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On quasi-identities of finite modular lattices. II

The existence of a finite identity basis for any finite lattice was established by R. McKenzie in 1970, but the analogous statement for quasi-identities is incorrect. So, there is a finite lattice that does not have a finite quasi-identity basis and, V.A. Gorbunov and D.M. Smirnov asked which finite lattices have finite quasi-identity bases. In 1984 V.I. Tumanov conjectured that a proper quasivariety generated by a finite modular lattice is not finitely based. He also found two conditions for quasivarieties which provide this conjecture. We construct a finite modular lattice that does not satisfy Tumanov's conditions but quasivariety generated by this lattice is not finitely based.

Keywords: lattice, finite lattice, modular lattice, quasivariety, variety, quasi-identity, identity, finite basis of quasi-identities, Tumanov's conditions.

Introduction

In 1970 R. McKenzie [1] proved that any finite lattice has a finite basis of identities. However the similar result for quasi-identities is not true. That is, there is a finite lattice that has no finite basis of quasi-identities (V.P. Belkin [2]). The problem "Which finite lattices have finite basis of quasi-identities?" was suggested by V.A. Gorbunov and D.M. Smirnov in [3]. V.I. Tumanov [4] found a sufficient condition consisting of two parts under which a locally finite quasivariety of lattices has no finite (independent) basis of quasi-identities. He also conjectured that a finite (modular) lattice has a finite (independent) basis of quasi-identities if and only if a quasivariety generated by this lattice is a variety. In general, the conjecture is not true. W. Dziobiak [5] found a finite lattice that generates finitely axiomatizable proper quasivariety. We also would like to point out that Tumanov's problem is still unsolved for modular lattices.

The main goal of the paper is to present a finite modular lattice that has no finite basis of quasi-identities and does not satisfy conditions of Tumanov's theorem.

1 Basic concepts and preliminaries

We recall some basic definitions and results for quasivarieties that we will refer to. For more information on the basic notions of universal algebra and lattice theory introduced below and used throughout this paper, we refer to [6] and [7].

A *quasivariety* is a class of algebras that is closed with respect to subalgebras, direct products, and ultraproducts. Equivalently, a quasivariety is the same thing as a class of algebras axiomatized by a set of quasi-identities. A *quasi-identity* means a universal Horn sentence with the non-empty positive part, that is of the form

$$(\forall \bar{x})[p_1(\bar{x}) \approx q_1(\bar{x}) \wedge \cdots \wedge p_n(\bar{x}) \approx q_n(\bar{x}) \to p(\bar{x}) \approx q(\bar{x})],$$

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where $p, q, p_1, q_1, \dots, p_n, q_n$ are terms. A *variety* is a quasivariety which is closed under homomorphisms. According to Birkhoff's theorem [8], a variety is a class of similar algebras axiomatized by a set of identities, where by an identity we mean a sentence of the form $(\forall\bar{x})[s(\bar{x}) \approx t(\bar{x})]$ for some terms $s(\bar{x})$ and $t(\bar{x})$.

By $\mathbf{Q}(\mathbf{K})$ ($\mathbf{V}(\mathbf{K})$) we denote the smallest quasivariety (variety) containing a class $\mathbf{K}$. If $\mathbf{K}$ is a finite family of finite algebras then $\mathbf{Q}(\mathbf{K})$ is called finitely generated. In case when $\mathbf{K} = \{A\}$ we write $\mathbf{Q}(A)$ instead of $\mathbf{Q}(\{A\})$. By Maltsev-Vaught theorem [9], $\mathbf{Q}(\mathbf{K}) = \mathbf{SPP}_u(\mathbf{K})$, where $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_u$ are operators of taking subalgebras, direct products and ultraproducts, respectively.

Let $\mathbf{K}$ be a quasivariety. A congruence α on an algebra A is called a $\mathbf{K}$*-congruence* or *relative congruence* provided $A/\alpha \in \mathbf{K}$. The set $\mathrm{Con}_{\mathbf{K}}A$ of all $\mathbf{K}$-congruences of A forms an algebraic lattice with respect to inclusion $\subseteq$ which is called a *relative congruence lattice*.

