DOI 10.31489/2023M3/47-58
UDC 510.67

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Geometry of strongly minimal hybrids of fragments of theoretical sets

In this article, strongly minimal geometries of fragment hybrids are considered. In this article, a new concept was introduced as a family of Jonsson definable subsets of the semantic model of the Jonsson theory T, denoted by $JDef(C_T)$. The classes of the Robinson spectrum and the geometry of hybrids of central types of a fixed $RSp(A)$ are considered. Using the construction of a central type for theories from the Robinson spectrum, we formulate and prove results for hybrids of Jonsson theories. A criterion for the uncountable categoricity of a hereditary hybrid of Jonsson theories is proved in the language of central types. The results obtained can be useful for continuing research on various Jonsson theories, in particular, for hybrids of Jonsson theories.

Keywords: Jonsson theory, semantic model, fragment, hybrid of Jonsson theories, Jonsson set, theoretical set, central type, pregeometry, Robinson theory, strongly minimal type.

Introduction

The current state of development of the conceptual and technical apparatus of model theory can be described without exaggeration as a set of syntactic and semantic concepts related to the consideration of most of the complete theories of first-order languages, on the other hand, due to the meager arsenal of the capabilities of the technical apparatus, the subject of study of incomplete theories. A special class of, generally speaking, incomplete theories is singled out in the study of Jonsson theories.

By virtue of the definition of the Jonsson theory, such a theory is, generally speaking, not complete. In the class of its models, there can be infinite and finite models, and isomorphic embeddings will also be used. Thus, we see that the transformation of certain results from complete theories to Jonsson's is complicated due to the different technical arsenal of the above theories. The reason for this problem is the replacement of elementary embeddings by isomorphic embeddings and the incompleteness of Jonsson theories. Thus, the universally homogeneous models that define the semantic model of Jonsson's theory are, generally speaking, not always saturated.

This fact clearly describes an example of group theory. The class of all groups has a Jonsson theory, a semantic model that is not saturated. In this regard, this class does not have a model companion, which makes it very difficult to apply the well-established technique of model companions to this class when studying the property of the center of this class.

Thus, the study of Jonsson theories is an important task.

In the works of the following authors, such as B. Jonsson [1], M. Morley and R. Vaught [2], A. Robinson [3], G. Cherlin [4], T.G. Mustafin [5], A.R. Yeshkeyev [6–8] gave a complete description of Jonsson theories and their companions. We would like to acknowledge the following authors with their publications, who played a great role in the study of this issue for Jonsson theories [9–12].

The notion of central type, which arises during signature enrichments, is one of the new concepts in Jonsson theories [13]. Thus, within the framework of the model theory of Jonsson theories, new relationships arise between classical concepts from the theory of models for complete theories.

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Also noteworthy is the emergence of a new method for constructing a Jonsson theory from two Jonsson theories. This is obtained using the concept of a hybrid of Jonsson theories, which was first defined in [14]. Various examples of algebraic objects and their constructions can be associated with this concept. In subsequent papers [15,16], results were obtained related to hybrids of Jonsson theories, which play an important role in model theory and in universal algebra.

The paper [17] considered Jonsson theories and their many syntactic and semantic properties of the first order in language enrichments that preserve the properties of Jonsson. Such Jonsson theories are called hereditary [12].

One of the classical methods of model theory is the method of interpreting a well-studied theory into a less-studied theory. Following the ideology of this method, a new method for studying Jonsson theories was defined, namely: using the concepts of syntactic and semantic similarity of Jonsson theories, new results were obtained in the framework of the classification of Jonsson theories.

1 Local properties of the geometry of strongly minimal sets

This article discusses the basic concepts of local properties of the geometry of strongly minimal sets on theoretical subsets of some existentially closed model. By studying the combinatorial properties of the pregeometry given on Jonsson sets, we have obtained results on relatively strongly minimal Jonsson sets. Minimal structures, pregeometries and geometries of minimal structures were defined. And also, for Jonsson theories, the concepts of dimension, independence and basis in Jonsson strongly minimal structures were considered.

First, let's define a hybrid of the first type and the second type.

Definition 1 ([14], p. 102). 1) Let T_1 and T_2 be some Jonsson theories of the countable language L of the same signature σ; C_1 and C_2 are their semantic models, respectively. In the case of common signature of Jonsson theories T_1, T_2, let us call a hybrid of Jonsson theories T_1 and T_2 of the first type the following theory $Th_{\forall\exists}(C_1 \diamond C_2)$ if that theory is Jonsson in the language of signature σ and denote it by $H(T_1, T_2)$, where the operation $\diamond \in \{\times, +, \oplus\}$ and $C_1 \diamond C_2 \in Mod\,\sigma$. Here $\times$ means cartesian product, $+$ means sum and $\oplus$ means direct sum. Herewith, the algebraic construction $(C_1 \diamond C_2)$ is called a semantic hybrid of the theories T_1, T_2.

