DOI 10.31489/2023M3/152-164

UDC 510.67

O.I. Ulbrikht*, N.V. Popova

Karaganda University of the name of academician E.A. Buketov,
Institute of Applied Mathematics, Karaganda, Kazakhstan
(E-mail: ulbrikht@mail.ru, dandn@mail.ru)

A fragment of a theoretical set and its strongly minimal central type

The paper defines a new class of algebras, the theory of which is a special case of Jonsson theories. This class applies to both varieties and Jonsson theories. The main results of this article are the following two results. In this article, an answer is obtained to the question of the equivalence of existential closure and algebraic closure of the model of the cosemantic class of a fixed spectrum of a Robinson hereditary variety. A criterion for strong minimality is obtained in the framework of the study of central types of central classes and fragments of a fixed spectrum.

Keywords: Jonsson theory, existentially closed model, algebraically closed model, cosemanticness, Robinson spectrum, Robinson hereditary variety, central type, Jonsson fragment, theoretical set, strongly minimal type.

Introduction

This article belongs to a fairly well-known topic in the field of model theory. Namely, this topic is related to the classification of theories regarding such an important concept as categoricity. As it is well known, only 4 combinations are possible with respect to the concept of categoricity: total categoricity; ω-categoricity and not ω_1-categoricity; ω_1-categoricity and not ω-categoricity; nowhere categoricity. The notion of strong minimality is closely related to the notion of ω_1-categoricity, that is, in all four combinations of the above concept of categoricity, the concept of strong minimality is either present or absent. Thus, the study of the strong minimality property is important in classifying complete theories.

The topic studied in this article is related to the study of Jonsson theories and their classes of models [1–6]. In papers earlier than this paper, the main methods used to study Jonsson theories [1,5,7,8] were considered. One of the methods for studying complete theories is to enrich the signature with symbols that allow one to obtain new information about the models of the old signature and their theories in the language of these symbols. In the works [9–11], related to the enrichment of Jonsson theories, the notion of a central type was introduced on the basis of the notion of heredity of Jonsson theory. The concept of heredity is closely related to the concept of the stability of the center of the Jonsson theory and the Jonsson stability of the Jonsson theory itself. As is well known, the best description among Jonsson theories lends itself to the study of perfect Jonsson theories due to the existence of a model companion of such theories. The concept of stability is closely related to the concept of categoricity, which plays an important role in the theory of classification of complete theories and, accordingly, incomplete theories. Due to the fact that the concept of heredity of Jonsson theory still does not have a complete description, this topic is relevant and modern in the framework of the study of the enrichment of Jonsson theories.

Jonsson theories, in their essence, are, generally speaking, incomplete theories. That is, the technical apparatus of the study of Jonsson theories, in comparison with complete theories, is less adapted to the transfer and adaptation of the concepts and achievements of complete theories.

*Corresponding author.
E-mail: ulbrikht@mail.ru

A valuable concept for operating on the properties of elements and subsets of a semantic model is the Jonsson set, that is, a definable set with the help of some existential formula, the definable closure of which defines some existentially closed submodel of the considered semantic model. An interesting and important special case of the Jonsson set is the notion of a theoretical set. In fact, within the framework of the study of the Jonsson set, we get a "new and special" Jonsson theory, the axioms of which are directly related to the given Jonsson set.

Until now, an unresolved problem is a problem of characterization of the concept heredity of Jonsson theory. The relevance of this problem is confirmed by the following important counterexample: the elementary theory of an algebraically closed field ceases to be Jonsson after enrichment with an unary predicate. In this regard, the study of the model-theoretic properties of central types in predicate enrichment is an important model-theoretic task for describing hereditary Jonsson theories.

The concepts of central type and Jonsson spectrum were first introduced by Yeshkeyev A.R., respectively, in [12, 13]. With the help of these concepts, complete descriptions of Jonsson Abelian groups [13] and Jonsson modules [14] with respect to the concept of cosemanticness were obtained, thereby starting a new study in the framework of model-theoretic algebra. Later, the study of the model-theoretic properties of these concepts was continued in the works [9, 11, 14–18].

Note that another method of studying Jonsson's theories is to study these theories using the concepts of syntactic and semantic similarities. In the papers [19–21], using these concepts, some results were obtained in the framework of the study of Jonsson theories and their centers, as well as definable subsets of the semantic model.

1 Basic necessary concepts and preliminaries

Let us give the main definitions of the concepts of model-theoretic concepts that you need to know in order to understand and be able to work in the framework of studying Jonsson theories and their classes of models. The following definitions and their model-theoretic properties are generators for that part of model theory that studies the basic properties of definable subsets of the semantic model of various fixed Jonsson theories.

Definition 1. [22; 80] A theory T is called a Jonsson theory if

1) theory T has an infinite model;

2) theory T is inductive, i.e. T is equivalent to the set of $\forall\exists$-sentences;

3) theory T has the joint embedding property (JEP), i.e. any two models A, B of the theory T are isomorphically embedded in some model C of the theory T;

4) theory T has the amalgamation property (AP), i.e. if for any A, B, $C \models T$ such that $f_1 : A \to B$, $f_2 : A \to C$ are isomorphic embeddings, there are $D \models T$ and isomorphic embeddings $g_1 : B \to D$, $g_2 : C \to D$, such that $g_1 f_1 = g_2 f_2$.

Examples of Jonsson theories are the theories of well-known classical algebras such as groups, Abelian groups, Boolean algebras, linear orders, fields of fixed characteristic, and polygons.

Note that Jonsson theories, generally speaking, are not complete.

