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A fractionally loaded boundary value problem two-dimensional in the spatial variable

In the paper, the boundary value problem for the loaded heat equation is solved, and the loaded term is represented as the Riemann-Liouville derivative with respect to the time variable. The domain of the unknown function is the cone. The order of the derivative in the loaded term is less than 1, and the load moves along the lateral surface of the cone, that is in the domain of the desired function. The boundary value problem is studied in the case of the isotropy property in an angular coordinate (case of axial symmetry). The problem is reduced to the Volterra integral equation, which is solved by the method of the Laplace integral transformation. It is also shown by direct verification that the resulting function satisfies the boundary value problem.

Keywords: loaded boundary value problem, heat equation, isotropy, Volterra integral equation, Laplace transformation.

Introduction

It is known [1] that, as a rule, mathematical models of nonlocal physical and biological fractal processes are based on loaded differential equations with fractional order partial derivatives. In monograph [2], A.M. Nakhushev gave a detailed bibliography on loaded equations, including various applications of loaded equations as a method for studying problems in mathematical biology, mathematical physics, mathematical modeling of nonlocal processes and phenomena, and continuum mechanics with memory. In [3, 4], a boundary value problem for a fractionally loaded one-dimensional heat equation is considered. The load moves at a variable velocity. The conditions for the unique solvability of the boundary value problem are established depending on the order of the fractional derivative. In this paper, we study the solvability of a boundary value problem that is two-dimensional in the spatial variable. In [5, 6], a boundary value problem for the heat equation is considered in a cone in Lebesgue and Sobolev spaces. The BVP is reduced to a Volterra type integral equation of the second kind, and the method of successive approximations is not applicable to it [5]. This fact follows from the incompressibility property of the integral operator [7, 8]. As a result, nonzero solutions of the homogeneous equation arise [9, 10]. Singular integral operators defined in a bounded domain of the hodograph plane are considered in [11]. In this paper, we show the unique solvability of the reduced integral equation and the boundary value problem posed in a certain functional class.

The paper is organized as follows: in Section 1 we introduce some necessary definitions and mathematical preliminaries of fractional calculus which will be needed in the forthcoming Section. In Section 2, the statement of a fractionally loaded BVP of heat conduction is given. The loaded term is represented as a fractional Riemann-Liouville derivative with respect to the time variable. Since the boundary value problem is studied in the case of the isotropy property in the angular coordinate (when passing to polar coordinates), the problem statement for this case is also given. In Section 3,

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the BVP is equivalently reduced to the Volterra integral equation, namely, to the generalized Abel equation. Section 4 contains solving the integral equation (homogeneous and nonhomogeneous) using the Laplace transform method. Further, the solution of the BVP in the case of axial symmetry is obtained. Also in this Section it is shown that the obtained solution satisfies the BVP. Finally, Section 5 presents the main results of the paper, namely, theorems on the solvability of the integral equation and the boundary value problem posed in Section 2.

Note that in this paper the order of the derivative in the loaded term is less than the order of the differential part of the equation. In [12], the order of the derivative is greater than two, and the boundary value problem was reduced to an integro-differential equation, which led to the non-uniqueness of the problem's solution.

Summing up the above analysis of studies, we can say that boundary value problems for loaded differential equations are well-posed in a number of cases in natural classes of functions, i.e., in this case, the loaded term is interpreted as a weak perturbation. In the case of violation of the uniqueness of the solution to a boundary value problem, the loaded term can be considered as a strong perturbation [13–15]. Everywhere linear equations are considered. An interesting method for studying semilinear equations in the [16].

1 Preliminaries

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann–Liouville fractional derivative.

Definition 1 ([17]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_rD^{\beta}_{a,t}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}d\tau,\ \beta, a \in R,\ n-1<\beta<n. \tag{1} $$

From formula (1) it follows that

$$ {}_rD^{0}_{a,t}f(t) = f(t), \quad {}_rD^{n}_{a,t}f(t) = f^{(n)}(t), \quad n \in N. $$

We study a boundary value problem for the loaded heat equation, that is two-dimensional in the spatial coordinate when the loaded term is represented in the form of a fractional derivative. The considered problem is reduced to an integral equation by inverting the integral part.

