DOI 10.31489/2023M2/84-92

UDC 517.95

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On one approximate solution of a nonlocal boundary value problem for the Benjamin-Bona-Mahony equation

The paper investigates a non-local boundary value problem for the Benjamin-Bona-Mahony equation. This equation is a nonlinear pseudoparabolic equation of the third order with a mixed derivative. To find a solution to this problem, an algorithm for finding an approximate solution is proposed. Sufficient conditions for the feasibility and convergence of the proposed algorithm are established, as well as the existence of an isolated solution of a non-local boundary value problem for a nonlinear equation. Estimates are obtained between the exact and approximate solution of this problem.

Keywords: partial differential equation, Benjamin-Bona-Mahony equation, algorithm, approximate solution.

Introduction

The paper considers a nonlocal boundary value problem for a third-order nonlinear partial differential equation or the Benjamin–Bona–Mahony equation. The Benjamin–Bona–Mahony equation or the regularized long-wavelength equation was studied in [1–4]. Modern studies on this topic can be found in [5–13]. In this article, introducing a new function, a non-local boundary value problem for a third-order nonlinear differential equation is reduced to a non-local boundary value problem for a hyperbolic equation. The resulting problem with different conditions was investigated in [14–18]. Similarly to the linear case [19–22], sufficient conditions for the unique solvability of the problem under consideration are established and an algorithm for finding an approximate solution is proposed.

1 Statement of the initial boundary problem

On $\Omega = [0, X] \times [0, Y]$ we consider a nonlocal boundary value problem for the nonlinear equation

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial w}{\partial y} + w\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}, \quad (x, y) \in \Omega, \tag{1}$$

$$w(x, 0) = \varphi(x), \quad x \in [0, X], \tag{2}$$

$$\frac{\partial w(0, y)}{\partial y} = \alpha(y)\frac{\partial w(X, y)}{\partial y} + \psi(y), \quad y \in [0, Y], \tag{3}$$

$$\frac{\partial w(0, y)}{\partial x} = \theta(y), \quad y \in [0, Y], \tag{4}$$

where the functions $\psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, the function $\varphi(x)$ is continuously differentiable on $[0, X]$, $\alpha(y) \neq 1$.

Let $C(\Omega, R)$ be the set of functions $w : \Omega \to R$ continuous on Ω.

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A function $u(x,y) \in C(\Omega,R)$, with partial derivatives $\frac{\partial u(x,y)}{\partial x} \in C(\Omega,R)$, $\frac{\partial u(x,y)}{\partial y} \in C(\Omega,R)$, $\frac{\partial^2 u(x,y)}{\partial x \partial y} \in C(\Omega,R)$, $\frac{\partial^2 u(x,y)}{\partial x^2} \in C(\Omega,R)$, $\frac{\partial^3 u(x,y)}{\partial x^2 \partial y} \in C(\Omega,R)$ is called a solution to problem (1)–(4) if it satisfies equation (1), for all $(x,y) \in \Omega$, and boundary conditions (2)–(4).

To find a solution to problem (1)–(4), we introduce the functions

$$w(0,y) = \lambda(y), \quad \widetilde{w}(x,y) = w(x,y) - \lambda(y),$$

the original problem can be written as

$$\frac{\partial^3 \widetilde{w}(x,y)}{\partial x^2 \partial y} = \frac{\partial \widetilde{w}(x,y)}{\partial y} + \frac{\partial \lambda(y)}{\partial y} + [\widetilde{w}(x,y) + \lambda(y)]\frac{\partial \widetilde{w}(x,y)}{\partial x} + \frac{\partial \widetilde{w}(x,y)}{\partial x},$$

$$\widetilde{w}(0,y) = 0, \quad x \in [0,X],$$

$$\widetilde{w}(x,0) + \lambda(0) = \varphi(x), \quad \lambda(0) = \varphi(0),$$

$$\frac{\partial \lambda(y)}{\partial y} = \alpha(y)\frac{\partial \widetilde{w}(X,y)}{\partial y} + \alpha(y)\frac{\partial \lambda(y)}{\partial y} + \psi(y), \quad y \in [0,Y],$$

$$\frac{\partial \widetilde{w}(0,y)}{\partial x} = \theta(y), \quad y \in [0,Y].$$