Definition 2. [23] Let $\kappa \geq \tau$. A model M of the theory T is called κ-universal for T, if each model of the theory T of cardinality strictly less than κ is isomorphically embeddable into M.

Definition 3. [23] Let $\kappa \geq \tau$. A model M of the theory T is called κ-homogeneous for T, if for any two models A and A_1 of the theory T, which are submodels of M, cardinality is strictly less than κ, and isomorphism $f : A \to A_1$, for each extension B of the model A, that is a submodel of M and the model of the theory T of cardinality is strictly less than κ there exists an extension B_1 of the model A_1, which is a submodel of M, and the isomorphism $g : B \to B_1$, continuing f.

A homogeneous-universal model for T is called a κ-homogeneous-universal model for T of cardinality κ, where $\kappa \geq \tau$.

The following concept is crucial when working with Jonsson theories.

Definition 4. [23] Let T be a Jonsson theory. A model C of the theory T is called a semantic model if it is τ^+–homogeneous and τ^+–universal simultaneously.

A semantic model plays an important role as a semantic invariant. Such a model always exists for any Jonsson theory.

The next important fact shows that any Jonsson theory is determined by its semantic model.

Fact 1. [23] Every Jonsson theory T has a κ^+-homogeneous-universal model of cardinality 2^κ. Conversely, if T is inductive, has an infinite model, and has a τ^+-homogeneous-universal model, then T is a Jonsson theory.

Theorem 1. [23] Let T be a Jonsson theory. Two κ-homogeneous-universal for T models A and B are elementarily equivalent.

Definition 5. [1; 161] Let C_T be a semantic model of the Jonsson theory T. Then the elementary theory $Th(C_T)$ of the model C_T is called the center of T and is denoted by T^*.

The following result makes it possible to describe a special subclass of Jonsson theories that have a model companion.

Fact 2. [23] Let T be the Jonsson theory. If T^* is model complete and $\kappa > \tau$, then κ-homogeneous universal models of T are κ-saturated; if T^* is not model complete, no semantic model of T is τ^+-saturated.

It follows from the Fact 2 and the mutual model compatibility of the Jonsson theory T and its center T^* that T^* is a model companion of the theory T.

Further in our article, the language will be countable, which means that $\tau = \omega$ and $\kappa > \omega$ or $\kappa \geqslant \omega$. From the Fact 2 for the notion of the perfectness of the Jonsson theory κ must be greater than τ.

Definition 6. [24] Let $\kappa > \tau$. Jonsson theory T is perfect if its semantic model is τ^+-saturated.

Thus, from the Fact 2 and the Definition 6 we can conclude that a perfect Jonsson theory is a Jonsson theory that has a model companion and it is equal to its center.

Recall that a model M of theory T is existentially closed in T if every existential sentence φ of L_M which holds in some model of T extending M holds in M.

The notion of an existentially closed model is a generalization of the notion of an algebraically closed field.

Lemma 1. [24] The semantic model C_T of the Jonsson theory T is T-existentially closed.

Proposition 1. [22; 97] If T is inductive theory, then every model of theory T can be extended to an existentially closed model.

Let us denote by E_T the class of all existentially closed models of the theory T.

Theorem 2. [24] If the Jonsson theory T is perfect, then $E_T = \mathrm{Mod}(T^*)$, where $T^* = \mathrm{Th}(C_T)$.

Definition 7. [25] Let $A \in \Sigma$, where Σ is a universal class in a countable language $L(\Sigma)$. Then A is algebraically closed if A has no proper algebraic extensions. An extension B of A is an algebraic closure of A if B is an algebraically closed algebraic extension of A.

The ability to compare complete theories is an important tool in classifying these theories. Mustafin T.G. a method of syntactic and semantic similarity was proposed for the classification of complete theories and their monster models [26]. Let us give the main definitions related to these concepts.

Let T be complete theory then $F(T) = \bigcup_{n<\omega} F_n(T)$, where $F_n(T)$ is Boolean algebra of formulas with n free variables.

Definition 8. [26] Let T_1 and T_2 be complete theories. We will say that T_1 and T_2 are syntactically similar ($T_1 \overset{S}{\bowtie} T_2$) if exists bijection $f : F(T_1) \to F(T_2)$ such that:

1) restriction f to $F_n(T_1)$ is isomorphism of Boolean algebras $F_n(T_1)$ and $F_n(T_2)$, $n < \omega$;
2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
3) $f(v_1 = v_2) = (v_1 = v_2)$.

Definition 9. [26] 1) $\langle A, \Gamma, \mathcal{M}\rangle$ is called the pure triple, where A is not empty, Γ is the permutation group of A and $\mathcal{M}$ is the family of subsets of A such that from $M \in \mathcal{M}$ follows that $g(M) \in \mathcal{M}$ for every $g \in \Gamma$.

2) If $\langle A_1, \Gamma_1, \mathcal{M}_1\rangle$ and $\langle A_2, \Gamma_2, \mathcal{M}_2\rangle$ are pure triples and $\psi : A_1 \to A_2$ is a bijection then ψ is an isomorphism if:

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;
(ii) $\mathcal{M}_2 = \{\psi(E) : E \in \mathcal{M}_1\}$.

Definition 10. [26] The pure triple $\langle C, Aut(\mathcal{C}), Sub(C)\rangle$ is called the semantic triple of complete theory T, where C is the universe of Monster model $\mathcal{C}$ of theory T, $Aut(\mathcal{C})$ is the automorphism group of $\mathcal{C}$, $Sub(C)$ is a class of all subsets of C each of which is a carrier of the corresponding elementary submodel of $\mathcal{C}$.