For a quasivariety $\mathbf{K}$, an algebra $A \in \mathbf{K}$ is said to be *subdirectly* $\mathbf{K}$*-irreducible* if the least congruence 0_A is completely meet irreducible in $\mathrm{Con}_{\mathbf{K}}A$. By Birkhoff's theorem for a quasivariety, every algebra of a quasivariety $\mathbf{K}$ is a subdirect product of subdirectly $\mathbf{K}$-irreducible algebras ([7,8]). By $\mathbf{K}_{SI}$ we denote the class of all subdirectly $\mathbf{K}$-irreducible algebras in $\mathbf{K}$. Since $\mathbf{Q}(\mathbf{K}) = \mathbf{SPP}_u(\mathbf{K}) = \mathbf{P}_s\mathbf{SP}_u(\mathbf{K})$, where $\mathbf{P}_s$ is operator of taking subdirect products, we have $\mathbf{K}_{SI} \subseteq \mathbf{SP}_u(\mathbf{K})$. Thus, for finitely generated quasivariety $\mathbf{Q}(A)$, every subdirectly $\mathbf{Q}(A)$-irreducible algebra is isomorphic to some subalgebra of A.

The least $\mathbf{K}$-congruence $\theta_{\mathbf{K}}(a,b)$ on an algebra $A \in \mathbf{K}$ containing pair $(a,b) \in A \times A$ is called a *principal* $\mathbf{K}$*-congruence* or a *relative principal congruence*. In case when $\mathbf{K}$ is a variety, relative congruence $\theta_{\mathbf{K}}(a,b)$ is a usual principal congruence that we denote by $\theta(a,b)$.

Let $(a] = \{x \in L \mid x \leq a\}$ ($[a) = \{x \in L \mid x \geq a\}$) be a principal ideal (coideal) of a lattice L. A pair $(a,b) \in L \times L$ is called *dividing* (*semi-dividing*) if $L = (a] \cup [b)$ and $(a] \cap [b) = \varnothing$ ($L = (a] \cup [b)$ and $(a] \cap [b) \neq \varnothing$).

For any semi-dividing pair (a,b) of a lattice M we define a lattice

$$M_{a-b} = \langle\{(x,0),(y,1) \in M \times 2 \mid x \in (a], y \in [b)\}; \vee, \wedge\rangle \leq_s M \times \mathbf{2},$$

where $\mathbf{2} = \langle\{0,1\}; \vee, \wedge\rangle$ is a two element lattice.

Theorem 1. (Tumanov's theorem [4])

Let $\mathbf{M}$, $\mathbf{N}$ ($\mathbf{N} \subset \mathbf{M}$) be locally finite quasivarieties of lattices satisfying the following conditions:

a) in any finitely subdirectly $\mathbf{M}$-irreducible lattice $M \in \mathbf{M}\backslash\mathbf{N}$ there is a semi-dividing pair (a,b) such that $M_{a-b} \in \mathbf{N}$;

b) there exists a finite simple lattice $P \in \mathbf{N}$ which is not a proper homomorphic image of any subdirectly $\mathbf{N}$-irreducible lattice.

Then the quasivariety $\mathbf{N}$ has no coverings in the lattice of subquasivarieties of $\mathbf{M}$. In particular, $\mathbf{N}$ has no finite (independent) basis of quasi-identities provided $\mathbf{M}$ is finitely axiomatizable.

A subalgebra B of an algebra A is called *proper* if $B \ncong A$. We will use the following folklore criterion of non-finite axiomatizability of quasivarieties (see [7]).

Lemma 1. A locally finite quasivariety $\mathbf{K}$ is not finitely axiomatizable if for any positive integer n there is a finite algebra L_n such that $L_n \notin \mathbf{K}$ and every proper subalgebra of L_n belongs to $\mathbf{K}$.

2 *Main result*

In this chapter we show that there are two locally finite quasivarieties of modular lattices $\mathbf{N}$ and $\mathbf{M}$, $\mathbf{N} \subset \mathbf{M}$, that do not satisfy conditions *a*) and *b*) of Tumanov's theorem, however, $\mathbf{N}$ is not finitely axiomatizable. Note that in our example we do not need to require that $\mathbf{M}$ be finitely axiomatizable. We also note that the first example of a finite lattice that does not satisfy the condition *b*) and has no finite basis of quasi-identities was provided in [10].