We introduce a new function $v(x,y) = \frac{\partial \widetilde{w}(x,y)}{\partial x}$, then $\widetilde{w}(x,y) = \int\limits_0^x v(\xi,y)d\xi$ and the problem goes to the equivalent problem

$$\frac{\partial^2 v(x,y)}{\partial x \partial y} = \int\limits_0^x \frac{\partial v(\xi,y)}{\partial y}d\xi + \frac{\partial \lambda(y)}{\partial y} + \left[\int\limits_0^x v(\xi,y)d\xi + \lambda(y)\right]v(x,y) + v(x,y), \tag{5}$$

$$v(x,0) = \varphi'(x), \quad x \in [0,X], \tag{6}$$

$$\frac{\partial \lambda(y)}{\partial y} = \frac{\alpha(y)}{1-\alpha(y)}\int\limits_0^X v(x,y)dx + \frac{\psi(y)}{1-\alpha(y)}, \quad \lambda(0) = \varphi(0), \tag{7}$$

$$v(0,y) = \theta(y), \quad y \in [0,Y]. \tag{8}$$

Integrating both parts of equation (5) with respect to the variable x and taking into account conditions (8) we get

$$\frac{\partial v(x,y)}{\partial y} = \theta'(y) + \int\limits_0^x \left(\int\limits_0^\xi \frac{\partial v(\xi_1,y)}{\partial y}d\xi_1 + \frac{\partial \lambda(y)}{\partial y} + \left[\int\limits_0^\xi v(\xi_1,y)d\xi_1 + \lambda(y)\right]v(\xi,y) + v(\xi,y)\right)d\xi. \tag{9}$$

Once again integrating over the variable y and using condition (6), we have [23]

$$v(x,y) = \varphi'(x) + \int\limits_0^y \left(\theta'(\eta) + \int\limits_0^x \int\limits_0^\xi \frac{\partial v(\xi_1,\eta)}{\partial \eta}d\xi_1 d\xi + \int\limits_0^x \frac{\partial \lambda(\eta)}{\partial \eta}d\xi + \right.$$

$$\left. + \int\limits_0^x \left(\left[\int\limits_0^\xi v(\xi_1,\eta)d\xi_1 + \lambda(\eta)\right]v(\xi,\eta) + v(\xi,\eta)\right)d\xi\right)d\eta. \tag{10}$$

In addition, from (12) it follows

$$\lambda(y)=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}\int\limits_0^X v(x,\eta)dx+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta. \qquad (11)$$

2 Main result

Setting $v(x,y)=\varphi'(x)$, from (7) and (11) we define

$$\frac{\partial\lambda^{(0)}(y)}{\partial y}=\frac{\alpha(y)}{1-\alpha(y)}\int\limits_0^X\varphi'(x)dx+\frac{\psi(y)}{1-\alpha(y)}=\frac{\alpha(y)}{1-\alpha(y)}[\varphi(X)-\varphi(0)]+\frac{\psi(y)}{1-\alpha(y)},$$

$$\lambda^{(0)}(y)=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}\int\limits_0^X\varphi'(x)dx+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta=$$

$$=\varphi(0)+\int\limits_0^y\left(\frac{\alpha(\eta)}{1-\alpha(\eta)}[\varphi(X)-\varphi(0)]+\frac{\psi(\eta)}{1-\alpha(\eta)}\right)d\eta.$$

Using equation (9), with $\lambda(y)=\lambda^{(0)}(y)$, we find

$$\frac{\partial v^{(0)}(x,y)}{\partial y}=\theta'(y)+\int\limits_0^x\left(\frac{\partial\lambda^{(0)}(y)}{\partial y}+\left[\int\limits_0^\xi\varphi'(\xi_1)d\xi_1+\lambda(y)\right]\varphi'(\xi)+\varphi'(\xi)\right)d\xi.$$

$$v^{(0)}(x,y)=\varphi'(x)+\int\limits_0^y\frac{\partial v^{(0)}(x,\eta)}{\partial\eta}d\eta.$$

