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Homogenization of Attractors to Ginzburg-Landau Equations in Media with Locally Periodic Obstacles: Critical Case

In this paper the Ginzburg-Landau equation is considered in locally periodic porous medium, with rapidly oscillating terms in the equation and boundary conditions. It is proved that the trajectory attractors of this equation converge in a weak sense to the trajectory attractors of the limit Ginzburg-Landau equation with an additional potential term. For this aim we use an approach from the papers and monographs of V.V. Chepyzhov and M.I. Vishik concerning trajectory attractors of evolution equations. Also we apply homogenization methods appeared at the end of the XX-th century. First, we apply the asymptotic methods for formal construction of asymptotics, then, we verify the leading terms of asymptotic series by means of the methods of functional analysis and integral estimates. Defining the appropriate axillary functional spaces with weak topology, we derive the limit (homogenized) equation and prove the existence of trajectory attractors for this equation. Then we formulate the main theorem and prove it with the help of axillary lemmas.

Keywords: attractors, homogenization, Ginzburg-Landau equations, nonlinear equations, weak convergence, perforated domain, strange term, porous medium.

Introduction

This work is connected with modelling of processes in perforated materials and porous media. Asymptotic analysis of solutions to problems in porous media is sufficiently complicated, especially in the case of a threshold value of sizes and a number of cavities with nontrivial Robin (Fourier) conditions on their boundaries, i.e. in the case of a singular perturbation of problems. In this situation the limit equation describing the effective behavior of the model, has a different structure if one compares it with the given one. We investigate the situation when an additional potential term appears in the limit Ginzburg-Landau equation and prove the Hausdorff convergence of attractors as the small parameter tends to zero. Thus, we construct the limit attractor and prove the convergence of the attractors of the given problem, to the attractor of the limit problem with an additional potential in the equation. Here we investigate the asymptotic behavior of attractors to an initial boundary value problem for complex Ginzburg-Landau equations in porous media. In many pure mathematical papers one can find the asymptotic analysis of problems in porous media (see, for example, [1–7]). Interesting homogenization results have been obtained for periodic, almost periodic and random structures. We want to mention here the basic frameworks [8–11], where one can find the detail bibliography.

About attractors see, for instance, [12–14] and the references in these monographs. Homogenization of attractors were studied in [14–17] (see also [18–21]).

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In this paper we present the proofs of weak convergence of the trajectory attractor $\mathfrak{A}_\epsilon$ to the Ginzburg-Landau equation in a perforated domain, as $\epsilon \to 0$, to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in some natural functional space. Here, the small parameter ϵ characterizes the linear size of cavities and the distance between them in porous medium. We prove the appearance of a so called "strange term" (the potential term) in the limit equation (for example see works [1, 2]).

1 Statement of the problem

We start by the definition of a perforated domain. Suppose $\Omega \subset \mathbb{R}^d, d \geq 2$, is a smooth bounded domain. Denote

$$\Upsilon_\epsilon = \{j \in \mathbb{Z}^d \, : \, \text{dist}\,(\epsilon j, \partial\Omega) \geq \epsilon\sqrt{d}\}, \quad \Box \equiv \{\xi : -\frac{1}{2} < \xi_j < \frac{1}{2}, \; j = 1, \dots, d\}.$$

Considering a smooth function $F(x,\xi)$ 1-periodic in ξ, which satisfies $F(x,\xi)\Big|_{\xi\in\partial\Box} \geq \text{const} > 0$, $F(x,0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \Box\backslash\{0\}$, we define $D_j^\epsilon = \{x \in \epsilon\,(\Box + j) \mid F(x, \frac{x}{\epsilon}) \leq 0\}$. The perforated domain now is defined in the following way:

$$\Omega_\epsilon = \Omega \backslash \bigcup_{j\in\Upsilon_\epsilon} D_j^\epsilon.$$

Denote by ω the set $\{\xi \in \mathbb{R}^d \mid F(x,\xi) < 0\}$, and by S the set $\{\xi \in \mathbb{R}^d \mid F(x,\xi) = 0\}$. The boundary $\partial\Omega_\epsilon$ consists of $\partial\Omega$ and the boundary of the holes $S_\epsilon \subset \Omega$, $S_\epsilon = (\partial\Omega_\epsilon) \cap \Omega$.

We study the asymptotic behavior of attractors to the problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} = (1+\alpha \mathrm{i})\Delta u_\epsilon + R(x, \dfrac{x}{\epsilon})\, u_\epsilon - \left(1 + \beta(x, \dfrac{x}{\epsilon})\mathrm{i}\right) |u_\epsilon|^2 u_\epsilon + g(x), & x \in \Omega_\epsilon, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_\epsilon}{\partial \nu} + \epsilon q(x, \dfrac{x}{\epsilon}) u_\epsilon = 0, & x \in S_\epsilon, t > 0, \\ u_\epsilon = 0, & x \in \partial\Omega, \\ u_\epsilon = U(x), & x \in \Omega_\epsilon, t = 0, \end{cases} \tag{1}$$

where α is a real constant, the vector ν is the outer unit vector to the boundary, $u = u_1 + \mathrm{i}u_2 \in \mathbb{C}$, $g(x) \in C^1(\Omega;\mathbb{C})$, a nonnegative 1-periodic in ξ function $q(x,\xi)$ belongs to $C^1(\Omega;\mathbb{R}^d)$. Suppose that

$$-\beta_1 \leq \beta(x,\xi) \leq \beta_2, \; -R_1 \leq R(x,\xi) \leq R_2 \;\; (\text{where } R_0, R_1, \beta_1, \beta_2 > 0), \tag{2}$$

for $x \in \Omega$, $\xi \in \mathbb{R}^d$ and the functions $R\,(x,\xi)$ and $\beta\,(x,\xi)$ can be averaged in $L_{\infty,*w}(\Omega)$. The averages are $\bar{R}(x)$ and $\bar{\beta}(x)$ respectively, i.e.,

$$\begin{aligned} \int_\Omega R\,(x,\xi)\,\varphi_1(x)dx \;&\to\; \int_\Omega \bar{R}(x)\varphi_1(x)dx, \\ \int_\Omega \beta\,(x,\xi)\,\varphi_1(x)dx \;&\to\; \int_\Omega \bar{\beta}(x)\varphi_1(x)dx \end{aligned} \tag{3}$$

for any $\varphi_1(x) \in L_1(\Omega)$, where $\xi = \dfrac{x}{\epsilon}$ as $\epsilon \to 0+$.

