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Compactness of Commutators for Riesz Potential on Local Morrey-type spaces

The paper considers Morrey-type local spaces from $LM^w_{p\theta}$ The main work is the proof of the commutator compactness theorem for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{q\theta}$. We also give new sufficient conditions for the commutator to be bounded for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{q\theta}$. In the proof of the commutator compactness theorem for the Riesz potential, we essentially use the boundedness condition for the commutator for the Riesz potential $[b, I_\alpha]$ in local Morrey-type spaces $LM^w_{p\theta}$, and use the sufficient conditions from the theorem of precompactness of sets in local spaces of Morrey type $LM^w_{p\theta}$. In the course of proving the commutator compactness theorem for the Riesz potential, we prove lemmas for the commutator ball for the Riesz potential $[b, I_\alpha]$. Similar results were obtained for global Morrey-type spaces $GM^w_{p\theta}$ and for generalized Morrey spaces M^w_p.

Keywords: Compactness, Commutators, Riesz Potential, Local Morrey-type spaces.

Introduction

First we give some definitions.

By $\mathfrak{M}(I)$ we denote the set of all measurable functions on I . The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f \in \mathfrak{M}(I)$ which are non-negative on I , while $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$ are used to denote the subset of those functions which are non-increasing and non-decreasing on I , respectively. When $I = (0,\infty)$, we write simply $\mathfrak{M}^+$, $\mathfrak{M}^\downarrow$ and $\mathfrak{M}^\uparrow$ instead of $\mathfrak{M}^+(I)$, $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$, accordingly. The family of all weight functions (also called just weights) on I , that is, locally integrable non-negative functions on $(0,\infty)$, is given by $\mathcal{W}(I)$.

For $p \in (0,\infty)$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\|\cdot\|_{p,w,I}$ on $\mathfrak{M}(I)$, by

$$\|f\|_{p,w,I} := \begin{cases} (\int_I |f(x)|^p w(x)dx)^{\frac{1}{p}}, & \text{if} \quad p < \infty; \\ ess \quad sup_I |f(x)|w(x), & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w,I)$ is given by

$$L^p(w,I) = f \in \mathfrak{M}(I) : \|f\|_{p,w,I} < \infty,$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,I}$. When $w \equiv 1$ on I , we write simply $L^p(I)$ and $\|\cdot\|_{p,I}$ instead of $L^p(w,I)$ and $\|\cdot\|_{p,w,I}$, respectively.

Let $1 \le p,\theta \le \infty$, w be a measurable non-negative function on $(0,\infty)$. The Local Morrey-type space $LM^w_{p\theta} \equiv LM^w_{p\theta}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{LM^w_{p\theta}} \equiv \left\| w(r) \, \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

where $B(t,r)$ the ball with center at the point t and of radius r.

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The space $LM^w_{p\theta}$ coincides with the known Morrey space M^λ_p at $w(r) = r^{-\lambda}, \theta = \infty$, where $0 \le \lambda \le \frac{n}{p}$, which, in turn, for $\lambda = 0$ coincides with the space $L_p(\mathbb{R}^n)$.

Following the notation of [1, 2], we denote by Ω_θ the set of all functions which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

Note that the space $LM^w_{p\theta}$ is non-trivial, that is consists not only of functions equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_\theta$.

In this paper we consider the Riesz Potential in the following form

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The Riesz Potential I_α plays an important role in the harmonic analysis and theory of operators.

For a function $b \in L_{loc}(\mathbb{R}^n)$ by M_b denote multiplier operator $M_b f = bf$, where f is measurable function. Then the commutator between I_α and M_b is defined by

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \frac{[b(x) - b(y)]\, f(y)}{|x-y|^{n-\alpha}} dy.$$

The commutators for Riesz Potential were investigated [3–9].

