DOI 10.31489/2023M1/168-176

UDC 519.62

M. Yücel[1,*], F.S. Muhtarov[2], O.Sh. Mukhtarov[2,3]

[1] *Hitit University, Çorum, Turkey;*
[2] *Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences, Baku, Azerbaijan;*
[3] *Tokat Gaziosmanpaşa University, Tokat, Turkey*
(E-mail: merve.yucel@outlook.com.tr, omukhtarov@yahoo.com, fahreddinmuhtarov@gmail.com)

Generalized differential transformation method for solving two-interval Weber equation subject to transmission conditions

The main goal of this study is to adapt the classical differential transformation method to solve new types of boundary value problems. The advantage of this method lies in its simplicity, since there is no need for discretization, perturbation or linearization of the differential equation being solved. It is an efficient technique for obtaining series solution for both linear and nonlinear differential equations and differs from the classical Taylor's series method, which requires the calculation of the values of higher derivatives of given function. It is known that the differential transformation method is designed for solving single interval problems and it is not clear how to apply it to many-interval problems. In this paper we have adapted the classical differential transformation method for solving boundary value problems for two-interval differential equations. To substantiate the proposed new technique, a boundary value problem was solved for the Weber equation given on two non-intersecting segments with a common end, on which the left and right solutions were connected by two additional transmission conditions.

Keywords: two-interval problems, the differential transformation method, Weber equation, transmission conditions.

Introduction

It is well known that two-dimensional elliptic equations often occur as a mathematical model of steady-state or equilibrium problems. For example, for a stationary flow of an incompressible inviscid fluid, the velocity potential satisfies the two-dimensional elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

the so-called Laplace's equation. Separation of variables method applied to the Laplace equation in parabolic coordinates leads to the Weber equation

$$y'' + (n + \frac{1}{2} - \frac{x^2}{4})y = 0,$$

where n is a constant. This equation was first studied by H. Weber in connection with the parabolic cylinder in the potential theory [1]. The Weber equation converts to the equation

$$u'' - xu' + nu = 0 \tag{1}$$

via the substitution $y = ue^{\frac{-x^2}{4}}$. Note that the solutions of the Weber equation are known as Weber-Hermite functions or parabolic cylinder functions. In the case when the constant n is a non-negative integer, the Weber equation (1) has the solution

$$u = e^{\frac{-x^2}{4}} H_n(x),$$

[*]Corresponding author.
E-mail: merve.yucel@outlook.com.tr

where $H_n(x)$ is the Hermite polynomial defined by the equality

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

In recent years, there has been increased interest in boundary value problems for many-interval differential equations with additional transmission conditions. [2–6]. Such type of transmission problems are motivated by the emergence of new and interesting applications in physics.

In this article, the Weber equation given on two non-intersecting intervals and satisfying supplementary transmission conditions between left and right solutions, will be solved by the differential transformation method (DTM, for short). The main idea of this method was first proposed by Zhou in connection with some problems of electrical circuits [7]. Using differential transform, the given differential equation and related initial and/or boundary conditions can be replaced by linear algebraic equations. Therefore this method is of great interest in physics, engineering and other natural sciences ([8–14]). For example, Sepasgozar et al. used DTM to solve the momentum and the heat transfer problems of non-Newtonian fluid flow in an axis-symmetric channel with porous wall [15]. Usman et al. applied differential transformation technique to investigate unsteady two phases on non-fluid flow and the heat transfer between moving parallel plates in the presence of the magnetic field [16].

In recent years, various modifications of the DTM have been used to solve many interesting problems that arise not only in theoretical mathematics, but also in applied sciences (see, for example [17–20] and references cited therein)

1 *Differential transformation and Differential inverse transformation*

Let $f = f(x)$ be an infinitely differentiable function on the real axis $R = (-\infty, \infty)$ and let $x_0 \in R$ be any point. Denote by $Y_{x_0}(f, n), \quad n = 0, 1, 2, ...$ the coefficient at the n. term of the Taylor series of the function f in the neighborhood of the point x_0, that is $Y_{x_0}(f, n) := \frac{1}{n!} f^{(n)}(x_0)$.

Definition 1. The sequence $Y_{x_0}(f) := (Y_{x_0}(f, 1), Y_{x_0}(f, 2), ...)$ is said to be differential transformation of the function f at the point x_0.

Definition 2. Let $A := (a_n)$ be any sequence, such that the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is convergent on the whole R. Then the function

$$Y_{x_0}^{-1}(A, x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is said to be the differential inverse transformation of the sequence $A := (a_n)$ at the point $x = x_0$.