Taking the functions $\lambda^{(0)}(y), v^{(0)}(x,y)$, numbers $\rho_1>0,\rho_2>0$ we construct the sets

$$S\left(\lambda^{(0)}(y),\rho_1\right)=\left\{\lambda(y)\in C([0,Y],R):\|\lambda(y)-\lambda^{(0)}(y)\|<\rho_1\right\},$$

$$S(v^{(0)}(x,y),\rho_2)=\left\{v(x,y)\in C(\Omega,R):\|v(x,y)-v^{(0)}(x,y)\|<\rho_2,(x,y)\in\Omega\right\},$$

$$G^0(\rho_1,\rho_2)=\left\{(x,y,w,v):(x,y)\in\Omega,\quad\left\|w(x,y)-\int\limits_0^x v^{(0)}(\xi,y)d\xi-\lambda^{(0)}(y)\right\|<\rho_1+\rho_2,\right.$$

$$\left.\|v(x,y)-v^{(0)}(x,y)\|<\rho_2\right\}.$$

Denote by $U(L_1,L_2,x,y)$ the collection $\left(\lambda^{(0)}(y),v^{(0)}(x,y),\rho_1,\rho_2\right)$, or which the function $f(x,y,w,v)$ in $G^0(\rho_1,\rho_2)$ has continuous partial derivatives $f'_w(x,y,w,v)$, $f'_v(x,y,w,v)$ and

$$\|f'_w(x,y,w,v)\|\le L_1,\quad\|f'_v(x,y,w,v)\|\le L_2,\quad L_1,L_2-const.$$

Based on the system $\{\lambda(y), \mu(y), v(x,y)\}$ we compose the triple $\{\lambda^{(0)}(y), \mu^{(0)}(y), v^{(0)}(x,y)\}$, which we take as the initial approximation of problem (5)–(8) and build successive approximations according to the following algorithm:

Step 1. Setting $v(x,y) = v^{(0)}(x,y)$, from (7) and (11) we determine $\frac{\partial\lambda^{(1)}(y)}{\partial y}$ and $\lambda^{(1)}(y)$. Using equation (9), with $\lambda(y) = \lambda^{(1)}(y)$, we find $\frac{\partial v^{(1)}(x,y)}{\partial y}$. Next, we find $v^{(1)}(x,y) = \varphi'(x) + \int\limits_0^y \frac{\partial v^{(1)}(x,\eta)}{\partial \eta} d\eta$.

Step 2. Taking $v(x,y) = v^{(1)}(x,y)$, from (7) and (11) we determine $\frac{\partial\lambda^{(2)}(y)}{\partial y}$ and $\lambda^{(2)}(y)$, respectively. Using equation (9), with $\lambda(y) = \lambda^{(2)}(y)$, we find $\frac{\partial v^{(2)}(x,y)}{\partial y}$. Let us find $v^{(2)}(x,y) = \varphi'(x) + \int\limits_0^y \frac{\partial v^{(2)}(x,\eta)}{\partial \eta} d\eta$.

Continuing the process, at the k-th step we obtain the system $\left\{\frac{\partial\lambda^{(2)}(y)}{\partial y}, \lambda_r^{(k)}(x), \frac{\partial v^{(k)}(x,y)}{\partial y}, v_r^{(k)}(x,t)\right\}$.

The conditions of the following statement ensure the feasibility and convergence of the proposed algorithm, as well as the solvability of problem (5)–(8).

Theorem 1. Let there exist $(\lambda^{(0)}(y), v^{(0)}(x,y), \rho_1, \rho_2) \in U(L_1, L_2, x, y)$, where $(x,y) \in \Omega$, $(\lambda(y), v(x,y)) \in S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x,y), \rho_2)$ and following conditions are satisfied:

1) the function $\varphi(x)$ is continuously differentiable on $[0, X]$,

2) the functions $a(y), \psi(y), \theta(y)$ are continuously differentiable on $[0, Y]$, $\alpha(y) \neq 1$,

3) $q = X\left(\frac{X}{2} + \frac{X\alpha}{1-\alpha} + \frac{XYL_1}{2} + \frac{XY^2}{2}\frac{L_1\alpha}{1-\alpha} + (L_2+1)Y\right) < 1$,