Definition 11. [26] Complete theories T_1 and T_2 are semantically similar if and only if their semantic triples are isomorphic.

Proposition 2. [26] If T_1 and T_2 are syntactically similar, then T_1 and T_2 semantically similar. The converse implication fails.

In what follows, we will denote the syntactic and semantic similarities of the complete theories T_1 and T_2 as $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$, respectively.

Let us recall the definition of semantic property.

Definition 12. [26] A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [26] it is known that:

The ability to compare complete theories with the help of syntactic and semantic similarity was useful in describing the most important properties of the theory of stability in the study of complete theories. The following result confirms the importance of syntactic and, accordingly, semantic similarity of complete theories.

Proposition 3. [26] The following properties and notions are semantic:

(1) type;
(2) forking;
(3) λ-stability;
(4) Lascar rank;
(5) Strong type;
(6) Morley sequence;
(7) Orthogonality, regularity of types;
(8) $I(\aleph_\alpha, T)$ – the spectrum function.

The following definition was introduced Yeshkeyev A.R. in the frame of Jonsson theories study [24].

Let T be an arbitrary Jonsson theory, then $E(T) = \bigcup_{n<\omega} E_n(T)$, where $E_n(T)$ is a lattice of $\exists$-formulas with n free variables, T^* is a center of Jonsson theory T, i.e. $T^* = Th(\mathcal{C})$, where $\mathcal{C}$ is semantic model of Jonsson theory T in the sense of [23].

Definition 13. [24] Let T_1 and T_2 are arbitrary Jonsson theories. We say that T_1 and T_2 are Jonsson syntactically similar ($T_1 \overset{S}{\rtimes} T_2$) if it exists a bijection $f : E(T_1) \longrightarrow E(T_2)$ such that:

1) the restriction of f to $E_n(T_1)$ is an isomorphism of lattices $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_{n+1}(T)$, $n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$.

In particular, a criterion was obtained that connects fixed Jonsson theories and their centers, which are complete theories. Thus, a connection is found between the concepts of syntactic and semantic similarity of complete theories and the corresponding similarities of fixed Jonsson theories.

Theorem 3. [24] Let T_1 and T_2 are $\exists$-complete perfect Jonsson theories, then following conditions are equivalent:

1) $T_1 \overset{S}{\rtimes} T_2$;

2) $T_1^* \overset{S}{\bowtie} T_2^*$.

One of the important and useful concepts of model theory is the formulaic definability of fixed subsets of the models under consideration. In particular, when studying complete theories, there are axiomatic approaches to such subsets [27]. In this article, when passing to fixed subsets of the semantic model of a fixed Jonsson theory, the concept of special definable formulaic subsets of the semantic model is used. These concepts were defined by Yeshkeyev A.R. [28], where he defined the concept of the Jonsson set and its particular case, the theoretical set. This approach is a generalization of the well-known concept of a basis in linear algebra.

Definition 14. [28] Let T be some Jonsson theory in a fixed language and C_T is its semantic model. A subset $X \subseteq C_T$ is called a Jonsson set in the theory T, if it satisfies the following properties:

1) the set X is a $\exists$-definable subset of C_T (this means that there is a $\exists$-formula, the solution of which in the C_T is the set X);

2) $cl(X) = M$, $M \in E_T$, where cl is some closure operator defining a pregeometry [29; 289] over C (for example $cl = acl$ or $cl = dcl$).

Further in our article it is assumed that $acl = dcl$.

Consider a countable language L, a complete for existential sentences perfect Jonsson theory T in the language L and its semantic model C_T. Let X be a Jonsson set in T and M be an existentially closed submodel of the semantic model C_T, where $\mathrm{dcl}(X) = M$. Then let $\mathrm{Th}_{\forall\exists}(M) = Fr(X)$, where $Fr(X)$ is the *Jonsson fragment* of the Jonsson set X.

Definition 14. A set X is called a theoretical set, if X is Jonsson set, $\varphi(C) = X$ and the universal closure of the formula $\varphi(x)$ defines some finitely axiomatizable Jonsson theory.

The concept of strong minimality, both for sets and for theories, has played an important role in the description of uncountably categorical complete theories [32]. Recall the definition of a strongly minimal type.

Let M be a structure of language L. A subset X of M is called minimal if it is definable (with parameters in M), infinite, and if for any definable (with parameters in M) subset Y of M either $X \cap Y$ or $X \setminus Y$ is finite. A formula $\varphi(x)$ (in $L(M)$) is strongly minimal if it defines a minimal set in all elementary extensions of M. A non-algebraic type is strongly minimal if it contains a strongly minimal formula.

2 Main results

Definition 15. A Jonsson theory T is called Robinson theory if it is universally axiomatizable.

Definition 16. [10] An enrichment $\tilde{T}$ is called admissible if the ∇-type (this means that $\nabla \subset L_\sigma$ and any formula from this type belongs to ∇) in this enrichment is definable within the framework of $\tilde{T}_\Gamma$-stability, where Γ is the enrichment of the signature σ.

Definition 17. [10] A Robinson theory T is called hereditary if in any of its admissible enrichments any extension is a Robinson theory.

Let T be a Robinson theory, A be an arbitrary model of signature σ. The Robinson spectrum of the model A is the set:

$$RSp(A) = \{T|\ T \text{ is Robinson theory in the language of signature } \sigma \text{ and } A \in Mod(T)\}.$$

Definition 18 (T.G. Mustafin [1]). We say that the Jonsson theory T_1 is cosemantic to the Jonsson theory T_2 ($T_1 \bowtie T_2$), if $C_{T_1} = C_{T_2}$, where C_{T_i} is the semantic model of the theory T_i, $i = 1, 2$.