We define the following spaces: $\mathbf{H} := L_2(\Omega;\mathbb{C})$, $\mathbf{H}_\epsilon := L_2(\Omega_\epsilon;\mathbb{C})$, $\mathbf{V} := H_0^1(\Omega;\mathbb{C})$, $\mathbf{V}_\epsilon := H^1(\Omega_\epsilon;\mathbb{C};\partial\Omega)$ is a set of functions from $H^1(\Omega_\epsilon;\mathbb{C})$ with a zero trace on $\partial\Omega$, and $\mathbf{L}_p := L_p(\Omega;\mathbb{C})$, $\mathbf{L}_{p,\epsilon} := L_p(\Omega_\epsilon;\mathbb{C})$. These spaces have, respectively, the next norms

$$\|v\|^2 := \int_\Omega |v(x)|^2 dx, \qquad \|v\|_\epsilon^2 := \int_{\Omega_\epsilon} |v(x)|^2 dx, \qquad \|v\|_1^2 := \int_\Omega |\nabla v(x)|^2 dx,$$

$$\|v\|_{1\epsilon}^2 := \int_{\Omega_\epsilon} |\nabla v(x)|^2 dx, \quad \|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p dx, \quad \|v\|_{\mathbf{L}_p\,\epsilon}^p := \int_{\Omega_\epsilon} |v(x)|^p dx.$$

Let us denote that dual spaces to $\mathbf{V}$ by $\mathbf{V}' := H^{-1}(\Omega;\mathbb{C})$ and, moreover, $\mathbf{L}_q$ is the dual spaces of $\mathbf{L}_p$, where $q=\frac{p}{p-1}$, in analogous way $\mathbf{V}'_\epsilon$ and $\mathbf{L}_{q,\epsilon}$ are the dual spaces of $\mathbf{V}_\epsilon$ and $\mathbf{L}_{p,\epsilon}$.

As usually (see [14]) we investigate the behavior of weak solutions to initial boundary value problem (1), i.e., the functions

$$u_\epsilon(x,s) \in L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\epsilon)\cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\epsilon)\cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\epsilon})$$

which satisfy problem (1) in the sense of integral identity, i.e. for any function $\psi\in C_0^\infty(\mathbb{R}_+;\mathbf{V}_\epsilon\cap\mathbf{L}_{4,\epsilon})$ we have

$$-\int_0^\infty\int_{\Omega_\epsilon} u_\epsilon\frac{\partial\psi}{\partial t}\,dxdt+(1+\alpha\mathrm{i})\int_0^\infty\int_{\Omega_\epsilon}\nabla u_\epsilon\nabla\psi\,dxdt-\int_0^\infty\int_{\Omega_\epsilon}\Big(\Big(R\Big(x,\frac{x}{\epsilon}\Big)u_\epsilon- \\ -\Big(1+\beta\Big(x,\frac{x}{\epsilon}\Big)\mathrm{i}\Big)|u_\epsilon|^2u_\epsilon\Big)\Big)\psi\,dxdt+\epsilon\int_0^{+\infty}\int_{S_\epsilon} q\left(x,\frac{x}{\epsilon}\right)u_\epsilon\psi\,d\sigma dt=\int_0^\infty\int_{\Omega_\epsilon} g(x)\psi\,dxdt. \quad (4)$$

Since $u_\epsilon(x,t)\in L_4(0,M;\mathbf{L}_{4,\epsilon})$, one can get $R\left(x,\frac{x}{\epsilon}\right)u_\epsilon(x,t)-\left(1+\beta\left(x,\frac{x}{\epsilon}\right)\mathrm{i}\right)|u_\epsilon(x,t)|^2u_\epsilon(x,t)\in L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon})$. In addition, since $u_\epsilon(x,t)\in L_2(0,M;\mathbf{V}_\epsilon)$, we have $(1+\alpha\mathrm{i})\Delta u_\epsilon(x,t)+g(x)\in L_2(0,M;\mathbf{V}'_\epsilon)$. Consequently, for any weak solution $u_\epsilon(x,s)$ to problem (1) we obtain

$$\frac{\partial u_\epsilon(x,t)}{\partial t}\in L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon})+L_2(0,M;\mathbf{V}'_\epsilon).$$

Keeping in mind the Sobolev embedding theorem, we conclude $L_{4/3}(0,M;\mathbf{L}_{4/3,\epsilon})+L_2(0,M;\mathbf{V}'_\epsilon)\subset L_{4/3}\left(0,M;\mathbf{H}_\epsilon^{-r}\right)$. Here the space $\mathbf{H}_\epsilon^{-r}:=H^{-r}(\Omega_\epsilon;\mathbb{C})$ and $r=\max\{1,d/4\}$. Therefore, for an arbitrary weak solution $u_\epsilon(x,t)$ of (1) we get $\frac{\partial u_\epsilon(x,t)}{\partial t}\in L_{4/3}\left(0,M;\mathbf{H}_\epsilon^{-r}\right)$.

Remark 1.1. Using the standard approach from [13], one can prove the existence of weak solution $u(x,s)$ to the problem (1) for every $U\in\mathbf{H}_\epsilon$ and fixed ϵ, satisfying $u(x,0)=U(x)$.

It is possible to prove the following basic Lemma similarly to Proposition 3 from [20].

Lemma 1.1. Suppose that $u_\epsilon(x,t)\in L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\epsilon)\cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\epsilon})$ is a weak solution to (1). Then
(i) $u\in C(\mathbb{R}_+;\mathbf{H}_\epsilon)$;
(ii) the function $\|u_\epsilon(\cdot,t)\|_\epsilon^2$ is absolutely continuous on $\mathbb{R}_+$ and, moreover,

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon(\cdot,t)\|_\epsilon^2+\|\nabla u_\epsilon(\cdot,t)\|_\epsilon^2+\|u_\epsilon(\cdot,t)\|_{\mathbf{L}_{4,\epsilon}}^4-\int_{\Omega_\epsilon}R\left(x,\frac{x}{\epsilon}\right)|u_\epsilon(x,t)|^2dx+ \\ +\epsilon\int_{S_\epsilon}q\left(x,\frac{x}{\epsilon}\right)|u_\epsilon(x,t)|^2d\sigma=\int_{\Omega_\epsilon}Re\left(g(x)\bar{u}_\epsilon(x,t)\right)dx,$$

for almost every $t\in\mathbb{R}_+$.

We fix ϵ. Bellow, where it is natural, we omit the index ϵ in the notation of functional spaces. Now we use the approach described in Section 2 to construct the trajectory attractor of (1), which has the form (7) if we set $E_1=\mathbf{L}_p\cap\mathbf{V},\ E_0=\mathbf{H}^{-r},\ E=\mathbf{H}$ and $A(u)=(1+\alpha\mathrm{i})\Delta u+R(\cdot)\,u-(1+\beta(\cdot)\mathrm{i})\,|u|^2u+g(\cdot)$.