4) $\frac{\sigma}{1-q}\frac{\alpha}{1-\alpha}\frac{XY^2}{2} < \rho_1, \quad \frac{qY\sigma}{1-q} < \rho_2$,

where $\sigma = \theta' + \frac{X\psi}{1-\alpha}\left(1 + \max\limits_{x\in[0,X]} \|\varphi'(x)\| Y\right) + X \max\limits_{x\in[0,X]} \|\varphi'(x)\| \left(1 + \max\limits_{x\in[0,X]} \|\varphi(x)\|\right)$, $\alpha = \max\limits_{y\in[0,Y]} \|\alpha(y)\|$, $\psi = \max\limits_{y\in[0,Y]} \|\psi(y)\|$, $\theta = \max\limits_{y\in[0,Y]} \|\theta(y)\|$, then the nonlocal problem (5)–(8) for the nonlinear Benjamin-Bona-Mahony equation has a unique solution belonging to $S(\lambda^{(0)}(y), \rho_1) \times S(v^{(0)}(x,y), \rho_2)$ and estimates are made:

$$a)\|\lambda^*(y) - \lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=k}^{\infty} q^i\sigma, \quad b)\|v^*(x,y) - v^{(k)}(x,y)\| \leq Y \sum_{i=k+1}^{\infty} q^i\sigma.$$

Proof. From the zero step of the algorithm, the following estimates hold:

$$\|\lambda^{(0)}(y)\| \leq \varphi(0) + \frac{\psi Y}{1-\alpha}, \quad \left\|\frac{\partial\lambda(y)}{\partial y}\right\| \leq \frac{\psi}{1-\alpha},$$

$$\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq \theta' + X\frac{\psi}{1-\alpha} + X \max_{x\in[0,X]} \|\varphi(x)\| \max_{x\in[0,X]} \|\varphi'(x)\| +$$

$$+XY\frac{\psi}{1-\alpha}\max_{x\in[0,X]} \|\varphi'(x)\| + X \max_{x\in[0,X]} \|\varphi'(x)\| \leq$$

$$\leq \theta' + \frac{X\psi}{1-\alpha}\left(1 + \max_{x\in[0,X]} \|\varphi'(x)\| Y\right) + X \max_{x\in[0,X]} \|\varphi'(x)\| \left(1 + \max_{x\in[0,X]} \|\varphi(x)\|\right) = \sigma,$$

$$\|v^{(0)}(x,y) - \varphi'(x)\| \leq \int\limits_0^y \left\|\frac{\partial v^{(0)}(x,\eta)}{\partial \eta}\right\| d\eta < Y\sigma.$$

From the first step of the algorithm, for $v(x,y) = v^{(0)}(x,y)$, the following inequalities follow

$$\|\lambda^{(1)}(y) - \lambda^{(0)}(y)\| \le \frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X \|v^{(0)}(x,\eta) - \varphi'(x)\|dxd\eta < \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sigma < \rho_1,$$

$$\left\|\frac{\partial\lambda^{(1)}(y)}{\partial y} - \frac{\partial\lambda^{(0)}(y)}{\partial y}\right\| \le \frac{\alpha}{1-\alpha}\int\limits_0^X \left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\|dx,$$

$$\left\|\frac{\partial v^{(1)}(x,y)}{\partial y} - \frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le$$

$$\le \int\limits_0^x\int\limits_0^\xi \left\|\frac{\partial v^{(0)}(\xi_1,y)}{\partial y}\right\|d\xi_1 d\xi + \int\limits_0^x \left\|\frac{\partial\lambda^{(1)}(y)}{\partial y} - \frac{\partial\lambda^{(0)}(y)}{\partial y}\right\|d\xi +$$

$$+L_1\int\limits_0^x\int\limits_0^\xi \|v^{(0)}(\xi_1,y) - \varphi'(\xi_1)\|d\xi_1 d\xi + L_1\|\lambda^{(1)}(y) - \lambda^{(0)}(y)\|\int\limits_0^x d\xi +$$

$$+L_2\int\limits_0^x \|v^{(0)}(\xi,y) - \varphi'(\xi)\|d\xi + \int\limits_0^x \|v^{(0)}(\xi,y) - \varphi'(\xi)\|d\xi \le$$

$$\le q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le q\sigma.$$

$$\|v^{(1)}(x,y) - v^{(0)}(x,y)\| \le \int\limits_0^y \left\|\frac{\partial v^{(1)}(x,\eta)}{\partial\eta} - \frac{\partial v^{(0)}(x,\eta)}{\partial\eta}\right\|d\eta \le \int\limits_0^y q\sigma d\eta < \rho_2.$$