It is said that the function $b(x) \in L_\infty(\mathbb{R}^n)$ belongs to the space $BMO(\mathbb{R}^n)$, if

$$\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int\limits_Q |b(x) - b_Q|\, dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty,$$

where Q - cube $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|} \int\limits_{\mathbb{R}^n} f(y)dy$.

By $VMO(\mathbb{R}^n)$ we denote the BMO-closure $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support. Through the $\chi(A)$ denotes the characteristic function of the set $B \subset \mathbb{R}^n$, and ${}^cA$ denotes the complement of A.

The main purpose of this work is to find sufficient conditions for the compactness of commutators operators $[b, I_\alpha]$ on the Local Morrey-type space $LM^w_{p\theta}(\mathbb{R}^n)$.

We note that in the case of the Morrey space this question was investigated in [4]. The following well-known theorem gives necessary and sufficient conditions for the boundedness and compactness for $[b, I_\alpha]$ on the Local Morrey-type spaces $LM^w_{p\theta}(\mathbb{R}^n)$.

1 *Formulas and theorems*

To formulate the following theorem on the boundedness of the Hardy operator in weighted Lebesgue spaces, we introduce the notation.

Denote by

$$H^* g(t) := \int_t^\infty g(s)ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator.

$$W(t) := \int_0^t w(t)dw,$$

$$U_*(t) := \int_t^\infty u(t)du,$$
$$V_*(t) := \int_t^\infty v(t)dv.$$

Theorem 1. Let $0 < q, p \le \infty$. Assume that $u, v, w \in \mathcal{W}(0,\infty)$. Then inequality

$$\|H_u^*(f)\|_{q,w,(0;\infty)} \le c\|f_u^*\|_{p,w,(0;\infty)}, f \in \mathfrak{M}^\uparrow$$

with the best constant c holds if and only if the following holds:

$$A_0^* := sup_{t>0}(\int_t^\infty U_*^q(\tau)w(\tau)d\tau)^{\frac{1}{q}} V_*^{-\frac{1}{p}}(t),$$

$$A_1^* := sup_{t>0} W^{\frac{1}{q}}(t)(\int_t^\infty (\frac{U_*(\tau)}{V_*(\tau)})^{p'} v(\tau)d\tau)^{\frac{1}{p'}},$$

and in this case $c \approx A_0^* + A_1^*$.

Theorem 2. (see. [2]) Let $1 < p < q < \infty$, $0 < \alpha = n(\frac{1}{p} - \frac{1}{q})$, $0 < \theta < \infty$, (w_1, w_2) satisfy the following condition

$$\left\| w_2(r)\left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_\theta(0,\infty)} \le \|w_1(r)\|_{L_\theta(t,\infty)}. \tag{1}$$

Then the operator I_α is bounded from $LM_{p\theta}^{w_1}(\mathbb{R}^n)$ to $LM_{q\theta}^{w_2}(\mathbb{R}^n)$.

It is well known that the boundedness of such operators on Morrey space $LM_{p\theta}^\lambda(\mathbb{R})$ was considered in [1, 2].

The following theorem on sufficient conditions for the precompactness of sets on Local Morrey-type and other spaces was proved in [10–14].

Theorem 3. (see. [13]) Suppose that $1 \le p \le \theta \le \infty$ and $w \in \Omega_{p\theta}$. Suppose that a subset S of $LM_{p\theta}^w$ satisfies the following conditions:

$$\sup_{f\in S} \|f\|_{LM_{p\theta}^w} < \infty, \tag{2}$$

$$\lim_{u\to 0} \sup_{f\in S} \|f(\cdot + u) - f(\cdot)\|_{LM_{p\theta}^w} = 0, \tag{3}$$

$$\lim_{r\to\infty} \sup_{f\in S} \left\| f\chi_{^cB(0,r)} \right\|_{LM_{p\theta}^w} = 0. \tag{4}$$

Then S is a pre-compact set in $LM_{p\theta}^w(\mathbb{R})$.