It's known [18] the function

$$ G(r,\xi,t) = \frac{\xi}{2a^2t}\exp\left\{-\frac{r^2+\xi^2}{4a^2t}\right\} I_0\left(\frac{r\xi}{2a^2t}\right) $$

is a fundamental solution to the equation

$$ \frac{\partial w}{\partial t} = \frac{a^2}{r}\left(r\frac{\partial w}{\partial r}\right), $$

where

$$ I_\nu(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)}, \quad -\infty<\nu<\infty $$

is the modified Bessel function.

It's known ([18]; p. 76) that in the domain $\Omega_\infty = \{(r,t)\,\big|\, 0 \le r < +\infty;\ t>0\}$ the solution to the boundary value problem of heat conduction

$$ \frac{\partial w}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + F(r,t), $$

$$w|_{t=0} = w_0(r)$$

is defined by the formula

$$w(r,t) = \int_0^{+\infty} G(r,\xi,t)\, w_0(\xi)\, d\xi + \int_0^t \int_0^{+\infty} G(r,\xi,t-\tau)\, F(\xi,\tau)\, d\xi d\tau. \tag{2}$$

The Green function $G(x,\xi,t-\tau)$ satisfies the relation

$$\int_0^{+\infty} G(x,\xi,t-\tau)\, d\xi = 1. \tag{3}$$

Indeed,

$$\int_0^{+\infty} G(r,\xi,t)d\xi = \frac{1}{2a^2t}\int_0^{+\infty} \xi \exp\left(-\frac{r^2+\xi^2}{4a^2t}\right) I_0\left(\frac{r\xi}{2a^2t}\right) d\xi =$$
$$= \frac{1}{2a^2t}\exp\left(-\frac{r^2}{4a^2t}\right)\int_0^{+\infty} \xi \exp\left(-\frac{\xi^2}{4a^2t}\right) I_0\left(\frac{r\xi}{2a^2t}\right) d\xi.$$

From [19] (formula 2.15.5 (4) when $\alpha = 2;\ \nu = 0,\ c = \dfrac{r}{2a^2t};\ \rho = \dfrac{1}{4a^2t}$) we have

$$\int_0^{+\infty} G(r,\xi,t)d\xi = \frac{1}{2a^2t}\exp\left(-\frac{r^2}{4a^2t}\right) A_\nu^{\nu+2}.$$

Since $\nu = 0 => A_0^2 = A_\nu^{\nu+2}$. Then we get equality (3).

We assume that the right side of the BVP's equation vanishes at $t < 0$ and belongs to the class

$$\Phi(x,y;t) \in L_\infty(A) \cap C(B), \tag{4}$$

where $A = \{(x,y;t)\,|\,x>0,\ -\infty<y<+\infty,\ t\in[0,T]\}$, $B = \{(x,y;t)\,|\,x>0,\ -\infty<y<+\infty,\ t\ge 0\}$, $T - const > 0$.

The classes in which the problem is studied are determined from the natural requirement for the existence and convergence of improper integrals that arise in the study.

2 Problem setting

Problem 1. In a domain

$$G = \{(x,y;t)\big|\sqrt{x^2+y^2} \le t;\ t>0\} \tag{5}$$

we consider a boundary value problem, two-dimensional in the spatial variable for a fractionally loaded heat equation:

$$u_t = a^2\Delta u + \lambda\{{}_{RL}D_{0t}^{\beta} u(x,y;t)\}\Big|_{\sqrt{x^2+y^2}=t/2} + \Phi(x,y;t) \tag{6}$$

with the condition of solution's boundedness:

$$\lim_{\sqrt{x^2+y^2}\to+\infty} u(x,y;t) = 0, \tag{7}$$

and with the condition on the lateral surface of the cone:

$$u(x,y;t)\Big|_{\sqrt{x^2+y^2}=t} = g(t), \tag{8}$$

where $\Phi(x,y;t)$ is a given function belonging to the class (4), λ is a complex parameter, ${}_{RL}D^{\beta}_{0t}u(x,y;t)$ is the Riemann-Liouville derivative of the order β, $0<\beta<1$, i.e.

$$ {}_{RL}D^{\beta}_{0t}u(x,y;t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{u(x,y;\tau)}{(t-\tau)^{\beta}}d\tau. \tag{9}$$

Let's move on to polar coordinates:

$$x = r\,\cos\phi;\; y = r\,\sin\phi;\; 0\le\phi<2\pi;\; r\ge 0.$$

Since the problem (6)–(8) is considered in the case of the isotropy property in the angular coordinate ϕ (case of axial symmetry), we obtain the following problem.