To define the trajectory space $\mathcal{K}_\epsilon^+$ for (1), we use the general approaches of Section 2 and for every $[t_1,t_2]\in\mathbb{R}$ we have the Banach spaces

$$\mathcal{F}_{t_1,t_2}:=L_4(t_1,t_2;\mathbf{L}_4)\cap L_2(t_1,t_2;\mathbf{V})\cap L_\infty(t_1,t_2;\mathbf{H})\cap\left\{v\ \Big|\ \frac{\partial v}{\partial t}\in L_{4/3}\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\}$$

with the following norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}}:=\|v\|_{L_4(t_1,t_2;\mathbf{L}_4)}+\|v\|_{L_2(t_1,t_2;\mathbf{V})}+\|v\|_{L_\infty(0,M;\mathbf{H})}+\left\|\frac{\partial v}{\partial t}\right\|_{L_{4/3}(t_1,t_2;\mathbf{H}^{-r})}.$$

Setting $\mathcal{D}_{t_1,t_2} = L_q(t_1, t_2; \mathbf{H}^{-r})$ we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(s) \in \mathcal{F}_{t_1,t_2}$ we have $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. One considers now weak solutions to (1) as solutions of an equation in the general scheme of Section 2.

Consider the spaces

$$\mathcal{F}_+^{loc} = L_4^{loc}(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}_{\epsilon,+}^{loc} = L_4^{loc}(\mathbb{R}_+; \mathbf{L}_{4,\epsilon}) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}_\epsilon) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\epsilon) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}_\epsilon{}^{-r}) \right\}.$$

We introduce the following notation. Let $\mathcal{K}_\epsilon^+$ be the set of all weak solutions to (1). For any $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\epsilon^+$ such that $u(0) = U(x)$. Consequently, the space $\mathcal{K}_\epsilon^+$ to (1) is not empty and is sufficiently large.

It is easy to see that $\mathcal{K}_\epsilon^+ \subset \mathcal{F}_+^{loc}$ and the space $\mathcal{K}_\epsilon^+$ is translation invariant, i.e., if $u(s) \in \mathcal{K}_\epsilon^+$, then $u(h+s) \in \mathcal{K}_\epsilon^+$ for all $h \geq 0$. Hence, $S(h)\mathcal{K}_\epsilon^+ \subseteq \mathcal{K}_\epsilon^+$ for all $h \geq 0$.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $L_2(t_1, t_2; \mathbf{H})$. We get

$$\rho_{0,M}(u,v) = \left(\int_0^M \|u(s) - v(s)\|_{\mathbf{H}}^2 ds \right)^{1/2} \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

The topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\epsilon,+}^{loc}$ in $\mathcal{F}_{\epsilon,+}^{loc}$) is generated by these metrics. Let us recall that $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for each $M > 0$. Bearing in mind (8), we conclude that the topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\epsilon^+$ of (1). Also it can be seen that the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\epsilon^+$, is continuous in this topology.

Using the scheme of Section 2, one can define bounded sets in the space $\mathcal{K}_\epsilon^+$ by means of the Banach space $\mathcal{F}_+^b$. We naturally get

$$\mathcal{F}_+^b = L_4^b(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^b(\mathbb{R}_+; \mathbf{V}) \cap L_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+; \mathbf{H}^{-r}) \right\}$$

and the set $\mathcal{F}_+^b$ is a subspace of $\mathcal{F}_+^{loc}$.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\epsilon^+$, $S(t) : \mathcal{K}_\epsilon^+ \to \mathcal{K}_\epsilon^+$, $t \geq 0$.

Suppose that $\mathcal{K}_\epsilon$ is the kernel to (1), that consists of all weak complete solutions $u(s), \in \mathbb{R}$, to our system of equations, bounded in

$$\mathcal{F}^b = L_4^b(\mathbb{R}; \mathbf{L}_4) \cap L_2^b(\mathbb{R}; \mathbf{V}) \cap L_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}; \mathbf{H}^{-r}) \right\}.$$

Proposition 1.1. Problem (1) has the trajectory attractors $\mathfrak{A}_\epsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\epsilon$ is uniformly (w.r.t. $\epsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover, $\mathfrak{A}_\epsilon = \Pi_+\mathcal{K}_\epsilon$, the kernel $\mathcal{K}_\epsilon$ is non-empty and uniformly (w.r.t. $\epsilon \in (0,1)$) bounded in $\mathcal{F}^b$. Recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on ϵ.

To prove this proposition we use the approach of the proof from [14]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}) one can use Lemma 1.1 similar to [14].

It is easy to verify, that $\mathfrak{A}_\epsilon \subset \mathcal{B}_0(R)$ for all $\epsilon \in (0,1)$. Here $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. By means of Lemma 2.1 we have

$$\mathcal{B}_0(R) \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{5}$$

$$\mathcal{B}_0(R) \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{6}$$

Formula (5) immediately follows, if we take $E_0 = \mathbf{H}^{-r},\ E = \mathbf{H}^{1-\delta},\ E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_1 = 2$, $p_0 = 4/3$, keeping in mind the compact embedding $\mathbf{V} \Subset \mathbf{H}^{1-\delta}$. Formula (6) follows from the compact embedding $\mathbf{H} \Subset \mathbf{H}^{-\delta}$, if we take $E_0 = \mathbf{H}^{-\mathbf{r}}(D),\ E = \mathbf{H}^{-\delta},\ E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_0 = 4/3$.

Bearing in mind (5) and (6), the attraction to the constructed trajectory attractor can be strengthen.

Corollary 1.1. For any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subset \mathcal{K}_\epsilon^+$ we get

$$\mathrm{dist}_{L_2(0,M;\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\epsilon\right) \to 0\ (t \to \infty),$$
$$\mathrm{dist}_{C([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\epsilon\right) \to 0\ (t \to \infty),$$

where M is a positive constant.

2 Trajectory attractors of evolution equations

This section is devoted to the construction of trajectory attractors to autonomous evolution equations.

Consider an autonomous evolution equation of the form

$$\frac{\partial u}{\partial t} = A(u), \quad t \geq 0. \tag{7}$$

Here $A(\cdot) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. As an example one can consider $A(u) = (1 + \alpha \mathrm{i})\Delta u + R(\cdot)\, u - (1 + \beta(\cdot)\mathrm{i})\, |u|^2 u + g(\cdot)$.

We study weak solutions $u(s)$ to (7) as functions of parameter $s \in \mathbb{R}_+$ as a whole. To be precise we say that $s \equiv t$ denotes the time. The set of solutions of (7) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (7). Now, we describe the trajectory space $\mathcal{K}^+$ in detail.

Consider solutions $u(s)$ of (7) defined on $[t_1, t_2] \subset \mathbb{R}$. We consider solutions to problem (7) in a Banach space $\mathcal{F}_{t_1,t_2}$. The space $\mathcal{F}_{t_1,t_2}$ is a set $f(s), s \in [t_1, t_2]$ satisfying $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space, satisfying $E_1 \subseteq E \subseteq E_0$.