Theorem 4. Let $1 < p \le q < \infty$, $0 < \alpha < n$ and $b \in BMO(R^n)$. $1 < p < \frac{n}{\alpha}$ $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w_1, w_2 \in \Omega_\theta$. Then the condition

$$A_0^* := sup_{t>0}(\int_t^\infty \int_\tau^\infty (1 + ln\frac{\tau}{r})drw(\tau)d\tau)^{\frac{1}{q}} [\int_t^\infty v(t)dv]^{-\frac{1}{p}}, \tag{5}$$

$$A_1^* := sup_{t>0} W^{\frac{1}{q}}(t)(\int_t^\infty (\frac{U_*(\tau)}{V_*(\tau)})^{p'} v(\tau)d\tau)^{\frac{1}{p'}} < \infty. \tag{6}$$

Then the commutator $[b, I_\alpha]$ is the boundedness operator from $LM_{p\theta}^{w_1}$ to $LM_{q\theta}^{w_2}$.

Note that for the case of Morrey space $LM^{\lambda}_{p\theta}(0 < \lambda < 1)$ (i.e., if $w(r) = r^{-\lambda}$) this assertion was proved earlier in [4], and in the case of $\lambda = 0$ is - known Frechet-Kolmogorov theorem [15]. We note that the pre-compactness some sets in Banach function spaces were investigated in [16]. Theorem 4 is proved using theorem 5.4 from [17] and theorem 3.4 from [5].

Now we give theorem about the compactness of the operators $[b, I_\alpha]$ on Local Morrey-type space $LM^{w}_{p\theta}(\mathbb{R}^n)$.

Theorem 5. Let $1 < p \le q < \infty$, $0 < \alpha < n$ and $b \in VMO(\mathbb{R}^n)$. $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w_1, w_2 \in \Omega_\theta$ satisfy the conditions (1), (5), (6). Then the commutator $[b, I_\alpha]$ is a compact operator from $LM^{w_1}_{p\theta}$ to $LM^{w_2}_{p\theta}$.

To prove this theorem we need the following auxiliary assertions.

Lemma 1. Let $n \in \mathbb{N}$, $1 \le p \le \infty$, $1 \le q \le \infty$, $0 < \alpha < n\left(1 - \frac{1}{q}\right)$, $\beta > 0$. Then there exists $C > 0$, depending only on n, p, q, α, such that for some $f \in L_p(B(0,\beta))$ satisfying the condition $suppf \subset \overline{B(0,\beta)}$, and for some $\gamma \ge 2\beta$, $t \in \mathbb{R}^n$, $r > 0$

$$\left\|(I_\alpha f)\chi_{{}^cB(0,\gamma)}\right\|_{L_q(B(t,r))} \le C\gamma^{\alpha-n}\left(min\{\gamma, r\}\right)^{\frac{n}{q}} \|f\|_{L_pB(0,\beta)}\,. \tag{7}$$

Proof. From the definition of the operator I_α, we have

$$I = \left\|\left(I_\alpha f\right)\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))} =$$

$$= \left(\int\limits_{B(t,r)\cap{}^cB(0,\gamma)} \left|\int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}} \le$$

$$\le \left(\int\limits_{B(t,r)\cap{}^cB(0,\gamma)} \left|\int\limits_{B(0,\beta)} \frac{f(y)}{|x-y|^{n-\alpha}}dy\right|^q dx\right)^{\frac{1}{q}}.$$

It is clear that $\beta \le \frac{\gamma}{2}$ for $x \in^c B(0,\gamma), y \in B(0,\beta)$ we have

$$|x-y| \ge |x| - |y| \ge |x| - \beta = \frac{|x|}{2} + \frac{|x|}{2} - \beta \ge \frac{|x|}{2}. \tag{8}$$