Problem 2. In a domain $\Omega_{\infty} = \{(r,t)\mid r>0;\; t>0\}$ find a solution to the equation

$$\frac{\partial w}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial w(r,t)}{\partial r}\Big) + \lambda\left\{{}_{RL}D^{\beta}_{0t}w(r;t)\right\}\Big|_{r=\frac{t}{2}} + F(r,t), \tag{10}$$

that satisfies the conditions

$$\lim_{r\to\infty} w(r,t) = 0, \tag{11}$$

$$w(r,t)|_{r=t} = g(t). \tag{12}$$

Hear $w(r,t) = u(r\,\cos\phi;\; r\,\sin\phi;\; t)$ is unknown function, $F(r,t) = \Phi(r\,\cos\phi;\; r\,\sin\phi;\; t)$.

The temperature field is assumed to be axisymmetric, i.e., it is approximated by the functional dependence of the temperature only on the value of r. Note that due to the axisymmetric nature of the problem under consideration and the degeneracy of the definition domain (5) to a point at the initial time, conditions (8) and (12) implies the matching condition at the cone top $w|_{r=0} = w|_{t=0} = g(0)$.

Now we have the following boundary value problem.

Problem 3. In a domain $\Omega_{\infty} = \{(r,t)\mid r>0;\; t>0\}$ find a solution to the equation (10) that satisfies condition (11) and the initial condition

$$w(r,t)|_{t=0} = g(0). \tag{13}$$

3 Reducing the boundary value problem to an integral equation

We invert the differential part of problem (10), (11), (13) by formula (2):

$$w(r,t) = \int_0^{+\infty} G(r,\xi,t)g(0)\,d\xi + \lambda\int_0^t\int_0^{+\infty} G(r,\xi,t-\tau)\mu(\tau)\,d\xi d\tau + f(r,t), \tag{14}$$

where

$$\mu(t) = \left\{{}_{RL}D^{\beta}_{0t}w(r;t)\right\}\Big|_{r=\frac{t}{2}}, \tag{15}$$

$$f(r,t) = \int_0^t\int_0^{+\infty} G(\xi,r,t-\tau)\,F(\xi,\tau)\,d\xi d\tau. \tag{16}$$

Taking into account equality (3), representation (14) can be rewritten in the form

$$w(r,t) = g(0) + \lambda \int_0^t \mu(\tau)d\tau + f(r,t). \tag{17}$$

Applying to (17) the operator of fractional differentiation according to formula (9), substituting $r = \frac{t}{2}$ into the resulting expression, by virtue of notation (15) on the left in (17) we obtain the function $\mu(t)$.

Since

$$\Gamma(1-\beta)_{RL}D_{0t}^{\beta}\left\{\int_0^t \mu(\tau)d\tau\right\} = \frac{d}{dt}\int_0^t \frac{1}{(t-\tau)^\beta}\int_0^\tau \mu(\theta)d\theta d\tau = \frac{d}{dt}\int_0^t \mu(\theta)\int_\theta^t \frac{d\tau}{(t-\tau)^\beta}d\theta =$$
$$= \frac{d}{dt}\int_0^t \frac{\mu(\theta)(t-\theta)^{1-\beta}}{1-\beta}d\theta = \int_0^t \frac{\mu(\theta)}{(t-\theta)^\beta}\,d\theta$$

then from (17) after the above procedure we obtain an integral equation

$$\mu(t) = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta} + \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^\beta}d\tau + f_1(t), \quad 0<\beta<1,$$

where

$$f_1(t) = \left\{{}_{RL}D_{0t}^{\beta} f(r,t)\right\}\Big|_{r=\frac{t}{2}}. \tag{18}$$

Thus, problem (10), (11), (13) is reduced to solving the Volterra integral equation of the second kind, namely the generalized Abel equation:

$$\mu(t) - \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^\beta}d\tau = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta} + f_1(t), \quad 0<\beta<1, \tag{19}$$

where $f_1(t)$ is defined by formulas (18), (16).

