p}} \leq (2^{k+l})^{1-\frac{\beta}{p}} 2^{(-\frac{k+l}{2}-2)\beta} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) =$$

$$= 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right).$$

Summing up both sides of the resulting inequality, we have

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\sum_{m=2^k}^{2^{k+1}-1} \sum_{n=2^l}^{2^{l+1}-1} |a_{mn}(f)|^{\beta}\right) \leq$$

$$\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) \sum_{k=1}^{\infty} \sum_{l=l}^{\infty} |a_{mn}(f)|^{\beta} \leq$$

$$\leq C \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(k+l)\left(1-\frac{\beta}{p}-\frac{\beta}{2}\right)} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{2^k}, \frac{1}{2^l}\right) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{-\frac{\beta}{2}-\frac{\beta}{p}} \omega_{1-1/p}^{\beta}\left(f, \frac{1}{m}, \frac{1}{n}\right) =$$

$$= C\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=2^m}^{2^{m+1}-1}\sum_{l=2^n}^{2^{n+1}-1}(kl)^{-\frac{\beta}{p}-\frac{\beta}{2}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{k},\frac{1}{l}\right) \geq$$

$$\geq \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{-(n+m)}2^{-\frac{\beta}{2}-\frac{\beta}{p}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^m},\frac{1}{2^n}\right)2^{(n+m)} =$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}2^{(n+m)\left(1-\frac{\beta}{2}-\frac{\beta}{p}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^m},\frac{1}{2^n}\right).$$

Therefore

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{-\frac{\beta}{2}-\frac{\beta}{p}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{m},\frac{1}{n}\right) < \infty,$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}(f)|^{\beta} < \infty.$$

Now we consider the case 2)

$$\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma}|a_{mn}(f)|^{\beta} \leq \left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}|a_{mn}(f)|^{p}\right)^{\frac{1}{p}}\left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma q}\right)^{\frac{1}{q}} \leq$$

$$\leq \left(2^{(k+1)\gamma q}2^{(l+1)\gamma q}2^{(k+l)}\right)^{\frac{1}{q}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2} =$$

$$= 2^{\gamma-2}2^{(k+l)\left(\gamma+\frac{1}{q}-\frac{1}{2}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2}.$$

Summing up both sides

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{m=2^k}^{2^{k+1}-1}\sum_{n=2^l}^{2^{l+1}-1}(mn)^{\gamma}|a_{mn}(f)|^{\beta}\right) \leq$$

$$\leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2^{\gamma-2}2^{(k+l)\left(\gamma+\frac{1}{q}-\frac{1}{2}\right)}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{2^k},\frac{1}{2^l}\right)2^{-\frac{k+l}{2}-2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma}|a_{mn}(f)| \leq$$

$$\leq C\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\gamma-\frac{1}{p}-\frac{1}{2}}\omega_{1-1/p}^{\beta}\left(f,\frac{1}{m},\frac{1}{n}\right).$$

Theorem 3 is proved.

Theorem 3 is an extension to the two-dimensional case of the corresponding theorem from the work [8].

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Еселі Фурье-Хаар қатарының коэффициенттері және вариациялық үзіліссіздік модулі

Мақалада көп айнымалы функциялар үшін вариациялық үзіліссіздік модулінің ұғымы енгізілген, Фурье-Хаар коэффициенттерінен құрылған еселі қатарларды вариациялық үзіліссіздік модулі арқылы бағалау және Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталуының теоремалары дәлелденген. Авторлар Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың вариациялық үзіліссіздік модулі арқылы бағалануын және қарастырылып отырған функциялар класынан алынған Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталуының жеткілікті шартын дәлелдеген. Көп айнымалы функциялардың $(1-1/p)$ ретті вариациялық үзіліссіздік модуліне қандай шарттар қойғанда, Фурье-Хаар коэффициенттерінен құрылған еселі қатарлардың абсолютті жинақталу деген мәселе зерттелген.

Кілт сөздер: Фурье-Хаар қатары, вариациялық үзіліссіздік модулі, еселі Фурье-Хаар коэффициенттері.

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Коэффициенты кратного ряда Фурье–Хаара и вариационный модуль непрерывности

В статье введено понятие вариационного модуля непрерывности для функций многих переменных, приведены оценка суммы коэффициентов кратного ряда Фурье–Хаара через вариационный модуль непрерывности, и доказаны теоремы об абсолютной сходимости рядов, составленных из коэффициентов кратных рядов Фурье–Хаара. Авторами исследован вопрос об абсолютной сходимости кратных рядов, составленных из коэффициентов Фурье–Хаара функций многих переменных ограниченной p-вариации. Приведена оценка коэффициентов кратного ряда Фурье–Хаара через вариационный модуль непрерывности, и доказана теорема достаточности условия абсолютной сходимости рядов, составленных из коэффициентов Фурье–Хаара рассматриваемого класса функции. Здесь изучен вопрос: «При каких условиях, накладываемых на вариационный модуль непрерывности дробного порядка функций многих переменных, имеет место абсолютная сходимость кратных рядов, составленных из коэффициентов Фурье–Хаара?»

Ключевые слова: ряды Фурье–Хаара, вариационный модуль непрерывности, коэффициенты кратного ряда Фурье–Хаара.

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