From this it follows that

$$I \le 2^{n-\alpha}\left(\int\limits_{{}^cB(0,\gamma)} \frac{dx}{|x|^{(n-\alpha)q}}\right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)|dy \le$$

$$\le 2^{n-\alpha}\left(\delta_n\int\limits_{\gamma}^{\infty} \rho^{(n-\alpha)q+n-1}d\rho\right)^{\frac{1}{q}} \left(\upsilon_n\beta^n\right)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))} =$$

$$= 2^{n-\alpha}\left(\frac{\delta_n}{(n-\alpha)q-n}\right)^{\frac{1}{q}} \upsilon_n^{1-\frac{1}{p}}\beta^{n(1-\frac{1}{p})}\gamma^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_p(B(0,\beta))} \equiv$$

$$\equiv C_1\gamma^{\alpha-n(1-\frac{1}{p})} \|f\|_{L_p(B(0,\beta))}\,. \tag{9}$$

$\beta = \frac{\gamma}{2}$ for $x \in^c B(0,\gamma), y \in B(0,\beta)$, then using (8) we get $|x-y| \ge \frac{|x|}{2}$.

Next, we consider

$$I \leq 2^{n-\alpha}\gamma^{\alpha-n}\left(\int\limits_{B(t,r)} dx\right)^{\frac{1}{q}} \int\limits_{B(0,\beta)} |f(y)|dy \leq$$

$$\leq 2^{n-\alpha}\gamma^{\alpha-n}\left(\upsilon_n r^n\right)^{\frac{1}{q}}\left(\upsilon_n \beta^n\right)^{1-\frac{1}{p}} \|f\|_{L_p(B(0,\beta))} =$$

$$= C_2 \gamma^{\alpha-n} r^{\frac{n}{q}} \|f\|_{L_p(B(0,\beta))}. \tag{10}$$

From inequality (9) and (10) it follows (7), where $C = \max\{C_1, C_2\}$.

Lemma 2. Let $n \in \mathbb{N},\ 1 \leq p \leq \infty,\ 1 \leq q \leq \infty,\ 0 < \alpha < n\left(1-\frac{1}{q}\right),\ \beta > 0$. Then there exists $C > 0$, depending only on n, p, q, α such that for some $f \in L_p(B(0,\beta))$, $b \in L_\infty(\mathbb{R}^n)$, satisfying the condition $supp\ b \subset \overline{B(0,\beta)}$, and for some $\gamma \geq 2\beta$, $t \in \mathbb{R}^n$, $r > 0$

$$\left\|([b, I_\alpha]f)\chi_{^c B(0,\gamma)}\right\|_{L_q(B(t,r))} \leq C\gamma^{\alpha-n}\left(min\{\gamma, r\}\right)^{\frac{n}{q}} \|b\|_{L_\infty(\mathbb{R}^n)} \|f\|_{L_p B(0,\beta)}. \tag{11}$$

Proof. Let $\gamma > \beta$,$supp\ b \subset B(0,\beta)$, for $x \in^c B(0,\gamma), b(x) = 0$. Then

$$\left\|[b, I_\alpha]\, f\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))} =$$

$$= \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{\mathbb{R}^n} \frac{(b(x)-b(y))f(y)}{|x-y|^{n-\alpha}} dy\right|^q dx\right)^{\frac{1}{q}} \leq$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{\mathbb{R}^n} \frac{b(y)f(y)}{|x-y|^{n-\alpha}} dy\right|^q dx\right)^{\frac{1}{q}} \leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{B(0,\beta)} \frac{|b(y)||f(y)|}{|x-y|^{n-\alpha}} dy\right|^q dx\right)^{\frac{1}{q}} \leq$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{B(0,\beta)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy\right|^q dx\right)^{\frac{1}{q}} \|b\|_{L_\theta(\mathbb{R}^n)} \leq$$

$$\leq \left(\int\limits_{B(t,r)\cap^c B(0,\gamma)} \left|\int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy\right|^q dx\right)^{\frac{1}{q}} \|b\|_{L_\theta(\mathbb{R}^n)} = \left\|(I_\alpha f)\,\chi^c_{B(0,\gamma)}\right\|_{L_q(B(t,r))} \|b\|_{L_\theta(\mathbb{R}^n)}.$$

From this and from Lemma 1 we obtain the inequality (11).