At the second step of the algorithm, for $v(x,y) = v^{(1)}(x,y)$, the following estimates hold:

$$\|\lambda^{(2)}(y) - \lambda^{(1)}(y)\| \le \frac{\alpha}{1-\alpha}\int\limits_0^y\int\limits_0^X \|v^{(1)}(x,\eta) - v^{(0)}(x,\eta)\|dxd\eta \le \frac{\alpha}{1-\alpha}\frac{XY^2}{2}q\sigma,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y} - \frac{\partial v^{(1)}(x,y)}{\partial y}\right\| \le q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y} - \frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le q^2\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le q^2\sigma,$$

$$\|v^{(2)}(x,y) - v^{(1)}(x,y)\| \le \int\limits_0^y \left\|\frac{\partial v^{(2)}(x,\eta)}{\partial\eta} - \frac{\partial v^{(1)}(x,\eta)}{\partial\eta}\right\|d\eta \le Yq^2\sigma.$$

$$\|\lambda^{(2)}(y) - \lambda^{(0)}(y)\| \le \|\lambda^{(2)}(y) - \lambda^{(1)}(y)\| + \|\lambda^{(1)}(y) - \lambda^{(0)}(y)\| \le$$

$$\le \frac{\alpha}{1-\alpha}\frac{XY^2}{2}q\sigma + \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sigma \le \frac{\alpha}{1-\alpha}\frac{XY^2}{2}(1+q)\sigma < \rho_1,$$

$$\left\|\frac{\partial v^{(2)}(x,y)}{\partial y} - \frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le (1+q)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(1)}(x,y)}{\partial y} - \frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \le$$

$$\leq (q+q^2)\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(0)}(x,y)}{\partial y}\right\| \leq (q+q^2)\sigma.$$

$$\|v^{(2)}(x,y)-v^{(0)}(x,y)\| \leq \|v^{(2)}(x,y)-v^{(1)}(x,y)\| + \|v^{(1)}(x,y)-v^{(0)}(x,y)\| \leq Y(q^2+q)\sigma < \rho_2.$$

At the $(k+1)-$th step of the algorithm, for $v(x,y)=v^{(k)}(x,y)$, the following estimates hold:

$$\|\lambda^{(k+1)}(y)-\lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\int_0^y\int_0^X \|v^{(k)}(x,\eta)-v^{(k-1)}(x,\eta)\|dxd\eta. \tag{12}$$

$$\left\|\frac{\partial\lambda^{(k+1)}(y)}{\partial y}-\frac{\partial\lambda^{(k)}(y)}{\partial y}\right\| \leq \frac{\alpha}{1-\alpha}\int_0^X \left\|\frac{\partial v^{(k)}(x,y)}{\partial y}-\frac{\partial v^{(k-1)}(x,y)}{\partial y}\right\|dx, \tag{13}$$

$$\left\|\frac{\partial v^{(k+1)}(x,y)}{\partial y}-\frac{\partial v^{(k)}(x,y)}{\partial y}\right\| \leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{(k)}(x,y)}{\partial y}-\frac{\partial v^{(k-1)}(x,y)}{\partial y}\right\|, \tag{14}$$

$$\|v^{(k+1)}(x,y)-v^{(k)}(x,y)\| \leq \int_0^y \left\|\frac{\partial v^{(k+1)}(x,\eta)}{\partial \eta}-\frac{\partial v^{(k)}(x,\eta)}{\partial \eta}\right\|d\eta. \tag{15}$$

$$\|\lambda^{(k+1)}(y)-\lambda^{(0)}(y)\| \leq \frac{\alpha}{1-\alpha}\int_0^y\int_0^X \|v^{(k)}(x,\eta)-\varphi'(x)\|dxd\eta \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=0}^{k} q^i\sigma < \rho_1,$$

$$\|v^{(k+1)}(x,y)-v^{(0)}(x,y)\| \leq Y\sum_{i=1}^{k+1} q^i\sigma < \rho_2.$$

Thus, it follows from inequalities (12)–(15) and $q<1$ that the sequence $\{\lambda^{(k)}(y), v^{(k)}(x,y)\}$ as $k\to\infty$, converges to $\{\lambda^*(y), v^*(x,y)\}$ to the solution of problem (5)–(8) in $S(\lambda^{(0)}(y),\rho_1)\times S(v^{(0)}(x,y),\rho_2)$.