It is easy to see that the cosemantic relation on a set of Jonsson theories is an equivalence relation. Since the Robinsonian theory is a special case of the Jonsson theory, then we can consider the $RSp(A)/_{\bowtie}$ factor set of the Robinson spectrum of the model A with respect to $\bowtie$.

And one can define the Robinson spectrum $RSp(K)$ of the class K structures for arbitrary signature by analog with the Robinson spectrum $RSp(A)$:

$$RSp(K) = \{T \mid T \text{ is a Robinson theory in the language } K \subseteq \mathrm{Mod}(T)\}.$$

We can note that if $A \in K$ then $RSp(A) \supset RSp(K)$.

Let $[T] \in RSp(K)/_{\bowtie}$, then $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_\Delta$ is the class of all existentially closed models of class $[T]$.

We will call a class $[T] \in RSp(K)/_{\bowtie}$ perfect if every theory $\Delta \in [T]$ is perfect.

We will call the class $[T] \in RSp(K)/_{\bowtie}$ hereditary if each theory $\Delta \in [T]$ is hereditary.

In what follows, we will work with a special class of K structures called a variety.

Recall that identities are formulas of the form $(\forall x_1, \dots x_n)\varphi(x_1, \dots x_n)$, where $\varphi(x_1, \dots x_n)$ is an atomic formula of signature σ.

Definition 19. [30] A class K of systems of signature σ is called a variety if there exists a collection F of identities of signature σ such that K consists of those and only those systems of signature σ in which all formulas from F. The collection F is called the defining collection of the variety.

Note that every variety is an axiomatizable class of algebras.

Examples of varieties are the classes of all semigroups, all groups, Abelian groups, Boolean rings, nilpotent groups of steps $\leq s$.

Let us formulate the following the well-known classical result:

Theorem 4 (Birkhoff [30], p. 337). For a non-empty class K of algebraic systems to be a variety, it is necessary and sufficient that the following conditions be satisfied:

1) the Cartesian product of an arbitrary sequence of K-systems is a K-system;
2) any subsystem of an arbitrary K-system is a K-system;
3) any homomorphic image of an arbitrary K-system is a K-system;

i.e. it is necessary and sufficient that the class K be hereditary, multiplicatively, and homomorphically closed.

Definition 20. A class of structures K_R of signature σ will be called a Robinson class if $Th_\forall(K_R)$ is a Robinson theory.

Definition 21. We will call a variety K Robinson hereditary if every Robinson class $K_R \subseteq K$ is a subvariety of the class K.

In [25] the question was formulated about the coincidence of the concepts of algebraic closure and existential closure in classes of models of a fixed variety. This question in this context is relevant to universal algebra. The concepts of algebraic closure and existential closure in the theory of models have an independent meaning, since the theory, generally speaking, may not be connected with the concept of variety. In this paper, the following result gives a positive answer to the above Forrest question in the framework of studying the cosemanticness classes of a fixed Robinson spectrum of a Robinson hereditary variety.

Theorem 5. Let K be a Robinson hereditary variety, $[T] \in RSp(K)/_{\bowtie}$ is perfect class, then for any algebraically closed model $A \in Mod[T]$ it follows that $A \in E_{[T]}$.

Proof. Suppose the opposite. Let there exist a model $A \in Mod([T])$ such that A is algebraically closed, but $A \notin E_{[T]}$. Then there exists a sentence $\theta = \exists \overline{x} \neg \varphi(\overline{x})$ and a model $B \in Mod([T])$ such that $B \supseteq A$ and $B \models \theta$, but $A \nvDash \theta$. Then $A \models \neg\theta$, that is, $A \models \forall \overline{x} \varphi(\overline{x})$. Since any theory $\Delta \in [T]$, Δ is Robinson theory, then according Proposition 1, there exists $B' \in E_{[T]}$ such that $A \to B'$, $B' \to C$, where C is a semantic model of the class $[T]$. Since class $[T]$ is perfect, then $C \models \neg\theta$. On the other hand, if $B \in E_{[T]}$, then $B \equiv_{\forall\exists} B'$ and $B' \models \theta$. If $B \notin E_{[T]}$, then there exists $B'' \in E_{[T]}$, such that $B \to B''$ and $B'' \to C$. In both cases we have $C \models \theta$. We got a contradiction. So our assumption was wrong, therefore, $A \in E_{[T]}$.

The idea of a central type allows one to study classes of models of the center of hereditary Jonsson theory in an enriched language. In this context, in the considered enrichment, we use a one-place predicate and some constant symbols, and one constant symbol is fixed in terms of the location of the interpretation of this constant relative to an existentially closed submodel of a fixed semantic model, which is an interpretation of a one-place predicate symbol. Taking into account the fact that in the pregeometry that specifies the closure of the set of types under consideration, the definable closure and algebraic closure of which are equal to each other, it allows avoiding collisions of non-preservation of the notion of Jonsson property in this enrichment.

Consider the general scheme for obtaining the central type for a hereditary cosemanticness class of Robinson theories [6].