Proof of Theorem 5. To the proof of Theorem 5 it is sufficient to show that the conditions (2)–(4) of Theorem 3 are hold.

Let F be an arbitrary bounded subset of $LM^{w_1}_{p\theta}$. Since $C^\infty_c(\mathbb{R}^n)$ is dense in $VMO(\mathbb{R}^n)$ we only need to prove that the set $G = \{[b, I_\alpha]f : f \in F, b \in C^\infty_c\}$ is pre-compact in the $GM^{w_2}_p$. By Theorem 3, we only need to verify the conditions (2), (3) and (4) hold uniformly F for $b \in C^\infty_c\}$.

Suppose that

$$\|f\|_{LM^{w_1}_{p\theta}} \leq D.$$

Applying condition (1), we have

$$\|[b, I_\alpha]f\|_{LM^{w_2}_{p\theta}} \leq C \cdot \|b\|{*} \sup_{f\in F} \|f\|_{M^{w_1}_p} \leq C \cdot D\|b\|{*} < \infty.$$

This implies that the condition (2) of Theorem 3 is hold.

Now we prove that condition (4) of Theorem 3 also is hold, i. e.

$$\lim_{\gamma\to\infty}\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=0.$$

It follows from Lemma 2. Indeed

$$\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=$$

$$=\left\|w(r)\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{L_p(B(0,r))}\right\|_{L_\theta(0,\infty)}\le$$

$$\le C\gamma^{-n}\left\|b\right\|_{L_\theta(\mathbb{R}^n)}\left\|f\right\|_{L_pB(0,\beta)}\sup_{\substack{r>0,\\ x\in\mathbb{R}^n}}\left\|w_2(r)\left(min\{\gamma,r\}\right)^{\frac{n}{p}}\right\|_{L_\theta(0,\infty)}.$$

When $r<l<\gamma$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=r^{\frac{n}{p}}$. By condition $\left\|w_2(r)r^{\frac{n}{p}}\right\|_{L_\theta(l,\infty)}<\infty$.

When $\gamma<t<r$ we have $(min\{\gamma,r\})^{\frac{n}{p}}=\gamma^{\frac{n}{p}}$. By condition $\left\|w_2(r)\right\|_{L_\theta(0,t)}<\infty$.

Therefore

$$\lim_{\gamma\to\infty}\left\|([b,I_\alpha]\,f)\chi^c_{B(0,\gamma)}\right\|_{LM^{w_2}_{p\theta}}=0.$$

This implies the required condition (4).

Now we prove that condition (3) of Theorem 3 for the set $[b,I_\alpha](f)$, $f\in F$, is hold i.e. we show that for any $0<\varepsilon<\frac{1}{2}$ and if $|z|$ is sufficiently small depending only on ε, then for every $f\in F$.

$$\left\|[(b,I_\alpha f)(\cdot+z)]-[b,I_\alpha]\,f(\cdot)\right\|_{LM^{w_2}_{p\theta}}\le C\cdot\varepsilon.$$

Let ε arbitrary number such that $0<\varepsilon<\frac{1}{2}$. For $|z|\in\mathbb{R}^n$ we have, that

$$[f,I_\alpha]f(x+z)-[b,I_\alpha]f(x)=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy-\int\limits_{\mathbb{R}^n}\frac{[b(x)-b(y)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy-\int\limits_{\mathbb{R}^n}\frac{[b(x)+b(x+z)-b(x+z)-b(y)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy+\int\limits_{\mathbb{R}^n}\frac{[b(y)-b(x+z)]f(y)}{|x-y|^{n-\alpha}}dy+$$