4 Solving the integral equation

Solving the integral equation in the case of the homogeneous equation in BVP (6)–(8). Consider the corresponding problem for $\Phi(x,y,t) \equiv 0$ in equation (6), i.e. $F(r,t) = 0$ in equation (10). Then integral equation (10) will take the form:

$$\mu(t) - \frac{\lambda}{\Gamma(1-\beta)}\int_0^t \frac{\mu(\tau)}{(t-\tau)^\beta}d\tau = \frac{g(0)}{\Gamma(1-\beta)}t^{-\beta}, \tag{20}$$

where $f_1(t)$ is defined by formulas (18), (16).

Let $\Phi(s) = L[\mu(t)]$ be the Laplace image of the function $\mu(t)$. Applying the integral Laplace transform to equation (20) we obtain:

$$\Phi(s) - \frac{\lambda\Phi(s)}{s^{1-\beta}} = \frac{g(0)}{s^{1-\beta}}, \quad Re\, s > |\lambda|^{\frac{1}{1-\beta}}.$$

From here

$$\Phi(s) = \frac{g(0)}{s^{1-\beta}-\lambda}, \quad Re\, s > |\lambda|^{\frac{1}{1-\beta}}. \tag{21}$$

Applying the inverse Laplace transform, taking into account formula 1.80 [20]

$$L\left[t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})\right]=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}};\ Re\,s>|a|^{\frac{1}{\alpha}},$$

where $E_{a,b}(z)$ is the Mittag-Leffler function, i.e.

$$E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)},$$

from (21) we get

$$\mu(t)=g(0)t^{-\beta}E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right). \tag{22}$$

Due to the representation (17) of the solution to problem (10), (13) for $F(r,t)=0$ in the domain Ω_∞, taking into account (22) we get

$$w(r,t)=g(0)+\lambda g(0)\int_0^t\tau^{-\beta}E_{1-\beta;1-\beta}\left(\lambda\tau^{1-\beta}\right)d\tau.$$

Since [20] (formula 1.99)

$$\int_0^z E_{a,b}(\lambda t^a)t^{b-1}dt=z^bE_{a,b+1}(\lambda z^a);\ (b>0),$$

then

$$w(r,t)=g(0)+\lambda g(0)\,t^{-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right). \tag{23}$$

(23) is the solution to problem (10), (13) in the domain Ω_∞, since condition (12) takes the form (13). Thus, the solution to problem (6)–(8) for $\Phi(x,y,t)=0$ in the case of axial symmetry has the form:

$$u(x,y;t)=g(0)+\lambda g(0)\,t^{-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right), \tag{24}$$

where $0<\beta<1$.

Due to the formula

$$E_{a;\,b}(z)=z\,E_{a;\,a+b}(z)+\frac{1}{\Gamma(b)}$$

we have at $b=1$ and $z=\lambda t^{1-\beta}$

$$\lambda t^{1-\beta}E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right)=E_{1-\beta;1}\left(\lambda t^{1-\beta}\right)-1.$$

Then (24) will take the form:

$$u(x,y,t)=g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right), \tag{25}$$

since $E_{a,1}(z)=E_a(z)$.

It can be shown by direct verification that function (25) satisfies homogeneous equation (6) in the case of axial symmetry.

The case of BVP (6)–(8) at $\beta=1/2$ *when* $\Phi(x,y,t)=0$.

If $\beta = \dfrac{1}{2}$ in BVP (6)–(8) then expression (9) can be rewritten as

$$_{RL}D_{0t}^{\frac{1}{2}}u(x,y,t) = \frac{1}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{u(x,t,\tau)}{\sqrt{t-\tau}}d\tau.$$

Let

$$u\Big|_{t=0} = g(0),$$

where

$$g(t) = u(x,y,t)\Big|_{\sqrt{x^2+y^2}=t}$$

and $\Phi(x,y,t) = 0$.