Let A be an arbitrary model of signature σ, $[T] \in RSp(A)/_{\bowtie}$ be a hereditary class, $C_{[T]}$ be semantic model of class $[T]$. For each theory $\Delta \in [T]$, consider its enrichment $\bar{\Delta}$ in language of signature $\sigma_\Gamma = \sigma \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$, obtained as follows:

$$\bar{\Delta} = Th_{\forall}(C_{[T]}, a)_{a \in P(C_{[T]})} \cup Th_{\forall}(E_{\Delta}) \cup \{P(c)\} \cup \{"P \subseteq "\},$$

where $\{"P \subseteq"\}$ is an infinite set of sentences expressing the fact that the interpretation of the symbol P is an existentially closed submodel in the language of the signature σ_Γ. That is, the interpretation of the symbol P is a solution to the equation $P(C_{[T]}) = M \subseteq E_{\Delta}$ in the language σ_Γ. Due to the heredity of the theory Δ, the theory $\bar{\Delta}$ is a Robinson theory. Collecting all such theories $\bar{\Delta}$, we obtain the class $[\bar{T}]$ of Robinson theories. The center $[\bar{T}]^* = Th(C_{[\bar{T}]})$ of class $[\bar{T}]$ is one of the completions for each theory $\bar{\Delta} \in [\bar{T}]$. Restricting the signature σ_Γ to $\sigma \cup \{P\}$, due to the laws of first-order logic, since the constant c does not already belong to this signature, we can replace this constant with the variable x. Then the theory $[\bar{T}]^*$ will be a complete 1-type for the variable x. We will call this type the central type of the class $[\bar{T}]$ in the above enrichment and denote it $P^c_{[\bar{T}]}$.

In work [6] was obtained criterion of uncountable categoricity for the hereditary Jonsson theory in the language of central types.

Theorem 6. [6] Let $[T]$ be hereditary class from $RSp(A)/_{\bowtie}$, then the following conditions are equivalent:

1) any countable model from $E_{[\overline{T}]}$ has an algebraically prime model extension in $E_{[\overline{T}]}$;

2) $P^c_{[\overline{T}]}$ is the strongly minimal type, where $P^c_{[\overline{T}]}$ is the central type of $[\overline{T}]$.

To prove the main result, we need a well-known fact:

Theorem 7 (Morley [31]). A theory T is ω_1-categorical if and only if any of its countable models has a simple proper elementary extension.

Obviously, we can use Morley's uncountable categoricity theorem in connection with the existence of an algebraically simple model extension for the central type in the framework of the following theorem. This means the following: the central type obtained by enriching the corresponding hereditary Robinson theory is exactly the center of the enriched Jonsson theory. If we replace the variable x with a constant that defines the central type, then we get a complete theory, which is model complete due to the perfection of the enriched Jonsson theory of this center. Thus, due to the model completeness, an algebraically simple model extension will also be a simple model extension, which allows us to consider this center as an ω_1-categorical theory, in which there is a strongly minimal formula, by virtue of the above Morley theorem.

Theorem 8. Let K be a Robinson hereditary variety, $[T] \in RSp(K)/_{\bowtie}$ be hereditary class, $X \subseteq C_{[T]}$ be a theoretical set defined by some strongly minimal $\exists$-formula $\varphi(x)$, Δ is some $\exists$-complete finitely axiomatizable Jonsson theory defined by $\forall x\varphi(x)$, then the following conditions are equivalent :

1) $\Delta \overset{S}{\rtimes} Fr(X) \overset{S}{\rtimes} T$;

2) the central type $P^c_{[\overline{T}]}$ of class $[\overline{T}]$ is strongly minimal.

Proof. If $\Delta \overset{S}{\rtimes} Fr(X) \overset{S}{\rtimes} T$, then by Theorem 3 $\Delta^* \overset{S}{\bowtie} Fr^*(X) \overset{S}{\bowtie} T^*$. But then, according to Proposition 3, these theories preserve the Morley rank, and, accordingly, the ω_1-categoricity, which is expressed in terms of the Morley rank. Thus, we have obtained that the theories Δ^*, $Fr^*(X)$ and $[T]^*$ are ω_1-categorical, i.e. all semantic models of these theories are saturated, hence the theories Δ, $Fr(X)$ and T are perfect. This means that the class $[T]$ is also perfect.

Note that in Theorem 6 item 1) is equivalent to the fact that the class $[T]$ is ω_1-categorical (this follows from Morley's theorem), and therefore perfect. Then, from Theorem 6 it follows that the central type $P^c_{[\overline{T}]}$ of the class $[\overline{T}]$ is strongly minimal.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09260237).

References

1 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во Караганд.гос.ун-та, 2016. — 370 с.

2 Yeshkeyev A.R. The structure of lattices of positive existential formulae of (Δ-PJ)-theories / A.R. Yeshkeyev // ScienceAsia. — 2013. — 39. — SUPPL.1. — P. 19–24. https://doi.org/10.2306/SCIENCEASIA1513-1874.2013.39S.019

3 Yeshkeyev A.R. Small models of hybrids for special subclasses of Jonsson theories / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics Series. — 2019. — No. 3(95). — P. 68–73. https://doi.org/10.31489/2019M2/68-73

4 Yeshkeyev A.R. (2019). Chains of existentially closed models of positive (n_1, n_2)-Jonsson theories / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University. Mathematics Series. — 2019. — No. 4(96). — P. 69–74. https://doi.org/10.31489/2019M4/69-74

5 Yeshkeyev A.R. Connection of Jonsson theory with some Jonsson polygons theories / A.R. Yeshkeyev, G.A. Urken // Bulletin of the Karaganda University. Mathematics Series. — 2019. — No. 3(95). — P. 74–78. https://doi.org/10.31489/2019M2/74-78

6 Yeshkeyev A.R. Independence and simplicity in Jonsson theories with abstract geometry / A.R. Yeshkeyev, M.T. Kassymetova, O.I. Ulbrikht // Siberian Electronic Mathematical Reports. — 2021. — 18. — No. 1. — P. 433–455. https://doi.org/10.33048/semi.2021.18.030

7 Yeshkeyev A.R. On Jonsson stability and some of its generalizations / A.R. Yeshkeyev // Journal of Mathematical Sciences. — 2010. — 166. — No. 5. — P. 646–654.