$$+\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{\mathbb{R}^n}[b(y)-b(x+z)]\left(\frac{f(y)}{|x-y|^{n-\alpha}}-\frac{f(y)}{|x+z-y|^{n-\alpha}}\right)dy+\int\limits_{\mathbb{R}^n}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy=$$

$$=\int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}}\frac{[b(x+z)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy+$$

$$+\int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}}\left(\frac{f(y)}{|x-y|^{n-\alpha}}-\frac{f(y)}{|x+z-y|^{n-\alpha}}\right)[b(x+z)-b(y)]dy+$$

$$+ \int\limits_{|x-y|\le|z|e^{\frac{1}{\varepsilon}}} \frac{[b(y)-b(x)]f(y)}{|x-y|^{n-\alpha}}dy - \int\limits_{|x-y|\le|z|e^{\frac{1}{\varepsilon}}} \frac{[b(y)-b(x+z)]f(y)}{|x+z-y|^{n-\alpha}}dy =$$

$$= J_1 + J_2 + J_3 - J_4. \tag{12}$$

Since $b \in C_0^n(\mathbb{R}^n)$, we have

$$|b(x) - b(x+z)| \le |\nabla f(x)| \cdot |z| \le C|z|.$$

Then

$$|J_1| \le C|z| I_\alpha(|f|)(x).$$

By Theorem 5

$$||J_1||_{LM^{w_2}_{p\theta}} \le C|z| \parallel I_\alpha(f) \parallel_{LM^{w_2}_{p\theta}} \le C|z| \parallel f \parallel_{LM^{w}_{p\theta}} \le CD|z|. \tag{13}$$

For J_2 we have that

$$(b(x+z) - b(y)) \le 2 \parallel b \parallel_\infty \le C.$$

Therefore

$$|J_2| \le C|z| \int\limits_{|x-y|>|z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^n} dy \le C_\varepsilon I_\alpha(|f|)(x).$$

Again by the of Theorem 1 we get

$$\parallel J_2 \parallel_{LM^{w_2}_{p\theta}} \le c\varepsilon \parallel I_\alpha(f) \parallel_{LM^{w_1}_{p\theta}} \le c\varepsilon \parallel C \cdot D \cdot \varepsilon.$$

Consequently,

$$|J_3| \le C \int\limits_{|x-y|\le|z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^{n-\alpha}}dy \le C\varepsilon^{-1}|z| \int\limits_{|x-y|\le|z|e^{\frac{1}{\varepsilon}}} \frac{f(y)}{|x-y|^{n-\alpha}}dy \le C \cdot \varepsilon^{-1}|z| I_\alpha(|f|)(x).$$

Thus, we have

$$\parallel J_3 \parallel_{LM^{w_2}_{p\theta}} \le C \cdot \varepsilon^{-1}|z| \parallel I_\alpha(f) \parallel_{LM^{w_2}_{p\theta}} \le C \cdot \varepsilon^{-1}|z| \parallel f \parallel_{LM^{w_1}_{p\theta}} \le \varepsilon^{-1}|z|. \tag{14}$$

Similarly, using the estimate finally by

$$|b(x+z) - b(y)| \le C|x+z-y|,$$

we have

$$|J_4| \le C \int\limits_{|x-y|\le e^{\varepsilon^{-1}}(y)} |x+z-y|^{-n+1+\alpha}|b(y)|dy \le C(\varepsilon^{-1}|z| + |z|) I_\alpha(|f|)(x+z).$$

Therefore

$$\parallel J_4 \parallel_{LM^{w_2}_{p\theta}} \le C \cdot (\varepsilon^{-1}|z| + |z|) \parallel f \parallel_{LM^{w_1}_{p\theta}} \le C \cdot (\varepsilon^{-1}|z| + |z|). \tag{15}$$