Then the solution to BVP (6)–(8) has the form (see (24))

$$u(x,y;t) = g(0)E_{\frac{1}{2}}\left(\lambda\sqrt{t}\right),$$

when $\Phi(x,y,t) = 0$. Since [20] (formula 1.65)

$$E_{\frac{1}{2}}\left(\pm z^{\frac{1}{2}}\right) = e^z \operatorname{erfc}\left(\mp z^{\frac{1}{2}}\right),$$

then

$$u(x,y;t) = g(0)e^{\lambda^2 t}\operatorname{erfc}\left(-\lambda^2 t\right),$$

where

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$$

is the complementary error function.

Solving the integral equation (19). Consider now equation (19). Let $L[f_1(t)] = F_1(s)$. Then, in the space of Laplace images, equation (19) takes the form:

$$\Phi(s) - \frac{\lambda\Phi(s)}{s^{1-\beta}} = \frac{g(0)}{s^{1-\beta}} + F_1(s).$$

From hear

$$\Phi(s) = \frac{g(0)}{s^{1-\beta}} + F_1(s) + \lambda\frac{F_1(s)}{s^{1-\beta}-\lambda}.$$

Applying the inverse Laplace transform, we get:

$$\mu(t) = g(0)t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right) + f_1(t) + \lambda f_1(t)\, t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right). \tag{26}$$

Then, taking into account function (26), representation (17) has the form:

$$w(r,t) = g(0) + \lambda\int_0^t \left(g(0)\tau^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right)d\tau + f_1(\tau)\right)d\tau +$$
$$+\lambda^2\int_0^t\int_0^\tau f_1(\theta)(\tau-\theta)^{-\beta}E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right)d\theta d\tau + f(r,t) =$$
$$= g(0) + \lambda g(0)t^{1-\beta}E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) +$$
$$+\lambda\int_0^t f_1(\tau)d\tau + \lambda^2\int_0^t f_1(\theta)d\theta\int_\theta^t (\tau-\theta)^{-\beta}E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right)d\tau + f(r,t)$$

that is

$$w(r,t) = g(0) + \lambda g(0) t^{1-\beta} E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t f_1(\tau) d\tau + \lambda^2 \int_0^t f_1(\theta) I(\theta;t) d\theta + f(r,t), \quad (27)$$

where

$$I(\theta;t) = \int_\theta^t (\tau-\theta)^{-\beta} E_{1-\beta,1-\beta}\left(\lambda(\tau-\theta)^{1-\beta}\right) d\tau = (t-\theta)^{1-\beta} E_{1-\beta,2-\beta}\left(\lambda(t-\theta)^{1-\beta}\right).$$

Then function (27) can be rewritten as:

$$w(r,t) = g(0) + \lambda g(0) t^{1-\beta} E_{1-\beta,2-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t f_1(\tau) d\tau +$$
$$+ \lambda^2 \int_0^t (t-\tau)^{1-\beta} E_{1-\beta,2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + f(r,t). \quad (28)$$

Due to the formula

$$E_{a,\,b}(z) = z\, E_{a;\,a+b}(z) + \frac{1}{\Gamma(\beta)}$$

we have at $b=1$ and $z = \lambda t^{1-\beta}$

$$\lambda t^{1-\beta} E_{1-\beta;\,2-\beta}\left(\lambda t^{1-\beta}\right) = E_{1-\beta}\left(\lambda t^{1-\beta}\right) - 1.$$

Then function (28) takes the form:

$$w(r,t) = g(0) E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t E_{1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau) d\tau + f(r,t), \quad (29)$$

where $f_1(\tau)$ and $f(r,t)$ are defined by formulas (18) and (16), respectively. (29) is a solution to BVP (10), (11), (13).

So, in the case of axial symmetry in the domain G, the function

$$u(x,y,t) = g(0) E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t E_{1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + f\left(\sqrt{x^2+y^2}, t\right)$$

is a solution to BVP (6)-(8), where $f_1(\tau)$ and $f(r,t)$ are defined by formulas (18) and (16), respectively, and $F(r,t) = \Phi(r\cos\phi,\ r\sin\phi;\ t)$.