2) If T_1 and T_2 are Jonsson theories of different signatures σ_1 and σ_2, then $H(T_1, T_2) = Th_{\forall\exists}(C_1 \diamond C_2)$ will be called a hybrid of the second type, if that theory is Jonsson in the language of signature $\sigma = \sigma_1 \cup \sigma_2$ where $C_1 \diamond C_2 \in Mod\,\sigma$.

Obviously that 1) is the particular case of 2).

Since the hybrid of two Jonsson theories is a Jonsson theory, in the case when this theory is perfect, we will say for brevity – a perfect hybrid of two Jonsson theories. As the center of the hybrid $H(T_1, T_2)$, we will mean the center of the Jonsson theory $Th_{\forall\exists}(C_1 \diamond C_2)$ and denote it by $H^*(T_1, T_2)$.

Let us define the Morley rank for existentially definable subsets of the semantic model.

We want to assign to each Jonsson subset of X of the semantic model an ordinal (or perhaps -1 or ∞) - its Morley rank, denoted by r_M.

Let T be a fragment of some Jonsson set, and it is a perfect Jonsson theory, C be a semantic model, Z be a definable set of C.

Definition 2. [6] $r_M(Z) \geq 0$, if and only if, Z is not empty; $r_M(Z) \geq \lambda$, if and only if, $r_M(Z) \geq \alpha$ for all $\alpha < \lambda$ (λ is limit ordinal); $r_M(Z) \geq \alpha + 1$, if and only if, in Z there is an infinite family Z_i of pairwise disjoint $\exists$-definable subsets such that $r_M(Z_i) \geq \alpha$ for all i.

Then the Morley rank of the set Z is $r_M(Z) = sup\{\alpha \mid r_M(Z)\} \geq \alpha$,

with the convention that $r_M(Z) = -1$ and $r_M(Z) = \infty$, if $r_M(Z) \geq \alpha$ for all α (in last case, we say that Z has no rank).

Definition 3. [6] The Morley degree $r_D(Z)$ of a Jonsson set Z, having Morley rank α, is the maximum length d of its decomposition $Z = Z_1 \cup ... \cup Z_n$ into disjoint existentially definable subsets of rank α.

If the rank is 0, then the degree of an existentially defined subset is the number of its elements. The Morley degree is also undefined if an existentially definable subset has no rank. In our case, we study Jonsson minimal sets. Note that a strongly minimal set is a set of rank 1 and degree 1.

Consider the closure operator, which is defined by an algebraic closure in the model-theoretic sense. A strongly minimal set that is equipped with the above closure operator is a pregeometry. A model of a strongly minimal theory is defined up to isomorphism by its dimension as a pregeometry. Completely categorical theories are controlled by a strongly minimal set; this remark is used in the proof of Morley's theorem. Boris Zilber considered the origin of pregeometry on vector spaces and algebraically closed fields.

Consider an example of an algebraic closure in Jonsson's strongly minimal theories, which is an existentially complete perfect Jonsson's theory in a countable language L.

If K is an algebraically closed field and $Z \subseteq K$, then $acl(Z)$ is an algebraically closed subfield generated by Z.

Consider the properties of the Jonsson algebraic closure that are true for any subset S of the semantic model of the Jonsson theory T.

Let M be some existentially closed submodel of the semantic model for a fixed theory in the language L, and $S \subseteq M$ be a Jonsson strongly minimal set.

Let $S \subseteq M^n$ be an infinite ∇-definable set, where $\nabla \subseteq L$ is the set of existential formulas of a given language.

Definition 4. [6] We say that S is Jonsson minimal in M if for any ∇-definable $Y \subseteq S$ either Y is finite or $S \backslash Y$ is finite.

Definition 5. [6] We say that S and φ are Jonsson strongly minimal if φ is Jonsson minimal in any existentially closed extension N from M.

Definition 6. [6] We say that a theory T is Jonsson strongly minimal if the formula $v = v$ is Jonsson strongly minimal (that is, if $M \in ModE_T$, then M is Jonsson strongly minimal)

Consider acl_S is an algebraic closure restricted to S.

For $Z \subseteq S$ let $acl_S(Z) = \{b \in S : \ b$ be a Jonsson algebraic over $Z\}$.