It is obvious that any analytic function $f(x)$ satisfies the following equality

$$Y_{x_0}^{-1}(Y_{x_0}(f), x) = f(x).$$

Let $C^\infty(R)$ be the set of all infinitely differentiable functions defined on the real axis R. It is easy to verify that the following properties are valid

(i) $Y_{x_0}(f + g, n) = Y_{x_0}(f, n) + Y_{x_0}(g, n), \quad f, g \in C^\infty(R), \quad n = 0, 1, 2, ...;$
(ii) $Y_{x_0}(\lambda f, n) = \lambda Y_{x_0}(f, n), \quad \lambda \in R, \quad f \in C^\infty(R);$
(iii) $Y_{x_0}(\frac{d^s f}{dx^s}, n) = \frac{(s+n)!}{n!} Y_{x_0}(f, s + n), \quad s, n = 0, 1, 2, ..., \quad f \in C^\infty(R);$
(iv) $Y_{x_0}(fg, n) = \sum_{k=0}^{n} Y_{x_0}(f, k) Y_{x_0}(g, n-k)$, that is $Y_{x_0}(fg) = Y_{x_0}(f) * Y_{x_0}(g)$, where $Y_{x_0}(f) * Y_{x_0}(g)$ denotes the convolution of the sequences $Y_{x_0}(f)$ and $Y_{x_0}(g)$.

Remark 1. Let $A = (a_n)$ be any real sequence. If we denote the sequence $(a_1, a_2, ..., a_k, 0, 0, 0...)$. by A^k, then we have

$$\lim_{k\to\infty} Y_{x_0}^{-1}((Y_{x_0}(f))^k, x) = f(x)$$

provided that $f(x)$ is an analytic function on the real axis R. Hence for sufficiently large k we can take

$$f(x) \approx Y_{x_0}^{-1}((Y_{x_0}(f))^k, x)$$

in real applications.

2 *Solution of two-interval Weber equation using the modified differential transformation technique*

Example 1. Consider the two-interval Weber equation

$$y''(x) - xy'(x) + 2y(x) = 0, \quad x \in [1, \frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{2}$$

subject to the boundary conditions, given by

$$y(0) = 0, \quad y(1) = 1$$

and additional transmission conditions at the interior singular point $x = \frac{1}{2}$, given by

$$y\left(\frac{1}{2} - 0\right) - \gamma_1 y\left(\frac{1}{2} + 0\right) = 0, \tag{3}$$

$$y'\left(\frac{1}{2} - 0\right) - \gamma_2 y'\left(\frac{1}{2} + 0\right) = 0, \tag{4}$$

where γ_1 and γ_2 are real numbers that will be specified later. We will consider the equation (2) on the left side $[0, \frac{1}{2})$ and the right side $(\frac{1}{2}, 1]$ of the domain $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, separately.

We will denote by $Y_0(y*, k)$ and $Y_1(y**, k)$ the differential transformation of $y(x)$ at the left end-point $x = 0$ and the right end-point $x = 1$, respectively. Applying the differential transformation to the differential equation (2) in the left interval $[0, \frac{1}{2})$, we have the following linear algebraic equations

$$Y_0(y*, k+2) = \frac{1}{(k+2)(k+1)}\left[\sum_{r=0}^{k}(k-n+1)Y_0(y*, k-r+1)\delta(r-1) - 2Y_0(y*, k)\right], \tag{5}$$

where $Y_0(y*, k) = \frac{1}{k!}\frac{d^k y*(x)}{dx^k}|_{x=0}$. The differential inverse transformation in the left interval has the following form:

$$y*(x) = Y_0(y*, 0) + xY_0(y*, 1) + ... + x^n Y_0(y*, n) + \dots$$

The first boundary condition $y(0) = 0$ becomes $Y_0(y*, 0) = 0$. Denoting $Y_0(y*, 1) = A$, (5) we have where A is unknown number that will be calculated later, and then substituting in the recursive relation $Y_0(y*, 3) = \frac{-A}{6}$, $Y_0(y*, 4) = 0$, $Y_0(y*, 5) = \frac{-A}{120}$, $Y_0(y*, 6) = 0$, $Y_0(y*, 7) = \frac{-A}{1680}$, ...