Let's establish the inequalities

$$\|\lambda^{(k+p)}(y)-\lambda^{(k)}(y)\| \leq \frac{\alpha}{1-\alpha}\frac{XY^2}{2}\sum_{i=k}^{k+p-1} q^i\sigma, \qquad \|v^{(k+p)}(x,y)-v^{(k)}(x,y)\| \leq Y\sum_{i=k+1}^{k+p} q^i\sigma,$$

as $p\to\infty$ we obtain estimates a), b) of Theorem 1.

Let's prove uniqueness.

Let there be two solutions $(\lambda^*(x), v^*(x,y))$, $(\lambda^{**}(x), v^{**}(x,y))$ in $S(\lambda^{(0)}(x),\rho_1)\times S(v^{(0)}(x,y),\rho_2)$ of problem (10)–(14). Similar to relations (12)–(15) for the differences $\lambda^{**}(y)-\lambda^*(y)$, $\frac{\partial\lambda^{**}(y)}{\partial y}-\frac{\partial\lambda^{*}(y)}{\partial y}$, $\frac{\partial v^{**}(x,y)}{\partial y}-\frac{\partial v^{*}(x,y)}{\partial y}$, $v^{**}(x,y)-v^*(x,y)$, for all $(x,y)\in\Omega$, we get:

$$\|\lambda^{**}(y)-\lambda^*(y)\| \leq \frac{\alpha}{1-\alpha}\int_0^y\int_0^X \|v^{**}(x,\eta)-v^*(x,\eta)\|dxd\eta,$$

$$\left\|\frac{\partial\lambda^{**}(y)}{\partial y}-\frac{\partial\lambda^{*}(y)}{\partial y}\right\| \leq \frac{\alpha}{1-\alpha}\int_0^X \|v^{**}(x,y)-v^*(x,y)\|dx,$$

$$\left\|\frac{\partial v^{**}(x,y)}{\partial y}-\frac{\partial v^{*}(x,y)}{\partial y}\right\|\leq q\max_{(x,y)\in\Omega}\left\|\frac{\partial v^{**}(x,y)}{\partial y}-\frac{\partial v^{*}(x,y)}{\partial y}\right\|,$$

$$\|v^{**}(x,y)-v^{*}(x,y)\|\leq\int\limits_0^y\left\|\frac{\partial v^{**}(x,\eta)}{\partial\eta}-\frac{\partial v^{*}(x,\eta)}{\partial\eta}\right\|d\eta.$$

Whence it follows that $\lambda^{**}(x)=\lambda^{*}(x)$, $v^{**}(x,y)=v^{*}(x,y)$. Theorem 1 is proved.

The function $w^{(k)}(x,y), k=1,2,...,$ is defined by the equality

$$w^{(k)}(x,y)=\int\limits_0^x v^{(k)}(\xi,y)d\xi+\lambda^{(k)}(y)$$

and denote by $S(w^{(0)}(x,y),\rho_1+\rho_2)$ the set of continuously differentiable with respect to y and twice with respect to x functions $w:\Omega\to R$ satisfying the inequalities

$$\|w(x,y)-\int\limits_0^x v^{(0)}(\xi,y)d\xi-\lambda^{(0)}(y)\|<\rho_1+\rho_2.$$

In view of the equivalence of problems (1)–(4) and (5)–(8), Theorem 1 implies.