In our case, the properties of the [18] algebraic closure are true for the Jonsson algebraic closure of any subset S of the semantic model of the theory.

Lemma 1. [6]

1 $acl(acl(Z)) = acl(Z) \supseteq Z$.

2 If $Z \subseteq B$, then $acl(Z) \subseteq acl(B)$.

3 If $z \in acl(Z)$, then $z \in acl(Z_0)$ for some finite $Z_0 \subseteq Z$.

Lemma 2 (Exchange). [6] Suppose that $S \subset M$ is Jonsson strongly minimal, $Z \subseteq S$ and $z, b \in S$. If $z \in acl(Z \cup \{b\}) \backslash acl(Z)$, then $b \in acl(Z \cup \{z\})$.

The concept of linear independence in vector spaces is one of the important concepts of algebra, and the concept of independence generalizes linear independence in vector spaces and in algebraically closed fields. In turn, algebraic independence is defined in the Jonsson strongly minimal set we are considering.

Let $M \in Mod\, E_T$, and S be a Jonsson strongly minimal set in M.

Definition 7. [6] We will call $Z \subseteq S$ is Jonsson independently if $a \notin acl(Z \backslash \{z\})$ for all $z \in Z$. If $C \subset S$, we say that Z is Jonsson independent over S if $z \notin acl(C \cup (Z \backslash \{z\}))$ for all $z \in Z$.

Definition 8. [6] We will call Z is a Jonsson basis for $Y \subseteq S$ if $Z \subseteq Y$ is Jonsson independent and $acl(Z) = acl(Y)$.

Note that any maximal Jonsson independent subset of Y is a Jonsson basis for Y.

Definition 9. [6] If $Y \subseteq S$, then the Jonsson dimension of the set Y is the cardinality of the Jonsson basis for Y.

Let $J\dim Y$ denote the Jonsson dimension of Y.

If S is uncountable, then $J-\dim(S) =\mid S \mid$, since the language is countable and $acl(A)$ is countable for any countable $Z \subseteq D$.

A J-pregeometry (X, cl) is a subset X of the semantic model of some fixed Jonsson theory with operator $cl : P(X) \to P(X)$ on the set of subsets X and if the following conditions are satisfied:

1) if $A \subseteq X$, then $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$;

2) if $A \subseteq B \subseteq X$, then $cl(A) \subseteq cl(B)$;

3) (exchange) $A \subseteq X$, $a, b \in X$ and $a \in cl(A \cup \{b\})$, then $a \in cl(A)$, $b \in cl(A \cup \{a\})$;

4) (finite character) If $A \subseteq X$ and $a \in cl(A)$, then there is a finite $A_0 \subseteq A$, such that $a \in cl(A_0)$.

We say that $A \subseteq X$ is *closed*, if $cl(A) = A$.

Since D is a Jonsson strongly minimal set, the Jonsson pregeometry is defined as follows $cl(A) = acl(A) \cap D$ for $A \subseteq D$ (by Theorem 12 from [6] and Lemma 1).

Definition 10. [6] If (X, cl) is a Jonsson pregeometry, we will call A is Jonsson independent if $a \notin cl(A \setminus \{a\})$ for all $a \in A$, and B is a J-basis for Y if $B \subseteq Y$ is J-independent and $Y \subseteq acl(B)$.

If $A \subseteq X$, we also consider the localization $cl_A(B) = cl(A \cup B)$.

If (X, cl) is a J-predgeometry, then we will call $Y \subseteq X$ is *Jonsson independent over* A, if Y is Jonsson independent in (X, cl_A).

$\dim(Y/A)$ is the dimension of Y in the localization (X, cl_A), $\dim(Y/A)$ is called the dimension of Y over A.

Definition 11. [6] We will call a J-pregeometry (X, cl) is a J-geometry if $cl(\varnothing) = \varnothing$ and $cl(\{x\}) = \{x\}$ for any $x \in X$.

For further study, we denote some important properties of pregeometry.

Definition 12. [6] Let (X, cl) be a J-predgeometry. We will call (X, cl) is trivial if $cl(A) = \bigcup_{a \in A} cl(\{a\})$ for any $A \subseteq X$. We will call (X, cl) is modular if, for any finite-dimensional closed sets $A, B \subseteq X$, holds $Jdim(A \cup B) = Jdim(A) + Jdim(B) - Jdim(A \cap B)$.

(X, cl) is locally modular if (X, cl_a) is modular for some $a \in X$.