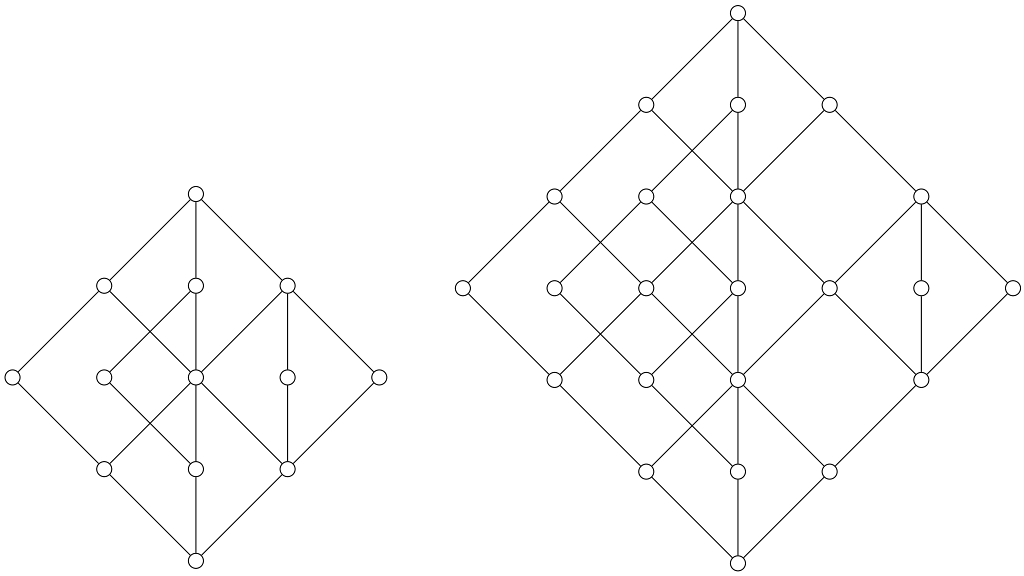

Figure 1. Lattices A' and A

Let A' and A are the modular lattices displayed in Figure 1. And let $\mathbf{Q}(A)$ and $\mathbf{V}(A)$ are quasivariety and variety generated by A, respectively. Since every subdirectly $\mathbf{Q}(A)$-irreducible lattice is a sublattice of A, and A' is simple, a homomorphic image of A and is not a sublattice of A we have $pA' \in \mathbf{V}(A) \backslash \mathbf{Q}(A)$, that is $\mathbf{Q}(A)$ is a proper quasivariety. One can check that A' has no semi-dividing pair. Thus, the condition $a)$ of Tumanov's theorem does not hold on the quasivariety $\mathbf{Q}(A)$. It is easy to see that M_3 is a unique simple lattice in $\mathbf{Q}(A)_{SI}$ and it is a homomorphic image of A. Hence, the condition $b)$ of Tumanov's theorem is not valid for quasivarieties $\mathbf{Q}(A)$ and $\mathbf{V}(A)$. Thus, to establish our main result we have to prove.

Theorem 2. Quasivariety $\mathbf{Q}(A)$ generated by the lattice A is not finitely based.

Proof. To prove the theorem we modify the proof of the second part of Theorem 3.4 from [11] (also see [10]).

According to Lemma 1 we will construct an infinite set $\{L_n \mid n \geq 0\}$ of finite modular lattices such that $L_n \in \mathbf{V}(A) \backslash \mathbf{Q}(A)$ and every n-generated subalgebra of L_n belongs to $\mathbf{Q}(A)$.

Let S be a non-empty subset of a lattice L. Denote by $\langle S \rangle$ the sublattice of L generated by S.

We define a modular lattice L_n by induction:

$n = 0$. $L_0 \cong M_{3-3}$ and $L_0 = \langle \{a_0, b_0, c_0, a^0, b^0, c^0\} \rangle$ (see Fig. 2).

$n = 1$. L_1 is a modular lattice generated by $L_0 \cup \{a_1, b_1, c_1, a^1, b^1, c^1\}$ such that $\langle \{a_1, b_1, c_1, a^1, b^1, c^1\} \rangle \cong M_{3-3}$, and $c_0 = a^1$, $a^0 \wedge b^0 = c_0 \vee b^1 = c_0 \vee c_1$ (see Fig. 3).

$n > 1$. L_n is a modular lattice generated by the set $L_{n-1} \cup \{a_n, b_n, c_n, a^n, b^n, c^n\}$ such that $\langle \{a_n, b_n, c_n, a^n, b^n, c^n\} \rangle \cong M_{3-3}$, and $c_{n-1} = a^n$, $a^0 \wedge b^0 = c_0 \vee b^n = c_0 \vee c_n$ (see Fig. 4).