8 Yeshkeyev A.R. Method of the rheostat for studying properties of fragments of theoretical sets / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics Series. — 2020. — No. 4(100). — P. 152–159. https://doi.org/10.31489/2020M4/152-159

9 Yeshkeyev A.R. Central types of convex fragments of the perfect Jonsson theory / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University. Mathematics Series. — 2019. — No. 1(93). — P. 95–101. https://doi.org/10.31489/2019M1/95-101

10 Yeshkeyev A.R. The J-minimal sets in the hereditary theories / A.R. Yeshkeyev, M.T. Omarova, G.E. Zhumabekova // Bulletin of the Karaganda University. Mathematics Series. — 2019. — No. 2(94). — P. 92–98. https://doi.org/10.31489/2019M2/92-98

11 Yeshkeyev A.R. An algebra of the central types of the mutually model-consistent fragments / A.R. Yeshkeyev, N.M. Mussina // Bulletin of the Karaganda University. Mathematics Series. — 2021. — No. 1(101). — P. 111–118. https://doi.org/10.31489/2021M1/111-118

12 Ешкеев А.Р. О форкинге для Δ-PM-теории и стабильность центральных типов Δ-PJ-теорий в обогащённой сигнатуре / А.Р. Ешкеев // Вестн. Казах. нац. ун-та. Сер. Математика, механика, информатика. — 2009. — 62. — № 3. — С. 8–15.

13 Ешкеев А.Р. JSp-косемантичность и JSB – свойство абелевых групп / А.Р. Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2016. — 13. — С. 861–874. https://doi.org/10.17377/semi.2016.13.068

14 Ешкеев А.Р. JSp-косемантичность R-модулей / А.Р. Ешкеев, О.И. Ульбрихт // Сиб. электрон. мат. изв. — 2019. — 16. — С. 1233–1244. https://doi.org/10.33048/semi.2019.16.084

15 Ешкеев А.Р. Свойства моделей центра и центрального типа некоторых позитивных йонсоновских теорий в допустимых обогащениях сигнатуры / А.Р. Ешкеев, О.И. Ульбрихт, Р.М. Оспанов // Вестн. Караганд. ун-та. Сер. Математика. — 2013. — № 4(72). — С. 40–48.

16 Yeshkeyev A.R. The properties of central types with respect to enrichment by Jonsson set / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics Series. — 2017. — No. 1(85). — P. 36–40. https://doi.org/10.31489/2017M1/36-40

17 Yeshkeyev A.R. Model-theoretical questions of the Jonsson spectrum / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics Series. — 2020. — No. 2(98). — P. 165–173. https://doi.org/10.31489/2020M2/165-173

18 Yeshkeyev A.R. The Number of Fragments of the Perfect Class of the Jonsson Spectrum / A.R. Yeshkeyev, O.I. Ulbrikht, M.T. Omarova // Lobachevskii Journal of Mathematics. — 2022. — 43. — P. 3658–3673.

19 Yeshkeyev A.R. Similarity of classes from Jonsson spectrum / A.R. Yeshkeyev, G.A. Urken // The 16th Asian logic conference, 14th International Conference on Computability and Randomness (CCR 2019). Program and Abstracts. — Nur-Sultan, Kazakhstan. — 2019. — P. 44–45.

20 Urken G.A. Syntactic similarity of definable closures of Jonsson sets / G.A. Urken // Bulletin of the Karaganda University. Mathematics Series. — 2018. — No. 2(90). — P. 119–123. https://doi.org/10.31489/2018M2/119-123

21 Ешкеев А.Р. Синтаксическое подобие некоторых йонсоновских теорий и их связь с допустимостью / А.Р. Ешкеев, Г.А. Уркен // Современная математика: проблемы и приложения: Сб. тр. Вторых Международных научных Таймановских чтений, посвящ. 100-летию акад.

А.Д. Тайманова. — Кызылорда, 2017. — С. 125–130.

22 Бравайс Дж. Справочная книга по математической логике. — Ч. I. Теория моделей / Дж. Бравайс. — М.: Наука, 1982. — 392 с.

23 Mustafin Y.T. Quelques proprietes des theories de Jonsson / Y.T. Mustafin // The Journal of Symbolic Logic. — 2002. — Vol. 67. – No. 2. — P. 528–536.

24 Ешкеев А.Р. Йонсоновские теории / А.Р. Ешкеев. — Караганда: Изд-во Караганд. гос. ун-та, 2009. — 250 с.

25 Forrest W. Model theory for universal classes with the amalgamation property: a study in the foundations of model theory and algebra / W. Forrest // Annals of Mathematical Logic, 1977. — 11. — P. 263–366.

26 Mustafin T.G. On similarities of complete theories / T.G. Mustafin // Logic Colloquium '90, Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. — Helsinki, 1990. — P. 259–265.

27 Бускаран Э. Теория моделей и алгебраическая геометрия / Э. Бускаран. — М.: МЦНМО, 2008. — 280 с.

28 Ешкеев А.Р. Теоретико-модельные свойства йонсоновских фрагментов / А.Р. Ешкеев // Вестн. Караганд. ун-та. Сер. Математика. — 2014. – № 4(76). — С. 37–41.