Theorem 1. [6] For a J-predgeometry (X, cl) the following are equivalent:

1 (X, cl) is modular;

2 if $A \subseteq X$ is closed and non-empty, $b \in X$, $x \in cl(A, b)$, then $\exists a \in A$, such that $x \in cl(a, b)$;

3 if $A, B \subseteq X$ are closed and non-empty, $x \in cl(A, B)$, then $\exists a \in A$ and $\exists b \in B$, such that $x \in cl(a, b)$.

Proof. Similarly to the proof of Lemma 8.1.13 from [19].

2 Model-theoretical properties of the Robinson spectrum

This section is devoted to the study of the model-theoretic properties of the Robinson spectrum of an arbitrary model of an arbitrary signature. The study of ω-categorical universals by specialists in model theory and universal algebra is well known ([20], § 5 of the appendix). In this section, we will deal with Robinson theories. The Robinson theory is a special case of the Jonsson theory, namely the Jonsson universal. To study the above theory, an algorithm for working with central types of a fixed spectrum was used. The elements of this spectrum are Jonsson universals. The result will be a central type enriched with additional constants and a unary predicate. Thus, we have obtained a criterion for the uncountable categoricity of the Robinson spectrum class in the language of central types.

In [21], Hrushovski E. defined Robinson theories of a theory as universal theories admitting a quantifier separation. In the study of Robinson theories, quantifier-free types occupy the main place. In our case, we are using central types.

Considering the structure of Jonsson sets, one can easily see that they have a very simple structure in the sense of the Morley rank, i.e. elements from the set-theoretic difference (hole) of the closure and the set have rank 0, i.e., they are all algebraic.

Another advantageous point for us in considering the concept of a Jonsson set is that we can obtain some existentially closed models by closing the Jonsson set.

This fact is necessary for studying the Morley rank for an arbitrary fragment of the set under consideration. Saturation for complete theories is a condition for the correctness of the definition of the Morley rank. Imperfect Jonsson theories require saturation with existential types in the semantic model. In the case of Jonsson sets, when studying elements from the set-theoretic difference, one can consider $\forall\exists$-consequences that are true in the closures of the Jonsson set. Based on this, we can conclude that the considered set of sentences will be Jonsson theory. In this section, strongly minimal Jonsson sets have been considered and described. The basic concepts associated with the notion of strong minimality for complete theories have been carried over to Jonsson theories. In particular, the notion of strong minimality is considered for fixed formula subsets of the semantic model of the Jonsson theory. In this case, the semantic model must be saturated in its power, i.e. the theory under consideration must be perfect. As is known, Jonsson's theory has a semantic model C of sufficiently large power. The semantic models of the perfect Jonsson theory are uniquely determined by their power. In our case, we will consider Jonsson subsets.

Definition 13. [6] A Jonsson theory T is called Robinson theory if it is universally axiomatizable.

Let T be a Robinson theory, A be an arbitrary model of signature σ. The Robinson spectrum of the model A is the set:

$$RSp(A) = \{T |\ T \text{ is Robinson theory in the language of signature } \sigma \text{ and } A \in Mod(T)\}.$$

Consider $RSp(A)/_{\bowtie}$ the factor set of the Robinson spectrum of the model A with respect to $\bowtie$.

If T is an arbitrary Robinson theory in the language of signature σ, then $E_{[T]} = \bigcup\limits_{\Delta \in [T]} E_{\Delta}$ is the class of all existentially closed models of class $[T] \in RSp(A)/_{\bowtie}$.

Let A be an arbitrary model of signature σ. Let $|RSp(A)/_{\bowtie}| = |K|$, K be some index set. We say that the class $[T] \in RSp(A)/_{\bowtie}$ is a $\aleph$-categorical if any theory $\Delta \in [T]$ is a $\aleph$-categorical and, respectively, the class $RSp(A)/_{\bowtie}$ will be called a $\aleph$-categorical if for each $j \in K$ the class $[T]_j$ is a $\aleph$-categorical.

Definition 14. [9] The set X is said to be Jonsson in the theory T if it satisfies the following properties:

1) X is the Σ-definable subset of C;

2) $dcl(X)$ is a support of some existentially closed submodel C.

Definition 15. [9] Let T be some Jonsson theory, C is the semantic model of the theory T, $X \subseteq C$. A set X is called theoretical set, if

1) X is Jonsson set, and let $\varphi(x)$ be the formula that defines the set X;

2) $\varphi(x) = \exists y \phi(x, y)$ and let θ be the universal closure of the formula $\varphi(x)$, i.e. θ is the sentence $\forall x \exists y \phi(x, y)$ defines some Jonsson theory.