For instance, $\mathcal{F}_{t_1,t_2}$ can be considered as the intersection spaces $C([t_1, t_2]; E)$, or $L_p(t_1, t_2; E)$, for $p \in [1, \infty]$. Suppose that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and $\|\Pi_{t_1,t_2}f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}\ \forall f \in \mathcal{F}_{\tau_1,\tau_2}$. Here $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto $[t_1, t_2]$, constant $C(t_1, t_2, \tau_1, \tau_2)$ does not depend on f.

Suppose that $S(h)$ for $h \in \mathbb{R}$ denotes the translation operator $S(h)f(s) = f(h+s)$. It is easy to see, that if the argument s of $f(\cdot)$ belongs to the segment $[t_1, t_2]$, then the argument s of $S(h)f(\cdot)$ belongs to $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. Suppose that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and $\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}_{t_1,t_2}}\ \forall f \in \mathcal{F}_{t_1,t_2}$. It is easy to see that this assumption is natural.

Suppose that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a Banach space, which is larger, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ and we suppose that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is a *solution* of (7), if $\frac{\partial u}{\partial t}(s) = A(u(s))$ in the sense of $D'((t_1, t_2); E_0)$.

Let us define the space $\mathcal{F}_+^{loc} = \{f(s),\ s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2}f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}$. For instance, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is a solution of (7), if $\Pi_{t_1,t_2}u(s) \in \mathcal{F}_{t_1,t_2}$ and $u(s)$ is a solution of (7) for every $[t_1, t_2] \subset \mathbb{R}_+$.

Let $\mathcal{K}^+$ be a set of solutions to (7) from $\mathcal{F}_+^{loc}$. Note, that $\mathcal{K}^+$ in general is not the set of *all* solutions from $\mathcal{F}_+^{loc}$. The set $\mathcal{K}^+$ consists on elements, which are *trajectories* and the set $\mathcal{K}^+$ is the *trajectory space* of the equation (7).

Suppose that the trajectory space $\mathcal{K}^+$ is *translation invariant*, i.e., if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for every $h \geq 0$.

Consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc} : S(h)f(s) = f(s + h),\ h \geq 0$. It is easy to see that the map $\{S(h), h \geq 0\}$ forms a semigroup in $\mathcal{F}_+^{loc} :\ S(h_1)S(h_2) = S(h_1 + h_2)$ for $h_1, h_2 \geq 0$ and in addition $S(0)$ is the identity operator. Next step is to change the variable h into the time variable t. The *translation semigroup* $\{S(t), t \geq 0\}$ maps the trajectory space $\mathcal{K}^+$ to itself: $S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+$ for all $t \geq 0$.

We investigate attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. Next step is to define a topology in the space $\mathcal{F}_+^{loc}$.

One can see, that metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ is defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1,t_2] \subset \mathbb{R}$. Suppose that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2}f, \Pi_{t_1,t_2}g) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}(f,g) \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$

$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g) \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

Now, we denote by Θ_{t_1,t_2} metric spaces on $\mathcal{F}_{t_1,t_2}$. For instance, ρ_{t_1,t_2} is metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of $\mathcal{F}_{t_1,t_2}$. At the other hand, in application ρ_{t_1,t_2} generates the topology Θ_{t_1,t_2} that is weaker than the strong one of the $\mathcal{F}_{t_1,t_2}$.

The *projective limit* of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, that is, by definition, a sequence $\{f_k(s)\} \subset \mathcal{F}_+^{loc}$ tends to $f(s) \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_k, \Pi_{t_1,t_2}f) \to 0$ as $k \to \infty$ for all $[t_1,t_2] \subset \mathbb{R}_+$. It is possible to show that the topology Θ_+^{loc} is metrizable. For this aim we use, for example, the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{8}$$

The translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This statement follows from the definition of Θ_+^{loc}.

We also define the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\},$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0} \|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}.$$

We remember that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We need from our Banach space $\mathcal{F}_+^b$ only one fact It should define bounded subsets in the trajectory space $\mathcal{K}^+$. For constructing a trajectory attractor in $\mathcal{K}^+$, instead of considering the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$, we use much weaker topology, i.e. the local convergence topology Θ_+^{loc}.

Assume that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, that is, every trajectory $u(s) \in \mathcal{K}^+$ of equation (7) has a finite norm. We define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 2.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ϵ-neighbourhood $O_\epsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\epsilon(\mathcal{P})$ for all $t \geq t_1$.

It is easy to see that the attracting property of $\mathcal{P}$ can be formulated equivalently: we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \longrightarrow 0 \quad (t \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. We remember that the Hausdorff semidistance is not symmetric, for any $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$.

Definition 2.2 ([14]). A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if

(i) $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc},
(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$,
(iii) $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$ we have

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Let us formulate the main assertion on the trajectory attractor for equation (7).

Theorem 2.1 ([13, 14]). Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation (7) is contained in $\mathcal{F}_+^b$. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.

Let us describe in detail, i.e., in terms of complete trajectories of the equation, the structure of the trajectory attractor $\mathfrak{A}$ to equation (7). We study the equation (7) on the time axis

$$\frac{\partial u}{\partial t} = A(u), \ t \in \mathbb{R}. \tag{9}$$

Note that the trajectory space $\mathcal{K}^+$ of equation (9) on $\mathbb{R}_+$ have been defined. We need this notion on the entire $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is a *complete trajectory* of equation (9) if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to $\mathbb{R}_+$.

We have $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. Let us define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \ \forall \ [t_1, t_2] \subseteq \mathbb{R}\}; \qquad \mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_k(s) \to f(s)$ $(k \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)$ $(k \to \infty)$ in Θ_{t_1,t_2} for each $[t_1, t_2] \subseteq \mathbb{R}$. It is easy to see that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 2.3. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (9) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (9) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (10), i.e.

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \leq C_u \quad \forall h \in \mathbb{R}.$$

Theorem 2.2. Assume that the hypotheses of Theorem holds. Then $\mathfrak{A} = \Pi_+\mathcal{K}$, the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

To prove this assertion one can use the approach from [14].

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0, M; E_1, E_0) = \left\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{p_1}(0, M; E_1), \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\right\},$$

$$W_{\infty,p_0}(0, M; E_1, E_0) = \left\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{\infty}(0, M; E_1), \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\right\},$$

(where $p_1 \geq 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s\in[0,M]\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0}.$$

Lemma 2.1 (Aubin-Lions-Simon, [22]). Assume that $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:

$$W_{p_1,p_0}(0,T;E_1,E_0) \Subset L_{p_1}(0,T;E), \qquad W_{\infty,p_0}(0,T;E_1,E_0) \Subset C([0,T];E).$$

In this paper we investigate evolution equations and their trajectory attractors depending on a small parameter $\epsilon > 0$.