Claim 1. For any $n > 0$, the lattice L_n does not belong to $\mathbf{Q}(A)$.

Proof. We prove by induction on $n > 0$.

$n = 1$. Assume that $L_1 \in \mathbf{Q}(A)$. At first we note that $M_{3,3}$ is a sublattice of $L_1/\theta(a_1, b_1)$ and $L_1/\theta(a_1, a^1 \wedge b^1)$. Hence, $(a_0, b_0) \in \theta_{\mathbf{Q}(A)}(a_1, b_1) \cap \theta_{\mathbf{Q}(A)}(a_1, a^1 \wedge b^1)$. One can also see that any non-trivial congruence contains (a_1, b_1) or $(a_1, a^1 \wedge b^1)$ or (a_0, b_0). Therefore, intersection of any two different non-trivial $\mathbf{Q}(A)$-congruences contains (a_0, b_0). It means that L_1 is subdirectly $\mathbf{Q}(A)$-irreducible. In this event, L_1 is a sublattice of A because $\mathbf{Q}(A)_{SI} \subseteq \mathbf{S}(A)$. One can check that L_1 is not a sublattice of A. Thus, L_1 does not belong to $\mathbf{Q}(A)$. Also we have $(a_0, b_0) \in \theta$ for any non-trivial $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_1$.

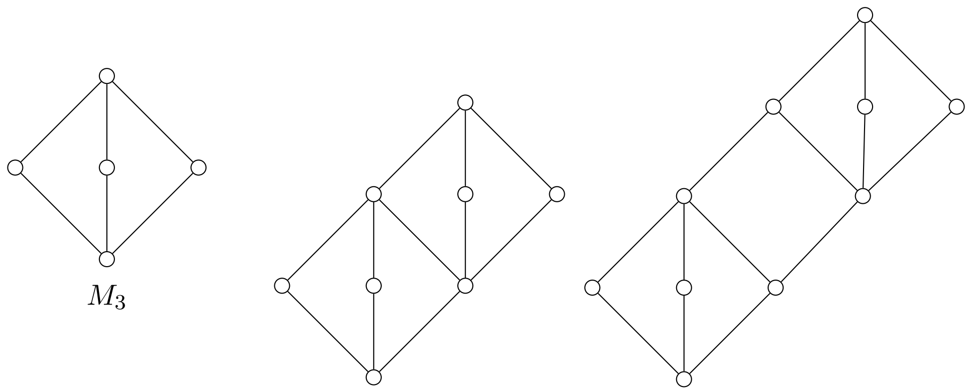

Figure 2. Lattices M_3, $M_{3,3}$ and M_{3-3}

$n > 1$. By induction, we have $L_{n-1} \notin \mathbf{Q}(A)$ and $(a_0, b_0) \in \theta$ for any non-trivial $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_{n-1}$.

Assume that $L_n \in \mathbf{Q}(A)$. We note that $M_{3,3}$ is a sublattice of $L_n/\theta(a_i, a^i \wedge b^i)$, L_{n-1} is a sublattice of $L_n/\theta(a_i, b_i) \leq_s L_{n-1} \times \mathbf{2}$ and $L_n/\theta(a_i, a^i) \cong L_{n-1}$, for all $0 < i \leq n$. Hence, $(a_i, a^i) \in \theta_{\mathbf{Q}(A)}(a_i, a^i \wedge b^i)$. It means that any homomorphic image of L_n that belongs to $\mathbf{Q}(A)$ is a homomorphic image of L_{n-1} or some $S \leq_s L_{n-1} \times \mathbf{2}$ or $L_n^- \cong L_n/\theta(a_0, b_0)$.

Let $\theta \in \mathrm{Con}_{\mathbf{Q}(A)} L_n$. If $(a_i, a^i) \in \theta$ then $\theta/\theta(a_i, a^i) \in \mathrm{Con}_{\mathbf{Q}(A)} L_n/\theta(a_i, a^i) \cong \mathrm{Con}_{\mathbf{Q}(A)} L_{n-1}$. By induction, $(a_0/\theta(a_i, a^i), b_0/\theta(a_i, a^i)) \in \theta/\theta(a_i, a^i)$. Since the congruence classes $a_0\theta(a_i, a^i)$ and $b_0\theta(a_i, a^i))$ consist of one elements $\{a_0\}$ and $\{b_0\}$, respectively, we get $(a_0, b_0) \in \theta$.