Definition 16. [23] We say that all $\forall\exists$-consequences of an arbitrary theory create a Jonsson fragment of this theory, if the deductive closure of these $\forall\exists$-consequences is a Jonsson theory.

Definition 17. [23] We say that all $\forall$-consequences of an arbitrary theory create a Robinson fragment of this theory, if the deductive closure of these $\forall$-consequences is a Robinson theory.

We say that a model $M \in E_T$ is *Jonsson minimal* if for any definable $X \subseteq M$ either X is finite or $M \backslash X$ is finite. We say that a theory T Jonsson strongly minimal, if every model $M \in E_T$ is minimal. A non-algebraic type containing a Jonsson strongly minimal formula is called Jonsson strongly minimal.

Theorem 2 ([22], p. 298). Let T be universal theory, complete for existential sentences, having a countably algebraically universal model. Then T has an algebraically prime model, which is (Σ, Δ)-atomic.

Definition 18. A relational structure $C_T =< C, (X_i)_{i\in I} >$ consists of a (non empty) set C, and a family $(X_i)_{i\in I}$ of subsets of $\bigcup_{n\geq 1} C_T^n$, that is, for each i, X_i is a subset of $C_T^{n_i}$ for some $n_i \geq 1$. We add the extra condition that the diagonal of C_T^2 is one of the X_i's.

Each X_i is called an basic subset of C_T.

Definition 19. Let $C_T =< C, (X_i)_{i\in I} >$ be a semantic model of the Jonsson theory in pure predicate language. We define the family of Jonsson definable subsets of the semantic model of the Jonsson theory T, denoted by $JDef(C_T)$. $Def(C_T)$ is the smallest family of subsets of $\bigcup_{n\geq 1} C_T^n$ with the following properties:

- For every $i \in I$, $B_i \in JDef(C_T)$
- $JDef(C_T)$ is closed under finite boolean combinations, i.e. if $M, N \subseteq C_T^n$, M, N are the Jonsson sets. $M, N \in JDef(C_T)$, then $M \cup N \in JDef(C_T), M \cap N \in JDef(C_T)$ and $C_T^n \setminus M \in JDef(C_T)$.
- $JDef(C_T)$ is closed under cartesian product, i.e. if $M, N \in JDef(C_T)$, $M \times N \in JDef(C_T)$.
- $JDef(C_T)$ is closed under projection, i.e. if $M \subset C_T^{n+m}$, $N \in JDef(C_T)$, if $\pi_n(M)$ is the projection of M on C_T^n, $\pi_n(N) \in JDef(C_T)$.
- $JDef(C_T)$ is closed under specialization, i.e. if $M \in Def(C_T)$, $M \subseteq C_T^{n+k}$ and if $\bar{m} \in C_T^n$ then

$$M(\bar{m}) = \{\bar{b} \in C_T^k; (\bar{m}, \bar{b}) \in M\} \in JDef(C_T).$$

- $JDef(C_T)$ is closed under permutation of coordinates, i.e. if $M \in JDef(C_T), M \subseteq C_T^n$, if σ is any permutation of $\{1, ..., n\}$,

$$\sigma(M) = \{(a_{\sigma(1)}, ..., a_{\sigma(n)};\ (a_1, ..., a_n) \in M\} \in JDef(C_T).$$

$cl : (P(C_T)) \to P(C_T)$. $P(C_T) = \{A \subseteq C_T \mid A \in JDef(C_T)\}$. When T perfect Jonsson theory, then T^* is the model complete, $\varphi(x) \in T$ follow that $\exists\psi(x),\ \ \psi(x) \in \Sigma_1$ such that $T^* \vdash \forall x(\varphi(x) \leftrightarrow \psi(x))$.

Definition 20. [17] An enrichment $\tilde{T}$ is called admissible if the ∇-type (this means that the ∇subset of the language L_σ and any formula from this type belongs to ∇) in this enrichment is definable within the framework of $\tilde{T}_\Gamma$-stability, where Γ is the enrichment of the signature σ.

Definition 21. [17] A Robinson theory T is called hereditary if, in any of its admissible enrichments, any extension is a Robinson theory. The class $[T] \in RSp(A)/_{\bowtie}$ will be called hereditary if each theory $\Delta \in [T]$ is hereditary.

Definition 22. [17] A model A is called the Δ-good algebraically prime model of the theory T if A is a countable model of the theory T and for each model B of the theory T, each $n \in \omega$ and all $a_0, \ldots, a_{n-1} \in A$, $b_0, \ldots, b_{n-1} \in B$ if $(A, a_0, \ldots, a_{n-1}) \equiv_\Delta (B, b_0, \ldots, b_{n-1})$, then for each $a_n \in A$ there is some $b_n \in B$, such that $(A, a_0, \ldots, a_n) \equiv_\Delta (B, b_0, \ldots, b_n)$.