Definition 2.4. We say that the trajectory attractors $\mathfrak{A}_\epsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\epsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\epsilon_1 \geq 0$ such that $\mathfrak{A}_\epsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\epsilon < \epsilon_1$, that is, for each $M > 0$ we have

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\epsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\epsilon \to 0).$$

3 Formal homogenization procedure

Let M_i be a solution to a problem

$$\begin{cases} \Delta_\xi M_i\,(x,\xi) = 0 \quad \text{in} \quad \omega, \\ \dfrac{\partial M_i(x,\xi)}{\partial\nu} = -\tilde{\nu}_i \quad \text{on} \quad S(x). \end{cases} \tag{11}$$

Denote by $\langle\cdot\rangle$ the integral over the set $\square\cap\omega$, and $Q(x) = \int_S q(x,\xi)\,d\sigma$.

The limit problem has the form

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - (1+\alpha\mathrm{i})\displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0}{\partial x_j}\right) - \\ \qquad -R(x)u_0 + (1+\beta(x)\mathrm{i})\,|u_0|^2\,u_0 + Q(x)u_0 = |\square\cap\omega|\,g(x), & x\in\Omega, \\ u_0 = 0, & x\in\partial\Omega, t>0, \\ u_0 = U(x), & x\in\Omega, t=0. \end{cases} \tag{12}$$

It is easy to see that system (12) also has trajectory attractor $\mathfrak{A}$ in the trajectory space $\mathcal{K}_+$ corresponding to problem (12) and $\mathfrak{A} = \Pi_+\mathcal{K}$, where $\mathcal{K}$ is the kernel of system (12) in $\mathcal{F}_+^b$.

The integral identity for problem (12) takes the form

$$-\int_{\mathbb{R}_+}\int_\Omega u_0\frac{\partial v}{\partial t}dtdx + (1+\alpha\mathrm{i})\int_{\mathbb{R}_+}\int_\Omega \sum_{i,j=1}^d \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle \frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_j}dtdx +$$
$$-\int_{\mathbb{R}_+}\int_\Omega\Big(R(x)u_0 - (1+\beta(x)\mathrm{i})\,|u_0|^2\,u_0 - Q(x)u_0\Big)v dtdx = \int_{\mathbb{R}_+}\int_\Omega |\square\cap\omega|\,g(x)\,v\,dtdx$$

for any function $v \in C_0^\infty(\mathbb{R}_+;\mathbf{V}\cap\mathbf{L}_4)$.

Remark 3.1. Note that $M_i(x,\xi)$ are not defined in the whole Ω. We can extend $M_i(x,\xi)$ into the interior of the cavities retaining the regularity of these functions by means of the technique of the symmetric extension, keeping the same notation for the extended functions.

4 Auxiliaries

We study the asymptotics of solution $u_\epsilon(x)$ as $\epsilon \to 0$ to the next boundary-value problem

$$\begin{cases} -(1+\alpha \mathrm{i})\Delta u_\epsilon = g(x) & \text{in} \quad \Omega_\epsilon, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_\epsilon}{\partial \nu_\epsilon} + \epsilon q\left(x, \dfrac{x}{\epsilon}\right) u_\epsilon = 0 & \text{on} \quad S_\epsilon, \\ u_\epsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

Here n_ϵ is the internal normal to the boundary of cavities and $q(x,\xi)$ is a sufficiently smooth 1-periodic in ξ function.

Definition 4.1. The function $u_\epsilon \in H^1(\Omega_\epsilon, \partial\Omega)$ is a solution of problem (13), if the following integral identity

$$(1+\alpha \mathrm{i}) \int_{\Omega^\epsilon} \nabla u_\epsilon(x)\, \nabla v(x)\, dx + \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right) u_\epsilon(x) v(x)\, ds = \int_{\Omega^\epsilon} g(x)\, v(x)\, dx$$

holds true for any function $v \in H^1(\Omega_\epsilon, \partial\Omega)$.

Here $H^1(\Omega^\epsilon, \partial\Omega)$ is the closure of the set of functions belonging to $C^\infty(\overline{\Omega}^\epsilon)$ and vanishing in a neighborhood of $\partial\Omega$, by the $H^1(\Omega^\epsilon)$ norm.

Here we derive the leading terms of the asymptotic expansion and, then, construct the homogranized problem. For this aim we consider the solution $u_\epsilon(x)$ to (13) as an asymptotic series

$$u_\epsilon(x) = u_0(x) + \epsilon u_1\left(x, \frac{x}{\epsilon}\right) + \epsilon^2 u_2\left(x, \frac{x}{\epsilon}\right) + \epsilon^3 u_3\left(x, \frac{x}{\epsilon}\right) + \dots \tag{14}$$

Substituting expression (14) in equation (13) and bearing in mind the relation

$$\frac{\partial}{\partial x}\zeta\left(x, \frac{x}{\epsilon}\right) = \left(\frac{\partial}{\partial x}\zeta(x,\xi) + \frac{1}{\epsilon}\frac{\partial}{\partial \xi}\zeta(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}},$$

we get the formula

$$\begin{aligned} -\frac{g(x)}{1+\alpha \mathrm{i}} = \Delta_x u_\epsilon(x) \cong \Delta_x u_0(x) + \epsilon\left(\Delta_x u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + 2\left(\nabla_x, \nabla_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + \frac{1}{\epsilon}\left(\Delta_\xi u_1(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon^2\left(\Delta_x u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + 2\epsilon\left(\nabla_x, \nabla_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + \left(\Delta_\xi u_2(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon^3\left(\Delta_x u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \\ + 2\epsilon^2\left(\nabla_x, \nabla_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon\left(\Delta_\xi u_3(x,\xi)\right)\Big|_{\xi=\frac{x}{\epsilon}} + \dots \end{aligned} \tag{15}$$

Similarily, substituting (14) into boundary conditions in (13), we get the relation

$$\begin{aligned} 0 = \frac{\partial u_\epsilon}{\partial \nu_\epsilon} + \epsilon\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha \mathrm{i}} u_\epsilon \cong \left(\nabla_x u_0, \nu_\epsilon\right) + \epsilon\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha \mathrm{i}} u_0 + \epsilon\left(\nabla_x u_1, \nu_\epsilon\right) + \\ + \left(\nabla_\xi u_1\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \epsilon^2\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha \mathrm{i}} u_1 + \epsilon^2\left(\nabla_x u_2, \nu_\epsilon\right) + \epsilon\left(\nabla_\xi u_2\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \\ + \epsilon^3\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha \mathrm{i}} u_2 + \epsilon^3\left(\nabla_x u_3, \nu_\epsilon\right) + \epsilon^2\left(\nabla_\xi u_3\big|_{\xi=\frac{x}{\epsilon}}, \nu_\epsilon\right) + \epsilon^4\frac{q\left(x, \frac{x}{\epsilon}\right)}{1+\alpha \mathrm{i}} u_3 + \dots, \end{aligned} \tag{16}$$

which means that it satisfies the boundary condition on S_ϵ.