Here C does not depend on z and ε. Finally from (12)–(15) taking a $|z|$ small enough we have

$$\|[b, I_\alpha(f)(\cdot + z)] - [b, I_\alpha] f(\cdot)\|_{LM^{w_2}_{p\theta}} \le \|J_1\|_{LM^{w_2}_{p\theta}} + \|J_2\|_{LM^{w_2}_{p\theta}} + \|J_3\|_{LM^{w_2}_{p\theta}} + \|J_4\|_{LM^{w_2}_{p\theta}} \le C \cdot D \cdot \varepsilon$$

i.e. the set $[b, I_\alpha](f),\ f \in F$ satisfies the condition (3) of Theorem 3. Then by Theorem 3, the set $[b, I_\alpha](f),\ f \in F$ is precompact in the $LM^{w_2}_{p\theta}$. Which completes the proof of the theorem.

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Локальді Морри типтес кеңістігінде Рисс потенциал коммутаторының компактылығы

Мақалада $LM^{w}_{p\theta}$ локальді Морри типті кеңістіктері қарастырылған. Негізгі жұмыс $LM^{w_1}_{p\theta}$-дан $LM^{w_2}_{q\theta}$-ге дейінгі локальді Морри типті кеңістіктердегі $[b, I_\alpha]$ Рисс потенциалы үшін коммутатордың компактылық теоремасын дәлелдеу. Сондай-ақ, Рисс потенциалы үшін коммутатордың $[b, I_\alpha]$ локальді Морри типті кеңістіктердегі $LM^{w_1}_{p\theta}$-дан $LM^{w_2}_{q\theta}$ шенелгендігі үшін жаңа жеткілікті шарттар берілген. Рисс потенциалы үшін коммутатордың компактылық теоремасын дәлелдеуде негізінен, $[b, I_\alpha]$ Рисс потенциалы үшін коммутатордың $LM^{w}_{p\theta}$ локальді Морри типті кеңістіктерінде шектелген шарты, сонымен қатар $LM^{w}_{p\theta}$ локальді Морри типті кеңістіктеріндегі жиындардың компактылық теоремасының жеткілікті шарттары пайдаланылған. Рисс потенциалы үшін коммутатордың жинақылық теоремасын дәлелдеу барысында $[b, I_\alpha]$ Рисс потенциалы үшін коммутатор шарының шенелген леммалары анықталған. Осыған ұқсас нәтижелер $GM^{w}_{p\theta}$ глобальды Морри типті кеңістіктер үшін және M^{w}_{p} жалпыланған Морри кеңістігі үшін де алынған.

Кілт сөздер: компактылы, коммутаторлар, Рисс потенциалы, локальді Морри типті кеңістіктер.

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Компактность коммутаторов для потенциала Рисса в локальных пространствах типа Морри

В статье рассмотрены локальные пространства типа Морри из $LM^{w}_{p\theta}$. Основной работой является доказательство теоремы компактности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри из $LM^{w_1}_{p\theta}$ в $LM^{w_2}_{q\theta}$. Приведены также новые достаточные условия ограниченности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри из $LM^{w_1}_{p\theta}$ в $LM^{w_2}_{q\theta}$. В доказательстве теоремы компактности коммутатора для потенциала Рисса существенно использованы условие ограниченности коммутатора для потенциала Рисса $[b, I_\alpha]$ в локальных пространствах типа Морри $LM^{w}_{p\theta}$, а также достаточные условия из теоремы предкомпактности множеств в локальных пространствах типа Морри $LM^{w}_{p\theta}$. В ходе доказательства теоремы компактности коммутатора для потенциала Рисса подтверждены леммы оценки по шару коммутатора для потенциала Рисса $[b, I_\alpha]$. Аналогичные результаты были получены для глобальных пространств типа Морри $GM^{w}_{p\theta}$ и для обобщенных пространств Морри M^{w}_{p}.

Ключевые слова: компактность, коммутаторы, потенциал Рисса, локальные пространства типа Морри.

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