Theorem 2. If the conditions of Theorem 1 are satisfied, then the sequence of functions $w^{(k)}(x,y)$, $k=1,2,...,$ is contained in $S(w^{(0)}(x,y),\rho_1+\rho_2)$ converges to the unique solution $w^{*}(x,y)$ of problem (1)–(4) in $S(w^{(0)}(x,y),\rho_1+\rho_2)$ and the inequality

$$\|w^{*}(x,y)-w^{(k)}(x,y)\|\leq\frac{\alpha}{1-\alpha}\frac{X^2Y^2}{2}\sum_{i=k+1}^{\infty}q^i\sigma+Y\sum_{i=k}^{\infty}q^i\sigma.$$

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grants No. AP09259780, 2021-2023).

References

1 Benjamin, T.B., Bona, J.L., & Mahony, J.J. (1972). Model Equations for Long Waves in Nonlinear Dispersive Systems. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 272*(1220), 47–78.

2 Goldstein, J.A., & Wichnoski, B.J. (1980). On the Benjamin–Bona–Mahony equation in higher dimensions. *Nonlinear Analysis, 4*(4), 665–675. https://doi.org/10.1016/0362-546X(80)90067-X

3 Olver, P.J. (1979). Euler operators and conservation laws of the BBM equation. *Mathematical Proceedings of the Cambridge Philosophical Society, 85*(1), 143–160. https://doi.org/10.1017/S0305004100055572

4 Bona, J.L., Pritchard, W.G., & Scott, L.R. (1980). *Solitary-wave interaction, Physics of Fluids, 23*(3), 438–441. https://doi.org/10.1063/1.863011

5 Coclite, G.M., & Di Ruvo, L. (2018). A note on convergence of the solutions of Benjamin-Bona-Mahony type equations. *Nonlinear Analysis-Real World Applications, 40*, 64–81. https://doi.org/10.1016/j.nonrwa.2017.07.014

6 Bautista, G.J., & Pazoto, A.F. (2019). On the controllability of a Boussinesq system for two-way propagation of dispersive waves. *Journal of Evolution Equations, 20*(2), 607–630. https://doi.org/10.1007/s00028-019-00541-5

7 Guo, Y.T., Wang, M., & Tang, Y.B. (2015). Higher regularity of global attractor for a damped Benjamin-Bona-Mahony equation on R. *Applicable Analysis, 94*(9), 1766–1783. https://doi.org/10.1080/00036811.2014.946561

8 Calvo, M.C., & Schratz, K. (2022). Uniformly accurate splitting schemes for the Benjamin-Bona-Mahony equation with dispersive parameter. *Bit Numerical Mathematics, 62*(4), 1625–1647. https://doi.org/10.1007/s10543-022-00925-z

9 Khater, M.M.A., & Salama, S.A. (2022). Semi-analytical and numerical simulations of the modified Benjamin-Bona-Mahony model. *Journal of Ocean Engineering and Science, 7*(3), 264–271. https://doi.org/10.1016/j.joes.2021.08.008

10 Qin, Y.M., Yang, X.G., & Liu, X. (2012). Pullback Attractors for the Non-Autonomous Benjamin-Bona-Mahony Equations in H-2. *Acta Mathematica Scientia, 32*(4), 1338–1348.

11 Kozhanov, A.I. (1999). Composite Type Equations and Inverse Problems (Inverse and Iii-Posed Problems). *VSP: Utrecht*. https://doi.org/10.1515/9783110943276

12 Kozhanov, A.I., & Lukina, G.A. (2019). Pseudoparabolic and pseudohyperbolic equations in noncylindrical time domains. *Mathematical notes of NEFU, 3*(26), 15–30. https://doi.org/10.25587/SVFU.2019.17.12.002

13 Ngoc, L.T.P., Triet, N.A., Duyen, P.T.M., & Long, N.T. (2023). General Decay and Blow-up Results of a Robin-Dirichlet Problem for a Pseudoparabolic Nonlinear Equation of Kirchhoff-Carrier Type with Viscoelastic Term. *Acta Math Vietnam*. https://doi.org/10.1007/s40306-023-00496-3

14 Dzhumabaev, D.S., Temesheva S.M. (2016). Bounder solution on a strip to a system of nonlinear hyperbolic equations with mixed derivatives. *Bulletin of the Karaganda University. Mathematics series, 84*(4), 35-45.