The normal vector ν_ϵ depends on x and $\frac{x}{\epsilon}$ in Ω_ϵ. Now, we consider x and $\xi = \frac{x}{\epsilon}$ as independent variables, and then we represent ν_ϵ in Ω_ϵ in the form

$$\nu_\epsilon(x, \frac{x}{\epsilon}) = \widetilde{\nu}(x,\xi)\Big|_{\xi=\frac{x}{\epsilon}} + \epsilon\nu'_\epsilon(x,\xi)\Big|_{\xi=\frac{x}{\epsilon}},$$

where $\widetilde{\nu}$ is a normal vector to $S(x) = \{\xi \mid F(x,\xi) = 0\}$,

$$\nu'_\epsilon = \nu' + O(\epsilon).$$

Collecting all the terms of order ϵ^{-1} in (15) and of order ϵ^0 in (16), we deduce the auxiliary problem

$$\begin{cases} \Delta_\xi u_1\,(x,\xi) = 0 & \text{in} \quad \omega, \\ \dfrac{\partial u_1(x,\xi)}{\partial \nu} = -\left(\nabla_x(u_0(x)), \tilde{n}\right) & \text{on} \quad S, \end{cases} \tag{17}$$

which we solve in the space of 1-periodic in ξ functions and here x is a parameter, $\omega := \{\xi \in \mathbb{T}^d \mid F(x,\xi) > 0\}$. This is the cell problem appearing in case of Neumann conditions on the boundary of cavities. It is easy to see that the compatibility condition $\int_{S(x)} (\nabla_x u_0(x), \tilde{\nu}(\xi))\ d\sigma = 0$ of (17) is satisfied, and the solution of this problem is the first corrector in (14).

At the next step we collect all the terms of order ϵ^0 in (15) and of order ϵ^1 in (16). This gives us

$$\begin{cases} \Delta_\xi u_2\,(x,\xi) &=& -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\left(\nabla_\xi, \nabla_x u_1(x,\xi)\right) \quad \text{in} \quad \omega, \\ \dfrac{\partial u_2(x,\xi)}{\partial \nu} &=& -\left(\nabla_x u_1(x,\xi), \tilde{\nu}\right) - \left(\nabla_\xi u_1(x,\xi), \nu'\right) - \\ && -\left(\nabla_x u_0(x), \nu'\right) - \dfrac{q(x,\xi)}{1+\alpha \mathrm{i}} u_0(x) \quad \text{on} \quad S(x). \end{cases} \tag{18}$$

The 1-periodic in ξ solution of the latter problem is the second term of the internal asymptotic expansion of $u_\epsilon(x)$.

It is easy to see that for our analysis it is convenient to represent the solution $u_1(x,\xi)$ of problem (17) in the following form:

$$u_1(x,\xi) = \left(\mathrm{grad}_x u_0(x), M(x,\xi)\right),$$

where 1-periodic vector–function $M(x,\xi) = (M_1(x,\xi),\ldots,M_d(x,\xi))$ is a solution to (11).

Now, (18) can be rewritten as follows

$$\begin{cases} \Delta_\xi u_2\,(x,\xi) &=& -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\ \frac{\partial M_i(x,\xi)}{\partial \xi_j} - \\ && -2\displaystyle\sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\ \frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\,\partial x_j} \quad \text{in} \quad \omega, \\ \dfrac{\partial u_2(x,\xi)}{\partial \nu} &=& -\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j} M_i(x,\xi)\nu_j - \sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\ \frac{\partial M_i(x,\xi)}{\partial x_j}\ \nu_j - \\ && -\dfrac{q(x,\xi)}{1+\alpha \mathrm{i}} u_0(x) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial u_0(x)}{\partial x_i}\ \left(\frac{\partial M_i(x,\xi)}{\partial \xi_j} + \delta_{ij}\right)\nu'_j \quad \text{on} \quad S(x). \end{cases}$$

Writing down the solvability condition in the last problem, we derive the equation:

$$\int_{\square\cap\omega}\left(\frac{g(x)}{1+\alpha\mathrm{i}}+\Delta_x u_0(x)+2\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j}+2\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial^2 M_i(x,\xi)}{\partial \xi_j\,\partial x_j}\right)d\xi=$$

$$=\int_Q\left(\sum_{i,j=1}^{d}\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}M_i(x,\xi)\nu_j+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{x_j}\,\nu_j+\right.$$

$$\left.+\sum_{i,j=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\frac{\partial M_i(x,\xi)}{\partial \xi_j}\,\nu'_j+\sum_{i=1}^{d}\frac{\partial u_0(x)}{\partial x_i}\,\nu'_i+\frac{q(x,\xi)}{1+\alpha\mathrm{i}}u_0(x)\right)d\sigma. \quad (19)$$

From (19) by the Stokes formula we derive the equation

$$|\square\cap\omega|\Delta_x u_0(x)+\sum_{i,j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}+\sum_{i,j=1}^{d}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}+$$

$$+|\square\cap\omega|\frac{g(x)}{1+\alpha\mathrm{i}}=\frac{Q(x)}{1+\alpha\mathrm{i}}u_0(x)+\sum_{i=1}^{d}U_i(x)\frac{\partial u_0(x)}{\partial x_i}, \quad (20)$$

which is the limit equation in Ω. We denoted by $<\cdot>$ the integral over $\square\cap\omega$, and $Q(x)=\int_{S(x)}q(x,\xi)\,d\sigma$. Moreover, $U_i(x)=\int_{S(x)}\left(\frac{\partial M_i(x,\xi)}{\partial \xi_j}\nu'_j+\nu'_i\right)d\sigma$.

It is not necessary to calculate $U_i(x)$, since by the selfadjointness of the operators of the given problems and the convergence of the corresponding belinear forms, we get that the G–limit operator is necessary selfadjoint. Therefore, the limit equation (20) takes the form:

$$(1+\alpha\mathrm{i})\sum_{i,j=1}^{d}\frac{\partial}{\partial x_j}\left(\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle\frac{\partial u_0(x)}{\partial x_i}\right)+|\square\cap\omega|g(x)=Q(x)u_0(x) \quad (21)$$

and, consequently,

$$U_i(x)=\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left\langle\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle-\sum_{j=1}^{d}\left\langle\frac{\partial^2 M_i(x,\xi)}{\partial x_j\,\partial \xi_j}\right\rangle.$$

It is easy to see that $\left\langle\delta_{ij}+\frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle$ is a smooth positively defined matrix (see [9]).

The next statement is about the limit behavior of the solution to (13).

Theorem 4.1. Suppose that $g(x)\in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small ϵ problem (13) has the unique solution and the following convergence

$$\|u_0-u_\epsilon\|_{H^1(\Omega_\epsilon)}\longrightarrow 0$$

takes place, where u_0 is a solution of equation (21) with zero Dirichlet conditions on $\partial\Omega$.

Remark 4.1. In fact, in the formulation of Theorem 4.1 the condition $q(x,\xi)\geq 0$ can be replaced by the weaker condition $Q(x)\geq 0$.