If $(a_i, b_i) \in \theta$ then $L_n/\theta(a_i, b_i) \leq_s L_{n-1} \times \mathbf{2}$. Since $L_{n-1} \leq L_n/\theta(a_i, b_i)$ then $L_{n-1}/(\theta \cap L_{n-1}^2) \in \mathbf{Q}(A)$. By induction, $(a_0/(\theta \cap L_{n-1}^2), b_0/(\theta \cap L_{n-1}^2)) \in (\theta \cap L_{n-1}^2)$. By argument above, $(a_0, b_0) \in \theta$.

Thus, we have that $(a_0, b_0) \in \theta$ for each non-trivial $\mathbf{Q}(A)$-congruence θ. It means that L_n is relative subdirectly irreducible. Hence, $L_n \leq A$. Contradiction. Therefore, $L_n \notin \mathbf{Q}(A)$.

Let L_n^- be a sublattice of L_n generated by the set $\{a_i, b_i, c_i, a^i, b^i, c^i \mid 0 < i \leq n\}$. One can see that $L_n^- \cong L_n/\theta(a_0, b_0)$ and $L_n^- \leq_s M_{3-3}^n$. Hence, $L_n^- \in \mathbf{Q}(A)$.

Claim 2. Every proper sublattice of L_n belongs to $\mathbf{Q}(A)$.

Proof. It is enough to prove the claim for arbitrary maximal proper sublattices of L_n. Since L_n is generated by the set of double irreducible elements $S = \{a_0, b_0, b^0, c^0, c_n\} \cup \{b_i, b^i \mid 0 < i \leq n\}$ then every maximal proper sublattices L of L_n generated by $S - \{x\}$ for some $x \in S$, that is $L = \langle S - \{x\}\rangle$.

Suppose that $x \in \{a_0, b_0, b^0, c^0\}$. Then the lattice $\langle\{a_0, b_0, c_0, a^0, b^0, c^0\} - \{x\}\rangle/\theta(c_0, a^0 \wedge b^0)$ is a homomorphic image of L with the kernel $\alpha = \theta(a_1, c_n)$ and belongs to $\mathbf{Q}(A)$.

One can see that for $\beta = \theta(a_0, b_0)$ if $x \in \{b^0, c^0\}$ and $\beta = \theta(b^0, c^0)$ if $x \in \{a_0, b_0\}$, L/β is isomorphic to L_n^- or $L_n^- \times \mathbf{2}$ and both these lattices belong to $\mathbf{Q}(A)$. Thus, α and β are $\mathbf{Q}(A)$-congruences. One can check that $\alpha \cap \beta = 0$. Hence $L \leq_s L/\alpha \times L/\beta$. Therefore, $L \in \mathbf{Q}(A)$.

Suppose that $x \in \{b_i, b^i \mid 0 < i \leq n\} \cup \{c_n\}$. For sake of brevity, we assume that $x = b_n$. Let $\alpha = \theta(c_0, c_{n-1})$. Then L/α is isomorphic to the sublattice S of L_1 generated by the set $\{a_0, b_0, c_0, a^0, b^0, c^0, a_1, b_1, b^1\}$. Since the lattice $P = \langle\{a_0, b_0, c_0, a^0, b^0, c^0, b^1, c^1\}\rangle$ is a sublattice of A and $S \leq_s P \times \mathbf{2}^2$ we get $S \in \mathbf{Q}(A)$. On the other hand, $L/\theta(a_0, b_0)$ is a sublattice of L_n^-. Since $L_n^- \in \mathbf{Q}(A)$ then $L/\theta(a_0, b_0) \in \mathbf{Q}(A)$. One can see that $\alpha \cap \theta(a_0, b_0) = 0$. Hence, L is a subdirect product of two lattices from $\mathbf{Q}(A)$. Therefore, $L \in \mathbf{Q}(A)$.

Thus, we obtain that $L_n \notin \mathbf{Q}(A)$ and every its proper sublattice belongs to $\mathbf{Q}(A)$. By Lemma 1, the quasivariety generated by the lattice A is not finitely based.