29 Marker D. Model Theory: In introduction / D. Marker. — Springer-Verlag New York. Inc., 2002. — 342 p.

30 Мальцев А.И. Алгебраические системы / А.И. Мальцев. — М.: Наука, 1970. — 392 с.

31 Сакс Дж.Е. Теория насыщенных моделей / Дж.Е. Сакс. — М.: Мир, 1976. — 191 с.

32 Baldwin J.T. On strongly minimal sets / J.T. Baldwin, A.H. Lachlan // The Journal of Symbolic Logic. — 1971. — 36. — No. 1. — P. 79–96.

О.И. Ульбрихт, Н.В. Попова

Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қазақстан;
Қолданбалы математика институты, Қарағанды, Қазақстан

Теоретикалық жиынның фрагменті және оның қатты минималды централдық типі

Жұмыста алгебралардың жаңа класы анықталған, оның теориялары йонсондық теориялардың дербес жағдайы болып табылады. Бұл класс көптүрліліктерге де, йонсондық теорияларға да қолданылады. Осы мақаланың негізгі нәтижелері келесі екі нәтиже болып табылады. Авторлар робинсон мұралық көптүрлілігінің бекітілген спектрінің косеманттылық класының моделінің экзистенциалды тұйықталуы мен алгебралық тұйықталуының эквиваленттілігі туралы сұраққа жауап алынған. Централдық кластардың централдық типтерін және бекітілген спектрдің фрагменттерін зерттеу аясында қатты минималдылық критерийі алынды.

Кілт сөздер: йонсондық теория, экзистенциалды тұйық модель, алгебралық тұйық модель, косеманттылық, робинсон спектрі, робинсон мұралық көптүрлілігі, централдық тип, йонсондық фрагмент, теоретикалық жиын, қатты минималдық тип.

О.И. Ульбрихт, Н.В. Попова

Карагандинский университет имени академика Е.А. Букетова,
Институт прикладной математики, Караганда, Казахстан

Фрагмент теоретического множества и его сильно минимальный центральный тип

В работе определён новый класс алгебр, теория которых является частным случаем йонсоновских теорий. Данный класс относится и к многообразиям, и к йонсоновским теориям. Основными результатами настоящей статьи являются следующие два: авторами получены ответ на вопрос об эквивалентности экзистенциальной замкнутости и алгебраической замкнутости модели класса косемантичности фиксированного спектра робинсоновски наследственного многообразия, а также критерий сильной минимальности в рамках изучения центральных типов центральных классов и фрагментов фиксированного спектра.

Ключевые слова: йонсоновская теория, экзистенциально замкнутая модель, алгебраически замкнутая модель, косемантичность, робинсоновский спектр, робинсоновски наследственное многообразие, центральный тип, йонсоновский фрагмент, теоретическое множество, сильно минимальный тип.

References

1 Yeshkeyev, A.R., & Kassymetova, M.T. (2009). *Ionsonovskie teorii i ikh klassy modelei: monografiia [Jonsson theories and their classes of models: monograph].* Karaganda: Izdatelstvo Karagandinskogo gosudarstvennogo universiteta [in Russian].

2 Yeshkeyev, A.R. (2013). The structure of lattices of positive existential formulae of (Δ-PJ)-theories. *ScienceAsia, 39*(SUPPL.1), 19–24. https://doi.org/10.2306/SCIENCEASIA1513-1874.2013.39S.019

3 Yeshkeyev, A.R., & Mussina, N.M. (2019). Small models of hybrids for special subclasses of Jonsson theories. *Bulletin of the Karaganda University. Mathematics Series*, 3(95), 68–73. https://doi.org/10.31489/2019M2/68-73

4 Yeshkeyev, A.R., & Omarova, M.T. (2019). Chains of existentially closed models of positive (n_1, n_2)-Jonsson theories. *Bulletin of the Karaganda University. Mathematics Series*, 4(96), 69–74. https://doi.org/10.31489/2019M4/69-74

5 Yeshkeyev, A.R., & Urken, G.A. (2019) Connection of Jonsson theory with some Jonsson polygons theories. *Bulletin of the Karaganda University. Mathematics Series*, 3(95), 74–78. https://doi.org/10.31489/2019M2/74-78

6 Yeshkeyev, A.R., Kassymetova, M.T., & Ulbrikht, O.I. (2021). Independence and simplicity in Jonsson theories with abstract geometry. *Siberian Electronic Mathematical Reports, 18*(1), 433–455. https://doi.org/10.33048/semi.2021.18.030

7 Yeshkeyev, A.R. (2010). On Jonsson stability and some of its generalizations. *Journal of Mathematical Sciences, 166*(5), 646–654.

8 Yeshkeyev, A.R. (2020). Method of the rheostat for studying properties of fragments of theoretical sets. *Bulletin of the Karaganda University. Mathematics Series*, 4(100), 152–159. https://doi.org/10.31489/2020M4/152-159

9 Yeshkeyev, A.R. & Omarova, M.T. (2019) Central types of convex fragments of the perfect Jonsson theory. *Bulletin of the Karaganda University. Mathematics Series*, 1(93), 95–101. https://doi.org/10.31489/2019M1/95-101

10 Yeshkeyev, A.R., Omarova, M.T., & Zhumabekova, G.E. (2019). The J-minimal sets in the hereditary theories. *Bulletin of the Karaganda University. Mathematics Series*, 2(94), 92–98. https://doi.org/10.31489/2019M2/92-98

11 Yeshkeyev, A.R., & Mussina, N.M. (2021). An algebra of the central types of the mutually model-consistent fragments. *Bulletin of the Karaganda University. Mathematics Series*, 1(101), 111–118. https://doi.org/10.31489/2021M1/111-118

12 Yeshkeyev, A.R. (2009). O forkinge dlia Δ-PM-teorii i stabilnost tsentralnykh tipov Δ-PJ-teorii v obogashchennoi signature [On forking for Δ-PM-theories and the stability of central types of Δ-PJ-theories in an enriched signature]. *Vestnik Kazakhskogo natsionalnogo universiteta. Seriia Matematika, mekhanika, informatika — Bulletin of Kazakh National University. Series of Mathematics, Mechanics and Computer Science*, 3(62), 8–15 [in Russian].