Definition 23. [23] Let T_1 and T_2 are Jonsson theory. We will say, that T_1 and T_2 are J-syntactically similar, if there is bijection $f : E(T_1) \to E(T_2)$ such that:

1) restriction f to $E_n(T_1)$ is isomorphism lattice $E_n(T_1)$ and $E_n(T_2)$, $n < \omega$;

2) $f(\exists v_{n+1}\varphi) = \exists\varphi n + 1 f(\varphi), \varphi \in E_{n+1}(T), n < \omega$;

3) $f(v_1 = v_2) = (v_1 = v_2)$

Consider the general scheme for obtaining the central type for an arbitrary Robinson theory.

Let C_T be a semantic model of the theory T, $A \subseteq C_T$. Let $\sigma_\Gamma = \sigma \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$. Let $\bar{T} = Th_\forall(C_T, a)_{a \in P(C_T)} \cup Th_\forall(E_T) \cup \{P(c)\} \cup \{"P \subseteq"\}$, where $P(C_T)$ is an existentially closed submodel of C_T, $\{"P \subseteq"\}$ is an infinite set of sentences, demonstrating that P is an existentially closed submodel of signature σ_Γ. This means that P is a solution to the equation $P(C_T) = M \subseteq E_T$ of signature σ_Γ. Due to the heredity of T, the theory $\bar{T}$ is also a Jonsson theory. Consider all completions of the theory $\bar{T}$ of signature σ_Γ. Since the theory $\bar{T}$ is Jonsson's, it has its own center, denoted by $\bar{T}^*$. The above mentioned center is one of the completions of the $\bar{T}$ theory. When the signature σ_Γ is restricted to $\sigma \cup P$, the constant c does not belong to this signature. Therefore, we can replace this constant with the variable x. After that, this theory will be a complete 1-type for the variable x.

Let X_1, X_2 be the strongly minimal theoretical sets. $Fr(X_1) = T_1, Fr(X_2) = T_2$ are the Robinson fragments. $H(T_1, T_2) = Th_\forall(C_{T_1} \times C_{T_2})$, $cl(X_1) = M_1, cl(X_2) = M_2; M_1, M_2 \in E_T$. $Fr(X_1) = \Delta_1, Fr(X_2) = \Delta_2$. Δ_1, Δ_2 are Jonsson syntactical similar. By virtue of Jonsson syntactical similarity of this fact $Th_\forall(M_1) = T_1, Th_\forall(M_2) = T_2$ also Jonsson syntactical similar. T_1, T_2 are the Jonsson strongly minimal theories. Then since $\bar{T}_1$ is a Jonsson theory, it has its own center, let us denote it by $\bar{T}_1^*$, this center is one of the above completions of the theory $\bar{T}_1$. Accordingly $\bar{T}_2$ is a Jonsson theory, it has its own center, let us denote it by $\bar{T}_2^*$, this center is one of the above completions of the theory $\bar{T}_2$. In the theorem we consider the hybrid $H(\bar{T}_1, \bar{T}_2)$ of the Jonsson theries T_1, T_2.

R_1 is every existential formula $\varphi(\bar{x})$ consistent with T is implied by some Δ formula $\theta(\bar{x})$ consistent with T.

Theorem 3. Let $[T]$ be class from $RSp(A)/_{\bowtie}$, complete for existential sentences, admitting R_1. Let $T_1, T_2 \in [T]$. Then the following conditions are equivalent:

1 $H(T_1, T_2)$ has an algebraically prime model;

2 $H(T_1, T_2)$ has $(\exists, \Delta)$-atomic model;

3 $H(T_1, T_2)$ has $(\Delta, \exists)$-atomic model;

4 $H(T_1, T_2)$ has a Δ-good algebraically prime model;

5 $H(T_1, T_2)$ has a single algebraically prime model.

Proof. Let $T_1, T_2 \in [T]$ satisfies the conditions of Theorem 3, then by virtue of the theorem 4.1 ([22], p.309) the $H(T_1, T_2)$ also satisfies the conditions this theorem.