4.1 Preliminary Lemmas

Here we give some technical propositions, which we use in the further analysis. Some of these propositions have been proved in [3, 23]. We omit their proofs.

Lemma 4.1. If the conditions of Theorem are satisfied, then the Friederichs type inequality

$$\int_{\Omega_\epsilon} |\nabla v|^2 dx + \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right) v^2 ds \geq C_1 \|v\|^2_{H^1(\Omega^\epsilon, \partial\Omega)}$$

is valid for any $v \in H^1(\Omega_\epsilon, \partial\Omega)$, where C_1 is independent of ϵ.

Now we formulate a modified version of Lemma 5 from [23].

Lemma 4.2. If we suppose

$$\frac{1}{|\square \cap \omega|} \int_{\square\cap\omega} Q(x)\ d\xi - \int_{S(x)} q(x,\xi)\ d\sigma \equiv 0,$$

then the following inequality

$$\left| \frac{1}{|\square \cap \omega|} \int_{\Omega^\epsilon} Q(x)\, v(x)\ dx - \epsilon \int_{S_\epsilon} q\left(x, \frac{x}{\epsilon}\right)\ v(x)\ d\sigma \right| \leq C_2 \epsilon \|v\|_{H^1(\Omega^\epsilon)}$$

holds for any $v(x) \in H^1(\Omega_\epsilon, \partial\Omega)$; the constant C_2 is independent of ϵ.

Proof. The proof of this assertion can be found in [24].

Lemma 4.3. If y_ϵ is a solution to

$$\begin{cases} -(1+\alpha \mathrm{i})\Delta y_\epsilon = h^\epsilon(x) & \text{in} \quad \Omega_\epsilon, \\ (1+\alpha \mathrm{i})\dfrac{\partial y_\epsilon}{\partial \nu_\epsilon} + \epsilon q\left(x, \dfrac{x}{\epsilon}\right) y_\epsilon = 0 & \text{on} \quad S_\epsilon, \\ y_\epsilon = 0 & \text{on} \quad \Omega, \end{cases}$$

where $h^\epsilon(x) = g(x)$ for $x \in \Omega_\epsilon$ and 0 otherwise, then

$$\|y_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C_3 \epsilon.$$

The proposition, which is a modification of Lemma 5 from [23], formulated below.

Lemma 4.4. Suppose $w^\epsilon(x) \in L_\infty(\Omega)$, and let Π^ϵ belong to $\{x \in \Omega \,|\, \mathrm{dist}\,(x, \partial\Omega) \leq C_0\epsilon\}$. Then the following inequality

$$\left| \int_{\Pi^\epsilon} w^\epsilon(x) \Big|_{\xi=\frac{x}{\epsilon}} \nabla_x u_0(x)\, v(x)\ dx \right| \leq C_4 \epsilon^{\frac{3}{2}} \|w\|_{L_\infty(\Omega)} \|v\|_{H^1(\Omega_\epsilon)}$$

holds for any $v(x) \in H^1(\Omega_\epsilon, \partial\Omega)$; the constant C_4 is independent of ϵ.

Proof of the Theorem 4.1. The proof of this assertion can be found in [23].

5 The main assertion

Here formulate the main proposition concerning the Ginzburg-Landau equation.

Theorem 5.1. The following limit holds in the topological space Θ^{loc}_+

$$\mathfrak{A}_\epsilon \to \overline{\mathfrak{A}} \ \text{ as } \epsilon \to 0+. \tag{22}$$

Moreover,

$$\mathcal{K}_\epsilon \to \overline{\mathcal{K}} \text{ as } \epsilon \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{23}$$

Remark 5.1. The functions belonging the sets $\mathfrak{A}_\epsilon$ and $\mathcal{K}_\epsilon$ are defined in the perforated domains Ω_ϵ. But, all these functions can be extended insides the cavities remaining their norms in the spaces $\mathbf{H}, \mathbf{V}$, and $\mathbf{L}_p$ (without perforation) with the constants independent of the small parameter (the prolongation of functions defined in perforated domains, see, for instance, in [10; Ch.VIII]). Hence, in Theorem 5.1, we have all the distances in the spaces without perforation.

Proof. It is easy to see that (23) implies (22). Hence, it is sufficient to prove (23), i.e., for every neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there exists $\epsilon_1 = \epsilon_1(\mathcal{O}) > 0$, such that

$$\mathcal{K}_\epsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \quad \text{for} \quad \epsilon < \epsilon_1. \tag{24}$$

Assume that (24) is not true. Then there exists a neighborhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\epsilon_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\epsilon_k}(\cdot) = u_{\epsilon_k}(s) \in \mathcal{K}_{\epsilon_k}$, such that

$$u_{\epsilon_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all} \quad k \in \mathbb{N}.$$

The function $u_{\epsilon_k}(s), s \in \mathbb{R}$ is a solution to

$$\begin{cases} \dfrac{\partial u_{\epsilon_k}}{\partial t} = (1+\alpha \mathrm{i})\Delta u_{\epsilon_k} + R\left(x, \dfrac{x}{\epsilon_k}\right) u_{\epsilon_k} - \left(1 + \beta\left(x, \dfrac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}|^2 u_{\epsilon_k} + g\left(x\right), & x \in \Omega_{\epsilon_k}, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_{\epsilon_k}}{\partial \nu} + \epsilon_k q\Big(x, \dfrac{x}{\epsilon_k}\Big) u_{\epsilon_k} = 0, & x \in S_{\epsilon_k}, t > 0, \\ u_{\epsilon_k} = 0, & x \in \partial\Omega, \\ u_{\epsilon_k} = U(x), & x \in \Omega_{\epsilon_k}, t = 0. \end{cases}$$

on the axis $t \in \mathbb{R}$. To get the uniform in ϵ estimate of the solution we use the following Lemmas (see [25; Ch. III, §5] and [26] respectively).

By means of integral identity (4) and Lemma 1.1 we derive the estimate, the sequence $\{u_{\epsilon_k}(x,s)\}$ is bounded in $\mathcal{F}^b$, i.e.,

$$\|u_{\epsilon_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\epsilon_k}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(s)\|_1^2 ds\right)^{1/2} +$$
$$+ \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\epsilon_k}(s)\|_{\mathbf{L}_4}^4 ds\right)^{1/4} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\|\frac{\partial u_{\epsilon_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^{4/3} ds\right)^{3/4} \le C \ \text{ for all } k \in \mathbb{N}. \tag{25}$$

The constant C is independent of ϵ.