From the proof of Theorem 2 we have more general result:

Theorem 3. Suppose that $\mathbf{K}$ is a locally finite quasivariety and

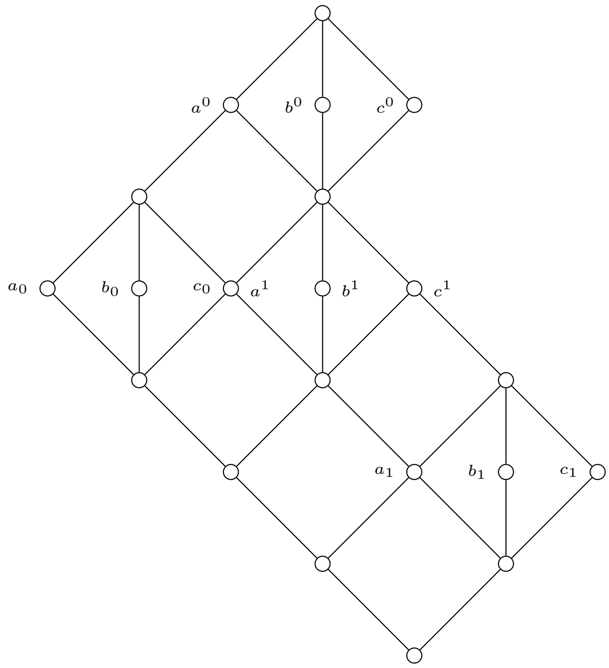

Figure 3. Lattice L_1

a) $M_{3,3} \notin \mathbf{K}$,
b) every proper sublattice of L_n belongs to $\mathbf{K}$,
c) $L_n \notin \mathbf{K}$ for all $n > 0$.
Then the quasivariety $\mathbf{K}$ is not finitely axiomatizable.

Corollary 1. There is an infinite number of finite lattices which do not satisfy conditions of Tumanov's theorem and have no finite basis of quasi-identities.

Indeed, the lattice A completed by n atoms $e_1, \ldots, e_n$ such that $e_i \vee e_j = a_0 \vee b_0$, $i \neq j \leq n$, satisfies the conditions of Theorem 2.

We note that the variety lattice of a variety $\mathbf{V}(A)$ is finite because it contains a finite number of subdirectly irreducible lattices by Johnson's Lemma [12]. G. Grätzer and H. Lasker [13] shown that the quasivariety lattice of a variety $\mathbf{V}(M_{3,3})$ is continuum. Since $M_{3,3} \in \mathbf{V}(A)$ we have that the quasivariety lattice of $\mathbf{V}(A)$ is continuum.

We would like to point out that V.I. Tumanov also provided that in his theorem the quasivariety $\mathbf{N}$ has no independent basis. Our proof does not allow to prove that $\mathbf{Q}(A)$ has no independent basis of quasi-identities. On the other hand, our proof holds on $\mathbf{K}$ that is not necessarily included in the finitely axiomatizable locally finite quasivariety.