Theorem 4. Let $[T]$ be hereditary class from $RSp(A)/_{\bowtie}$, $T_1, T_2 \in [T]$, then the following conditions are equivalent:

1 any countable model from $E_{H(\bar{T}_1, \bar{T}_2)}$ has an algebraically prime model extension in $E_{H(\bar{T}_1, \bar{T}_2)}$;

2 $P^c_{H(\bar{T}_1, \bar{T}_2)}$ is the strongly minimal type, where $P^c_{H(\bar{T}_1, \bar{T}_2)}$ is the central type of $H(\bar{T}_1, \bar{T}_2)$.

Proof. (1)⇒(2). For convenience of the proof, we denote $H(\bar{T}_1, \bar{T}_2) = \mathbb{T}$. Consider a semantic model $C_\mathbb{T}$ of the class $[T]$. The $C_\mathbb{T}$ model is ω-universal by virtue of the definitions of κ-universality and κ-homogeneity. In our case, the power is uncountable. Therefore, consider a countable elementary submodel D of the $C_\mathbb{T}$ model. The elementary submodel D is existentially closed since $C_\mathbb{T}$ is existentially closed by virtue of (Lemma [23], p. 162). Therefore, the elementary submodel D is countably algebraically universal. We apply the 2 theorem, according to which every theory $\overline{\Delta} \in \mathbb{T}$ has an algebraically simple model A_0. We define $A_{\delta+1}$ by induction, which is an algebraically simple extension of the A_δ model and $A_\lambda = \bigcup\{A_\delta \mid \delta < \lambda\}$. Then let $\overline{A} = \bigcup\{A_\delta | \delta < \omega_1\}$. Suppose $B \models \Delta$ and $cardB = \omega_1$. Let us show that

$B \approx A$, for this we decompose B into a chain $\{B_\delta | \delta < \omega_1\}$ of countable models. Such a decomposition is possible due to the fact that the $\overline{\Delta}$ theory is Jonsson. We define the function $g: \omega_1 \to \omega_1$ and the chain $\{f_\delta : A_{g\delta} \to B_\delta \mid 0 < \delta < \omega_1\}$ of isomorphisms by the formula induction on δ:

1) $g0 = 0$ and $f_0 : A_0 \to B_0$;

2) $g\lambda = \bigcup\{g\delta | \delta < \lambda\}$ and $f_\lambda = \bigcup\{f_\delta | \delta < \lambda\}$;

3) $f_{\delta+1}$ is equal to the union of the chain $\{f_\delta^\gamma | \gamma \leq \rho\}$, which is determined by induction on γ;

4) $f_{\delta+1}^0 = f_\delta$, $f_{\delta+1}^\lambda = \bigcup\{|f_{\delta+1}^\gamma | \gamma < \lambda\}$;

5) suppose that $f_1^\gamma : A_{g\delta+\gamma} \to B_{\delta+1}$. If $f_{\delta+1}^\gamma$ is a mapping onto, then $\rho = \gamma$. Otherwise, by virtue of the algebraic primeness of $A_{g\delta+\gamma+1}$, we can extend $f_{\delta+1}^\gamma$ to $f_{\delta+1}^{\gamma+1} : A_{g\delta+\gamma+1} \to B_{\delta+1}$;

6) $g(\delta+1) = g\delta + \rho$.

By virtue of $f = \bigcup\{| f_\delta \mid \delta < \omega_1\}$ $\overline{A}$ is mapped isomorphically to B. Now let's apply the theorem 3. B is an arbitrary model of the Δ theory. $\overline{A}$ is the only algebraic prime and existentially closed model. By virtue of the condition and construction, it follows that E_Δ for each $\overline{\Delta} \in \mathbb{T}$ has a unique model in uncountable cardinality. This condition means that the semantic model $C_\mathbb{T}$ is saturated, i.e. the class $\mathbb{T}$ will be perfect. Thus $Mod\mathbb{T}^* = E_\mathbb{T}$. Therefore, the theory $\mathbb{T}^*$ is ω_1-categorical. $\mathbb{T}^*$ has a strongly minimal formula according to the Lachlan-Baldwin theorem. Since we are dealing with a central type, we get a non-principal type that contains the Jonsson strongly minimal formula. This implies that the type is Jonsson strongly minimal.