Checking that function (29) is a solution to BVP (10), (11), (13).

We first rewrite function (29) in the form (28). Since

$$\frac{d}{dt}\left[t^{1-\beta} E_{1-\beta;2-\beta}\left(\lambda t^{1-\beta}\right)\right] = t^{-\beta} E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right),$$

then

$$\frac{d}{dt}\int_0^t (t-\tau)^{1-\beta} E_{1-\beta;2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau =$$
$$= \int_0^t \frac{d}{dt}\left[(t-\tau)^{1-\beta} E_{1-\beta;2-\beta}\left(\lambda(t-\tau)^{1-\beta}\right)\right] f_1(\tau)\, d\tau =$$
$$= \int_0^t (t-\tau)^{-\beta} E_{1-\beta;1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau.$$

Then from (28) we have

$$\frac{\partial w}{\partial t} = \lambda g(0)t^{-\beta}E_{1-\beta;1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda f_1(t) + \\ + \lambda^2 \int_0^t (t-\tau)^{-\beta}E_{1-\beta;1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + \frac{\partial f(r,t)}{\partial t}. \quad (30)$$

$$\frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r,t)}{\partial t}\right). \quad (31)$$

By virtue of notation (15) and equality (26), we have

$${}_{RL}D_{0t}^{\beta}w(r,t)\Big|_{r=\frac{t}{2}} = \mu(t) = g(0)t^{-\beta}E_{1-\beta,1-\beta}\left(\lambda t^{1-\beta}\right) + f_1(t) + \\ + \lambda \int_0^t (t-\tau)^{-\beta}E_{1-\beta;1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau. \quad (32)$$

Substituting (30)-(32) into equation (10) we get:

$$\frac{\partial f(r,t)}{\partial t} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f(r,t)}{\partial r}\right) + F(r,t). \quad (33)$$

By notation (16), we have

$$\frac{\partial f(r,t)}{\partial t} = \frac{\partial}{\partial t}\int_0^t\int_0^{+\infty} G(\xi,r,t-\tau)F(\xi,\tau)\, d\xi d\tau = \\ = \int_0^t\int_0^{+\infty} \frac{\partial G(\xi,r,t-\tau)}{\partial t}F(\xi,\tau)d\xi d\tau + \int_0^{+\infty} G(\xi,r;0)F(\xi,0)\, d\xi;$$

$$\frac{a^2}{r}\frac{\partial\left(rf_r(r,t)\right)}{\partial r} = \frac{a^2}{r}\frac{\partial}{\partial r}\left(\int_0^t\int_0^{+\infty} rG(\xi,r,t-\tau)F(\xi,\tau)\, d\xi d\tau\right).$$

It is known [21] that

$$e^{-z}I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{z}\right)\right)$$

when $|\arg z| < \frac{\pi}{2}$ and $|z| \to \infty$. Then $\lim_{t\to 0} G(\xi,r,t) = 0$. Therefore, equality (33) takes the form:

$$\int_0^t\int_0^{+\infty} \frac{\partial G(\xi,r,t-\tau)}{\partial t}F(\xi,\tau)\, d\xi d\tau = \int_0^t\int_0^{+\infty} \frac{a^2}{r}\frac{\partial\left(rG(\xi,r,t-\tau)\right)}{\partial r}F(\xi,\tau)d\xi d\tau + F(r,t)$$

or

$$\int_0^t\int_0^{+\infty} \Big[\frac{\partial G(\xi,r,t-\tau)}{\partial t} - \frac{a^2}{r}\frac{\partial(rG(\xi,r,t-\tau))}{\partial r}\Big]F(\xi,\tau)d\xi d\tau = F(r,t). \quad (34)$$

Since $G(\xi,r,t)$ is the fundamental solution of the heat equation in polar coordinates, then

$$\frac{\partial G}{\partial t} - \frac{a^2}{r}\frac{\partial(rG)}{\partial r} = \delta(\xi - r)\delta(t),$$

where δ is the Dirac function. Then equality (34) takes the form:

$$\int_0^{+\infty} \delta(\xi - r)\delta(t) * F(\xi, t)d\xi = F(r, t)$$

or

$$\int_0^{+\infty} \delta(\xi - r)F(\xi, t)d\xi = F(r, t).$$

Hence, function (29) satisfies equation (10). Function (29) obviously satisfies condition (11) due to the choice of classes for $F(r, t)$. Let us now show that function (29) satisfies condition (13). We have

$$w(r, t)|_{t=0} = g(0) + \lim_{t\to 0} f(r, t) = g(0)$$

due to equality (16).