Thus we have the following series expansion of the left solution:

$$y*(x) \quad = \quad Ax - \frac{A}{6}x^3 - \frac{A}{120}x^5 - \frac{A}{1680}x^7 + ... \tag{6}$$

Applying differential transformation in the neighborhood of the right end-point $x_0 = 1$ we have

$$Y_1(y**, k+2) = \frac{1}{(k+2)(k+1)}[(k+1)Y_1(y**, k+1) + kY_1(y**, k) - 2Y_1(y**, k)]. \quad (7)$$

Applying the differential inverse transformation in the right interval $(\frac{1}{2}, 1]$ gives

$$y**(x) = Y_1(y**, 0) + (x-1)Y_1(y**, 1) + ... + (x-1)^n Y_1(y**, n) + ...$$

The boundary condition $y(1) = 1$ becomes $Y_1(y**, 0) = 1$. Let $Y_1(y**, 1) = B$. Here B is unknown parameter that will be calculated later. Using the recursive relation (7) we have
$Y_1(y**, 2) = \frac{1}{2}(B-2)$, $Y_1(y**, 3) = \frac{-1}{3}$, $Y_1(y**, 4) = \frac{-1}{12}$, $Y_1(y**, 5) = \frac{-1}{30}$, $Y_1(y**, 6) = \frac{-1}{90}$, $Y_1(y**, 7) = \frac{-1}{252}$, ...

Then we have the following series expansion of the right solution

$$\begin{aligned} y**(x) &= 1 + B(x-1) + \frac{1}{2}(B-2)(x-1)^2 - \frac{1}{3}(x-1)^3 - \\ &- \frac{1}{12}(x-1)^4 - \frac{1}{30}(x-1)^5 - \frac{1}{90}(x-1)^6 - \frac{1}{252}(x-1)^7 + ... \end{aligned} \quad (8)$$

To find the unknown parameters A and B, we put the relations (6) and (8) into the transmission conditions (3)–(4). Then using "Mathematica"8, we can calculate approximate values of the unknown numbers A and B as $A = 1.21302$, $B = 0.550509$. Here we continued iterating up to the 7 th term in the series expansion for DTM-solutions $y*(x)$ and $y**(x)$. Below, Figure 1 shows the graph of the DTM-solution

$$y(x) = \begin{cases} y*(x) & for \quad x \in [0, \frac{1}{2}), \\ y**(x) & for \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

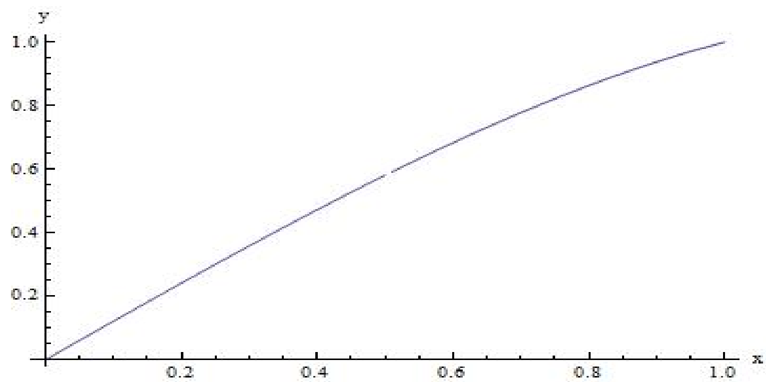

Figure 1. Approximate DTM- solution of the problem (2)–(4) for $\gamma_1 = \gamma_2 = 1$.

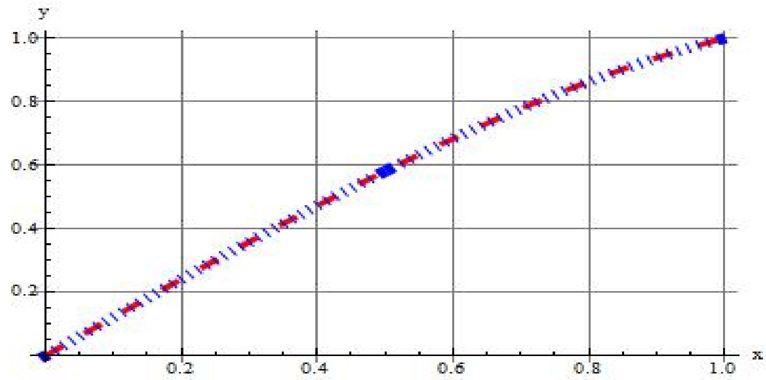

Figure 2. Comparison of the exact solution (red line) and the DTM-solution (blue line) of the problem (2)–(4) for $\gamma_1 = \gamma_2 = 1$.