(2) $\Rightarrow$ (1). Due to the fact that $P_\mathbb{T}^c$ is a strongly minimal type, when passing to the signature $\sigma_\Gamma = \sigma \cup \Gamma$, the type becomes $\mathbb{T}^*$ theory. As mentioned above, the theory is the center of the class $\mathbb{T}$, hence it is complete. Let us show that $\mathbb{T}^*$ is ω_1-categorical. By inductance, for any models $A, B \in Mod\mathbb{T}^*$ there are models $A', B' \in E_\mathbb{T}$ and isomorphic embeddings $f : A \to A'$, $g : B \to B'$. Suppose $|A'| = |B'| = \omega_1$. If $A \not\cong B$, then $A' \not\cong B'$. Therefore, there exists $\varphi(x) \in B(At)$ such that $A' \models \varphi(x)$ and $B' \models \neg\varphi(x)$. Since in our case $\mathbb{T}$ is an inherited class, then $\mathbb{T} \in RSp(A)/_{\bowtie}$. Due to the universal axiomatizability of this class and the fact that $A' \in Mod(\mathbb{T}^*)$ as an existentially closed model is isomorphically embedded into the semantic model C of the class $\mathbb{T}$. Since $\mathbb{T}^* = Th(C)$ is complete, $\mathbb{T}^* \vdash \exists x\varphi(x)$ follows. Since A' and B' are Jonsson minimal, either $\varphi(A')$ is finite or $A' \backslash \varphi(A')$ is finite. Let $\varphi(A')$ be finite, then there exists a $\forall\exists$-proposition ψ which shows that $\varphi(A')$ is finite and $\mathbb{T}^* \vdash \forall\exists(\varphi \& \psi)$ hence $B' \models \psi(x)$ but $B' \models \psi(x) \& \neg\varphi(x)$, but at the same time, since $A', B' \in E_\mathbb{T}$, $A' \equiv_{\forall\exists} B'$, then we got a contradiction with strongly minimality.

If the definable complement of the formula is finite in the model A' under consideration, then the proof is carried out in a similar way. Thus $\mathbb{T}$ is ω_1-categorical.

By virtue of Morley's uncountable categoricity theorem, $\mathbb{T}^*$ is ω_1-categorical, and hence this theory is perfect. Then, by virtue of the Jonsson theory completeness criterion $\mathbb{T}^*$ is a model complete theory and $Mod\mathbb{T}^* = E_\Delta$ for every $\overline{\Delta} \in \mathbb{T}$, i.e. $Mod\mathbb{T}^* = E_{(T)}$. If $\mathbb{T}^*$ is model complete, then any isomorphic embedding is elementary. Since $\mathbb{T}^*$ is a complete theory, by virtue of Morley's theorem we obtain what is required.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP09260237).

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Қатты минималды гибридтерінің фрагменттерінің теоретикалық жиындарының геометриясы

Мақала фрагмент гибридтерінің қатты минималды геометрияларын зерттеуге арналған. Авторлар $JDef(C_T)$ деп белгіленген «T йонсондық теорияның семантикалық моделінің йонсондық анықталған ішкі жиындарының үйірі» деген жаңа тұжырымдама енгізген. Робинсон спектрінің кластары және бекітілген $RSp(A)$ централдық типтерінің гибридтерінің геометриясы қарастырылған. Робинсон спектріндегі теориялар үшін централдық типті құруды пайдалана отырып, йонсондық теориялардың гибридтері үшін нәтижелерді тұжырымдалған және дәлелденген. Йонсондық теориялардың мұралық гибридінің саналымсыз категориялық критерийі централдық типтер тілінде дәлелденген. Алынған нәтижелер йонсондық әртүрлі теориялар бойынша, атап айтқанда, йонсондық теориялардың гибридтері бойынша зерттеулерді жалғастыру үшін пайдалы болуы мүмкін.

Кілт сөздер: йонсондық теория, семантикалық модель, фрагмент, йонсондық теориялардың гибриді, йонсондық жиын, теоретикалық жиын, централдық тип, алғашқы геометрия, робинсондық теория, қатты минималды тип.

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Геометрия сильно минимальных гибридов фрагментов теоретических множеств

Статья посвящена изучению сильно минимальных геометрий гибридов фрагментов. Авторами было введено новое понятие «семейство йонсоновских определимых подмножеств семантической модели йонсоновской теории T», обозначаемое через $JDef(C_T)$. Рассмотрены классы робинсоновского спектра и геометрия гибридов центральных типов фиксированного $RSp(A)$. С помощью построения центрального типа для теорий из робинсоновского спектра формулируются и доказываются результаты для гибридов йонсоновских теорий, в частности, критерий несчетной категоричности наследственного гибрида йонсоновских теорий на языке центральных типов. Полученные результаты могут быть полезны для продолжения исследований различных йонсоновских теорий, в частности, для гибридов йонсоновских теорий.

Ключевые слова: йонсоновская теория, семантическая модель, фрагмент, гибрид йонсоновских теорий, йонсоновское множество, теоретическое множество, центральный тип, предгеометрия, робинсоновская теория, сильно минимальный тип.

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