13 Yeshkeyev, A.R., & Ulbrikht, O.I. (2016). JSp-kosemantichnost i JSB svoistvo abelevykh grupp [JSp-cosemanticness and JSB property of Abelian groups]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 13,* 861–874 [in Russian]. https://doi.org/10.17377/semi.2016.13.068

14 Yeshkeyev, A.R., & Ulbrikht, O.I. (2019). JSp-kosemantichnost R-modulei [JSp-cosemanticness of R-modules]. *Sibirskie elektronnye matematicheskie izvestiia — Siberian Electronic Mathematical Reports, 16*, 1233–1244 [in Russian]. https://doi.org/10.33048/semi.2019.16.084

15 Yeshkeyev, A.R., Ulbrikht, O.I., & Ospanov, R.M. (2013). Svoistva modelei tsentra i tsentralnogo tipa nekotorykh pozitivnykh ionsonovskikh teorii v dopustimykh obogashcheniiakh signatury [Properties of models of a center and central type of some positive Jonsson theories in an allowed enrichment of signatures]. *Vestnik Karagandinskogo universiteta. Seriia Matematika — Bulletin of the Karaganda University. Mathematics Series*, 4(72), 40–48 [in Russian].

16 Yeshkeyev, A.R. (2017). The properties of central types with respect to enrichment by Jonsson set. *Bulletin of the Karaganda University. Mathematics Series*, 1(85), 36–40. https://doi.org/10.31489/2017M1/36-40

17 Yeshkeyev, A.R. (2020). Model-theoretical questions of the Jonsson spectrum. *Bulletin of the Karaganda University. Mathematics Series*, 2(98), 165–173. https://doi.org/10.31489/2020M2/165-173

18 Yeshkeyev, A.R., Ulbrikht, O.I., & Omarova, M.T. (2022). The Number of Fragments of the Perfect Class of the Jonsson Spectrum. *Lobachevskii Journal of Mathematics, 43,* 3658–3673.

19 Yeshkeyev A.R., & Urken G.A. (2019). Similarity of classes from Jonsson spectrum. *The 16th Asian logic conference, 14th International Conference on Computability and Randomness (CCR 2019). Program and Abstracts.* Nur-Sultan, Kazakhstan, 44–45.

20 Urken, G.A. (2018). Syntactic similarity of definable closures of Jonsson sets. *Bulletin of the Karaganda University. Mathematics Series*, 2(90), 119–123. https://doi.org/10.31489/2018M2/119-123

21 Yeshkeyev, A.R., & Urken, G.A. (2017). Sintaksicheskoe podobie nekotorykh ionsonovskikh teorii i ikh sviaz s dopustimostiu [Syntactic similarity of some Jonsson theories and their connection with admissibility]. *Sovremennaia matematika: problemy i prilozheniia: sbornik trudov Vtorykh Mezhdunarodnykh nauchnykh Taimanovskikh chtenii, posviashchennykh 100-letiiu akademika A.D. Taimanova — Modern Mathematics: Problems and Applications: Proceedings of the Second International Scientific Taimanov Readings, Dedicated to the 100th Anniversary of Academician A.D. Taimanov.* Kyzylorda, 125–130 [in Russian].

22 Brawise, J. (1982). *Spravochnaia kniga po matematicheskoi logike. Chast I. Teoriia modelei [Handbook of mathematical logic. Part 1. Model theory].* Moscow: Nauka [in Russian].

23 Mustafin, Y.T. (2002). Quelques proprietes des theories de Jonsson. *The Journal of Symbolic Logic, 67*(2), 528–536.

24 Yeshkeyev, A.R. (2009). *Ionsonovskie teorii [Jonsson theories].* Karaganda: Izdatelstvo Karagandinskogo gosudarstvennogo universiteta [in Russian].

25 Forrest, W. (1977). Model theory for universal classes with the amalgamation property: a study in the foundations of model theory and algebra. *Annals of Mathematical Logic, 11,* 263–366.

26 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90, Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic,* Helsinki, 259–265.

27 Bouscaren E. (2008). Teoriia modelei i algebraicheskaia geometriia [Model Theory and Algebraic Geometry]. Moscow: MTsNMO [in Russian].

28 Yeshkeyev, A.R. (2014). Teoretiko-modelnye svoistva ionsonovskikh fragmentov [Model-theoretic properties of Jonsson fragments]. *Vestnik Karagandinskogo universiteta. Seriia Matematika – Bulletin of the Karaganda University. Mathematics Series*, 4(76), 37–41 [in Russian].

29 Marker, D. (2002). *Model Theory: In introduction.* Springer-Verlag New York, Inc.

30 Mal'tsev, A.I. (1970). *Algebraicheskie sistemy [Algebraic Systems].* Moscow: Nauka [in Russian].

31 Sacks, G.E. (1976). *Teoriia nasyshchennykh modelei [Saturated model theory].* Moscow: Mir [in Russian].

32 Baldwin, J.T., & Lachlan A.H. (1971). On strongly minimal sets. *The Journal of Symbolic Logic, 36*(1), 79–96.