So, function (29) is a solution to BVP (10), (11), (13).

5 Main results

Theorem 1. Equation (19) is uniquely solvable in the class $\mu(t) \in C([0; T])$, for any function side $f_1(t) \in AC([0; T])$, and the solution to equation (19) is determined by formula (26).

Theorem 2. Let conditions (4) and $F(r, t) = \Phi(r\cos\phi,\ r\sin\phi;\ t) \in L_1(t \in [0; T])$ be satisfied for the function $\Phi(x, y; t)$, the function $\mu(t)$ is defined by formula (26). Then in the class $L_1(t \in [0; T])$ the boundary value problem (6)–(8) for the case of axial symmetry has a unique solution defined by formula

$$u(x, y, t) = g(0)E_{1-\beta}\left(\lambda t^{1-\beta}\right) + \lambda \int_0^t E_{1-\beta}\left(\lambda(t-\tau)^{1-\beta}\right) f_1(\tau)\, d\tau + f\left(\sqrt{x^2 + y^2}, t\right),$$

where $f_1(\tau)$ and $f(r, t)$ are defined by formulas (18) and (16), respectively.

Remark. Since equation (19) is a generalised Abel equation, its solution can be written as [22]

$$\mu(t) = f_1(t) + \int_0^t R(t-\tau)f_1(\tau)d\tau,$$

where

$$R(t-\tau) = \frac{d}{dt}\sum_{k=0}^{\infty} \frac{\left(\lambda(t-\tau)^{1-\beta}\right)^n}{\Gamma(1 + (1-\beta)n)}$$

or

$$R(t) = \frac{d}{dt}E_{1-\beta}\left(\lambda t^{1-\beta}\right).$$

After simple transformations, we get formula (26).

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Кеңістіктік айнымалыдағы екі өлшемді бөлшектік жүктемелі шеттік есеп

Жұмыста жүктемелі жылуөткізгіштік теңдеуі үшін шеттік есеп қарастырылды, жүктелген мүше уақыт айнымалысына қатысты Риман–Лиувилл туындысы ретінде берілген. Белгісіз функцияның анықталу облысы конус болып табылады. Жүктелген мүшедегі туындының реті 1-ден кіші, ал жүк конустың бүйір беті бойымен қозғалады және ізделінді функцияның анықталу облысына жатады. Шеттік есеп бұрыштық координаттағы изотропия қасиеті (осьтік симметрия жағдайы) жағдайында зерттелді. Есеп Вольтерра интегралдық теңдеуіне келтірілді және Лаплас интегралды түрлендіру әдісімен шешілді. Алынған функцияның шеттік есептерді қанағаттандыратыны тікелей тексеру арқылы көрсетілді.

Кілт сөздер: жүктелген шеттік есеп, жылуөткізгіштік теңдеуі, изотропия, Вольтерра интегралдық теңдеуі, Лаплас түрлендіруі.

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Дробно-нагруженная краевая задача, двумерная по пространственной переменной

В работе найдено решение краевой задачи для нагруженного уравнения теплопроводности, в котором нагруженное слагаемое представлено в виде производной Римана–Лиувилля по временной переменной. Область определения неизвестной функции — конус. Порядок производной в нагруженном члене меньше 1, и нагрузка движется по боковой поверхности конуса, который находится в области определения искомой функции. Краевая задача исследована в случае свойства изотропности по угловой координате (случай осевой симметрии). Задача сведена к интегральному уравнению Вольтерра, которое решается методом интегрального преобразования Лапласа. Непосредственной проверкой также показано, что полученная функция удовлетворяет поставленной задаче.

Ключевые слова: нагруженная краевая задача, уравнение теплопроводности, изотропность, интегральное уравнение Вольтерра, преобразование Лапласа.