Consequently, there exists a subsequence $\{u_{\epsilon'_k}(x,s)\} \subset \{u_{\epsilon_k}(x,s)\}$, such that $u_{\epsilon_k}(x,s) \to u(s)$ as $k \to \infty$ in Θ^{loc}. Here $u(x,s) \in \mathcal{F}^b$ and $u(s)$ are the solution to (25) with the same constant C. Because of (25) we get $u_{\epsilon_k}(x,s) \rightharpoonup u(x,s)$ $(k \to \infty)$ weakly in $L_2^{loc}(\mathbb{R};\mathbf{V})$, weakly in $L_4^{loc}(\mathbb{R};\mathbf{L}_4)$, $*$-weakly in $L_\infty^{loc}(\mathbb{R}_+;\mathbf{H})$ and $\frac{\partial u_{\epsilon_k}(x,s)}{\partial t} \rightharpoonup \frac{\partial u(x,s)}{\partial t}$ $(k \to \infty)$ weakly in $L_{4/3,w}^{loc}(\mathbb{R};\mathbf{H}^{-r})$. We claim that $u(x,s) \in \overline{\mathcal{K}}$. We have $\|u\|_{\mathcal{F}^b} \le C$. Hence, we have to establish that $u(x,s)$ is a weak solution to (12).

According to the auxiliary problem in the case $\theta = 1$ we have

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega_{\epsilon_k}} \nabla u_{\epsilon_k}\nabla\psi\, dxdt + \epsilon_k \int_{-M}^{M}\int_{S_{\epsilon_k}} q\Big(x, \frac{x}{\epsilon_k}\Big) u_{\epsilon_k}\psi d\sigma dt + \int_{-M}^{M}\int_{\Omega_{\epsilon_k}} g(x)\psi dxdt \longrightarrow$$

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega}\sum_{i,j=1}^{d}\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x,t)}{\partial x_i}\frac{\partial \psi}{\partial x_j} dxdt -$$
$$+ \int_{-M}^{M}\int_{\Omega} Q(x)u_0(x,t)\psi dxdt + \int_{-M}^{M}\int_{\Omega} |\Box \cap \omega|\, g(x)\psi dxdt$$

as $k \to \infty$.

The differentiation is continuous in the space of generalized functions, also $\dfrac{\partial u_\epsilon}{\partial t} \longrightarrow \dfrac{\partial u_0}{\partial t}$ as $\epsilon \to 0+$.

Now, we prove that

$$R\left(x, \frac{x}{\epsilon_k}\right) u_{\epsilon_k}(x,s) \rightharpoonup \bar{R}(x)u(x,s) \tag{26}$$

and

$$\left(1+\beta\left(x, \frac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) \rightharpoonup \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) \tag{27}$$

as $k\to\infty$ weakly in $L^{loc}_{4/3,w}\left(\mathbb{R};\mathbf{L}_{4/3}\right)$.

Fixing an arbitrary number $M>0$, we consider the sequence $\{u_{\epsilon_k}(x,s)\}$ bounded in $L_4\left(-M,M;\mathbf{L}_4\right)$ (see (25)). Hence, the sequence $\{|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s)\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$. Because $\{u_{\epsilon_k}(x,s)\}$ is bounded in $L_2(-M,M;\mathbf{V})$ and $\left\{\dfrac{\partial u_{\epsilon_k}(x,s)}{\partial t}\right\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{H}^{-r}\right)$ we suppose that $u_{\epsilon_k}(x,s)\to u(x,s)$ as $k\to\infty$ strongly in $L_2\left(-M,M;\mathbf{H}\right)$ and hence

$$u_{\epsilon_k}(x,s)\to u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M).$$

It follows that

$$|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s)\to |u(x,s)|^2 u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M). \tag{28}$$

We have

$$\left(1+\beta\left(x, \frac{x}{\epsilon_k}\right)\mathrm{i}\right)|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) =$$
$$= \left(1+\beta\left(x, \frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right) +$$
$$+\left(\left(1+\beta\left(x, \frac{x}{\epsilon_k}\right)\mathrm{i}\right) - \left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2 u(x,s). \tag{29}$$

We show that both terms in the right-hand side of (29) tends to zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence $\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right)$ converges to zero as $k\to\infty$ almost everywhere in $(x,s)\in\Omega\times(-M,M)$ (see (28)) and is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ (see (2)). Consequently using Lemma 1.3 from [27] we get $\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right)\left(|u_{\epsilon_k}(x,s)|^2 u_{\epsilon_k}(x,s) - |u(x,s)|^2 u(x,s)\right)$ $\rightharpoonup 0$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ as $k\to\infty$.

The sequence $\left(\left(1+\beta\left(x,\frac{x}{\epsilon_k}\right)\mathrm{i}\right) - \left(1+\bar{\beta}(x)\mathrm{i}\right)\right)|u(x,s)|^2 u(x,s)$ goes weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ to zero as $k\to\infty$, since by the assumption $\beta\left(x,\dfrac{x}{\epsilon}\right)\rightharpoonup\bar{\beta}(x)$ *-weakly in $L_{\infty,w}\left(-M,M;\mathbf{L}_\infty\right)$ as $k\to\infty$ (see (3)) and $|u(x,s)|^2 u(x,s)\in L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

We have proved (27). The convergence of (26) is proved similarly.

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Локальды периодты кеуектері бар орталарда Гинсбург-Ландау теңдеулерінің аттракторларын орташалау: критикалық жағдай

Жұмыста теңдеуде және шекаралық шарттарында тез тербелмелі мүшелері бар Гинсбург-Ландау теңдеуін тесік облыста қарастырылған. Бұл теңдеудің траекториялық аттракторлары әлсіз мағынада «оғаш мүшесі» (әлеуеті) бар орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторларына жуықтайтыны дәлелденеді. Ол үшін В.В. Чепыжовтың және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай–ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданылған, содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдалған. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтай отырып, шекті (орташаланған) теңдеуі алынған және осы теңдеудің траекториялық аттракторы бар екені дәлелденген. Содан кейін негізгі теорема тұжырымдалған, оны көмекші леммалардың көмегімен дәлелденген.

Кілт сөздер: аттракторлар, орташалау, Гинсбург-Ландау теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тесік облыс, «оғаш мүше», кеуекті орта.

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Усреднение аттракторов уравнений Гинзбурга-Ландау в средах с локально периодическими препятствиями: критический случай

Мы рассматриваем уравнение Гинзбурга-Ландау с быстро осциллирующими членами в уравнении и граничных условиях в перфорированной области. Доказываем, что траекторные аттракторы этого уравнения в слабом смысле сходятся к траекторным аттракторам усредненного уравнения Гинзбурга-Ландау со «странным членом» (потенциалом). Для этого используем подход из статей и монографий В.В. Чепыжова и М.И. Вишика о траекторных аттракторах эволюционных уравнений. Также мы применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее выверяем главные члены асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, выводим предельное (усредненное) уравнение и доказываем существование траекторного аттрактора для этого уравнения. Затем формулируем основную теорему и доказываем ее с помощью вспомогательных лемм.

Ключевые слова: аттракторы, усреднение, уравнения Гинзбурга-Ландау, нелинейные уравнения, слабая сходимость, перфорированная область, «странный член», пористая среда.