15 Assanova, A.T., & Tokmurzin, Z.S. (2020). Method of functional parametrization for solving a semi-periodic initial problem for fourth-order partial differential equations. *Bulletin of the Karaganda University. Mathematics series, 100*(4), 5-16. https://doi.org/10.31489/2020M4/5-16

16 Orumbayeva, N., & Shayakhmetova, B. (2016). On a method of finding a solution of semi-periodic boundary value problem for hyperbolic equations. *International Conference on Analysis and Applied Mathematics (ICAAM 2016), 1759*, 1759. https://doi.org/10.1063/1.4959735

17 Keldibekova, A.B. (2020). Solvability of a semi-periodic boundary value problem for a third order differential equation with mixed derivative. *Bulletin of the Karaganda University. Mathematics series, 98*(2), 84-99. https://doi.org/10.31489/2020M2/84-99

18 Orumbayeva, N.T., & Sabitbekova, G. (2017). A boundary value problem for nonlinear differential equation with arbitrary functions. *Bulletin of the Karaganda University. Mathematics series, 85*(1), 71–76.

19 Orumbayeva, N.T., & Tokmagambetova, T.D. (2021). On One Solution of the Boundary Value Problem for a Pseudohyperbolic Equation of Fourth Order. *Lobachevskii Journal of Mathematics, 42*(15), 3705–3714. https://doi.org/10.1134/S1995080222030167

20 Orumbayeva, N.T., & Keldibekova, A.B. (2020). On One Solution of a Periodic Boundary-Value Problem for a Third-Order Pseudoparabolic Equation. *Lobachevskii Journal of Mathematics, 41*(9), 1864–1872. https://doi.org/10.1134/S1995080220090218

21 Koshkarova, B.S., & Kussainova, L.K. (2019). Oscillatory and Nonoscillatory Conditions for a Second-Order Half-Linear Differential Equation. *Journal of Mathematical Sciences, 241*(3), 285–

289.

22 Orumbayeva, N.T., Tokmagambetova, T.D., & Nurgalieva, Zh.N. (2021). On the solvability of a semi-periodic boundary value problem for the nonlinear Goursat equation. *Bulletin of the Karaganda University. Mathematics series, 104*(4), 110–117. https://doi.org/10.31489/2021M4/110-117

23 Zhakupbekova, A.Zh., & Koshkarova, B.S. (2010). About the properties of certain integral operators in weighted Holder spaces. *Bulletin of the Karaganda University. Mathematics series, 57*(1), 25–28.

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Бенджамин-Бон-Махони теңдеуі үшін бейлокал шеттік есептің бір жуық шешімі жайында

Жұмыста Бенджамин-Бона-Махони теңдеуі үшін бейлокал шеттік есеп зерттелді. Қарастырылып отырған теңдеу аралас туындылы үшінші ретті сызықтық емес псевдопараболалық теңдеу болып табылады. Осы есептің шешімін табу үшін жуық шешімін табу алгоритмі ұсынылған. Ұсынылған алгоритмнің орындалуы мен жинақтылығының жеткілікті шарттары алынды, сондай-ақ, сызықтық емес теңдеу үшін бейлокал шеттік есебінің оқшауланған шешімінің бар болуы табылған. Қарастырылған есептің дәл және жуық шешімі арасындағы бағалаулары алынды.

Кілт сөздер: дербес туындылы дифференицалдық теңдеу, Бенджамин-Бона-Махони теңдеуі, алгоритм, жуық шешім.

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Об одном приближенном решении нелокальной краевой задачи для уравнения Бенджамина-Бона-Махони

В работе исследована нелокальная краевая задача для уравнения Бенджамина-Бона-Махони. Рассматриваемое уравнение является нелинейным псевдопараболическим уравнением третьего порядка со смешанной производной. Для нахождения решения данной задачи предложен алгоритм поиска приближенного решения. Установлены достаточные условия осуществимости и сходимости предложенного алгоритма, а также существование изолированного решения нелокальной краевой задачи для нелинейного уравнения. Получены оценки между точным и приближенным решениями рассматриваемой задачи.

Ключевые слова: дифференциальные уравнения в частных производных, уравнение Бенджамина-Бона-Махони, алгоритм, приближенное решение.