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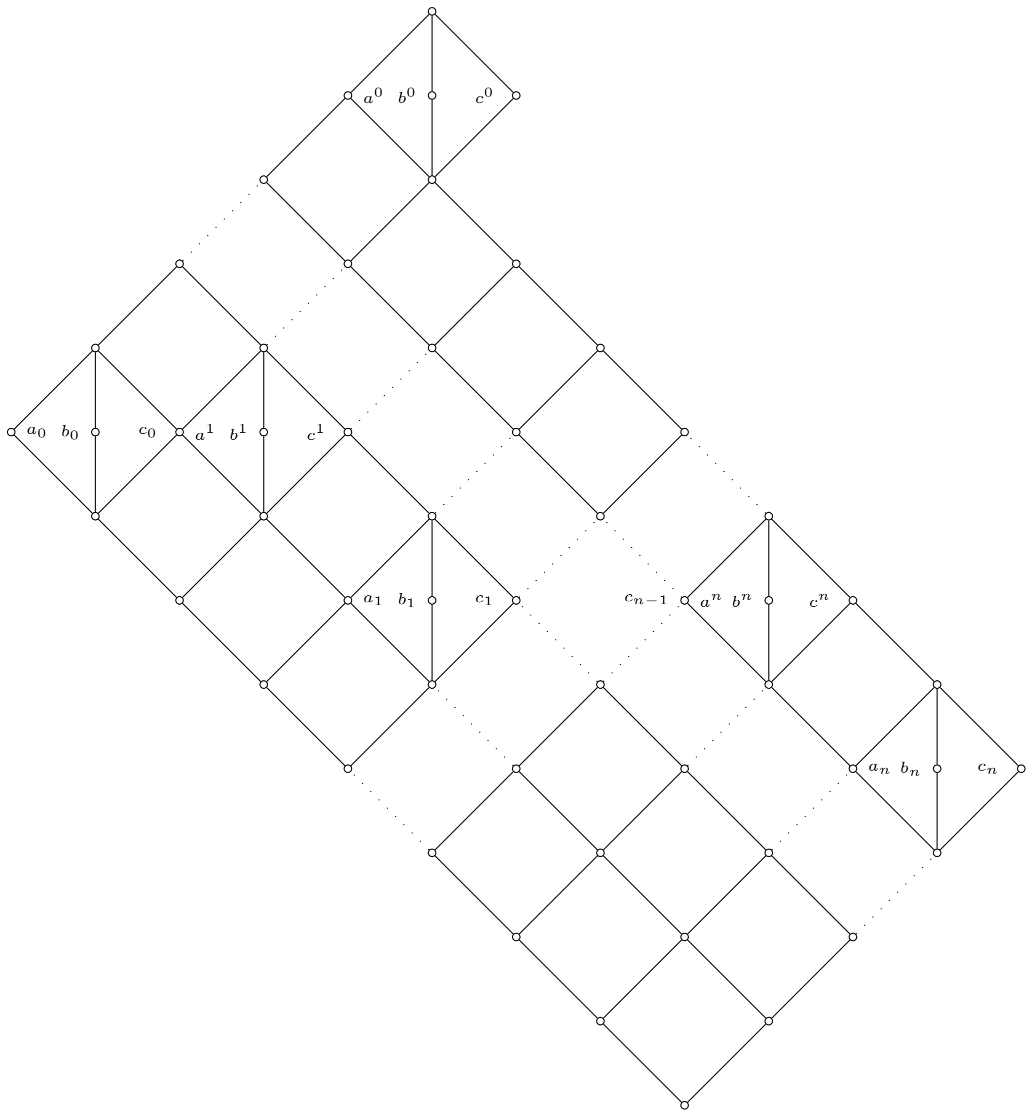

Figure 4. Lattice L_n, $n \geq 2$

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Соңғы модулярлық торлардың квазитепе-теңдіктері туралы. II

1970 жылы Р. Маккензи кез келген шекті тордың ақырлы базисті тепе-теңдіктері болатынын дәлелдеді. Дегенмен, квазитепе-теңдіктерге қатысты бұл мәлімдеме дұрыс емес. Сонымен, ақырлы базисі жоқ квазитепе-теңдіктерден шекті торлар бар. В.А. Горбунов пен Д.М. Смирнов келесі мәселені қозғады: «Ақырлы базисі бар квазитепе-теңдіктерден тұратын қандай шекті торлар бар?». 1984 жылы В.И. Туманов шекті модулярлы тордан туындаған өздік квазикөпбейненің ақырлы базисі жоқ деген болжам айтты. Ол сондай-ақ осы болжамды қамтамасыз ететін квазикөпбейнелердің екі шартын тапты. Ал біз Тумановтың шарттарын қанағаттандырмайтын шекті модулярлы торды құрастырдық, бірақ бұл тордан туындаған квазикөпбейненің ақырлы базисі жоқ.

Кілт сөздер: тор, соңғы тор, модулярлық тор, квазикөпбейне, көпбейне, квазитепе-теңдік, тепе-теңдік, квазитепе-теңдіктің соңғы базисі, Тумановтың шарттары.

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О квазитождествах конечных модулярных решеток. II

В 1970 г. Р. Маккензи доказал, что любая конечная решетка имеет конечный базис тождеств. Однако аналогичное утверждение для квазитождеств неверно. Итак, существуют конечные решетки, которые не имеют конечного базиса квазитождеств. В.А. Горбунов и Д.М. Смирнов озвучили следующую проблему: «Какие конечные решетки имеют конечный базис квазитождеств?» В 1984 г.

В.И. Туманов предположил, что собственное квазимногообразие, порожденное конечной модулярной решеткой, не является конечно базируемым. Он также нашел два условия для квазимногообразий, которые подтверждают эту гипотезу. Мы же построили конечную модулярную решетку, которая не удовлетворяет условиям Туманова, но квазимногообразие, порожденное этой решеткой, не является конечно базируемым.

Ключевые слова: решетка, конечная решетка, модулярная решетка, квазимногообразие, многообразие, квазитождество, тождество, конечный базис квазитождеств, условия Туманова.

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