Example 2. Now we shall consider a two-interval Weber equation with negative n, given by

$$y''(x) - xy'(x) - 4y(x) = 0, \qquad x \in [1, \frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{9}$$

together with the boundary conditions at the end points $x = 0$ and $x = 1$, given by

$$y(0) = 1, \quad y(1) = 0 \tag{10}$$

subject to the additional transmission conditions, given by

$$c_1 y\left(\frac{1}{2} - 0\right) - c_2 y\left(\frac{1}{2} + 0\right) = 0, \tag{11}$$

$$c_3 y'\left(\frac{1}{2} - 0\right) - c_4 y'\left(\frac{1}{2} + 0\right) = 0, \tag{12}$$

where c_1, c_2, c_3 and c_4 are real numbers that will be specified later. As above, we shall consider the differential equation (9) on the left side $[0, \frac{1}{2})$ and the right side $(\frac{1}{2}, 1]$ of the domain $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, separately.

As in above, $Y_0(y*, k)$ and $Y_1(y*, k)$ denotes the Y- transforms of $y(x)$ at the left end-point $x = 0$ and the right end-point $x = 1$, respectively. Using differential transformation in the left interval, i.e. in the neighborhood of the point $x_0 = 0$, we have

$$Y_0(y*, k+2) = \frac{1}{(k+2)(k+1)} \left[\sum_{r=0}^{k} (k - n + 1) Y_0(y*, k - r + 1)\delta(r - 1) + 4Y_0(y*, k) \right], \tag{13}$$

where $Y_0(y*, k) = \frac{1}{k!} \frac{d^k y*(x)}{dx^k}|_{x=0}$. The differential inverse transformation in the left interval has the following form:

$$y*(x) = Y_0(y*, 0) + xY_0(y*, 1) + ... + x^n Y_0(y*, n) + ...$$

The first boundary condition $y(0) = 1$ becomes $Y_0(y*, 0) = 1$. Denoting $Y_0(y*, 1) = K$, where K is unknown parameter that will be calculated later, and then substituting in the recursive relation (13), we have

$$Y_0(y*,2)=2,\ \ Y_0(y*,3)=\tfrac{5K}{6},\ \ Y_0(y*,4)=1,\quad Y_0(y*,5)=\tfrac{7K}{24},$$
$$Y_0(y*,6)=\tfrac{4}{15},\quad Y_0(y*,7)=\tfrac{K}{16},\quad Y_0(y*,8)=\tfrac{1}{21},\ ...$$

Thus we have the series expansion of the left solution $y*(x)$ in the form

$$y*(x) \;=\; 1+Kx+2x^2+\frac{5K}{6}x^3+x^4+\frac{7K}{24}x^5+\frac{4}{15}x^6+\frac{K}{16}x^7+\frac{1}{21}x^8+... \tag{14}$$

Now applying differential transformation to the equation (9) in the right interval, we have

$$Y_1(y**,k+2)=\frac{1}{(k+2)(k+1)}[(k+1)Y_1(y**,k+1)+kY_1(y**,k)-4Y_1(y**,k)]. \tag{15}$$

The differential inverse transformation in the right interval $(\frac{1}{2},1]$ has the following form:

$$y**(x)=Y_1(y**,0)+(x-1)Y_1(y**,1)+...+(x-1)^nY_1(y**,n)+...$$

The second boundary condition $y(1)=0$ becomes $Y_1(y**,0)=0$. Putting $Y_1(y**,1)=M$, where M is unknown parameter that will be calculated later, and using the recursive relation (15) we have

$$Y_1(y**,2)=\tfrac{M}{2},\ \ Y_1(y**,3)=M,\ \ Y_1(y**,4)=\tfrac{M}{2},$$
$$Y_1(y**,5)=\tfrac{9M}{20},\ \ Y_1(y**,6)=\tfrac{5M}{24},\quad Y_1(y**,7)=\tfrac{53M}{420},\ ...$$

Consequently we have the series expansion of the right solution $y**(x)$ in the form

$$\begin{aligned} y**(x) \;&=\; M(x-1)+\frac{M}{2}(x-1)^2+M(x-1)^3+\frac{M}{2}(x-1)^4+\frac{9M}{20}(x-1)^5\frac{5M}{24}(x-1)^6- \\ &-\;\frac{53M}{420}(x-1)^7+... \end{aligned} \tag{16}$$

Substituting (14)–(16) in the transmission conditions (11)–(12) we obtain two algebraic equation with respect to the variables K, M.

Finally, using «Mathematica 8», we can calculate approximate values of the parameters K and M as $K=-1.93316$, $M=-0.70003$. Here we were continued iterating up to the 7 th term in the series expansion for the DTM-solutions $y*(x)$ and $y**(x)$. The approximate DTM-solution of the problem (9)–(10) is presented graphically in Figure 3 and Figure 4.

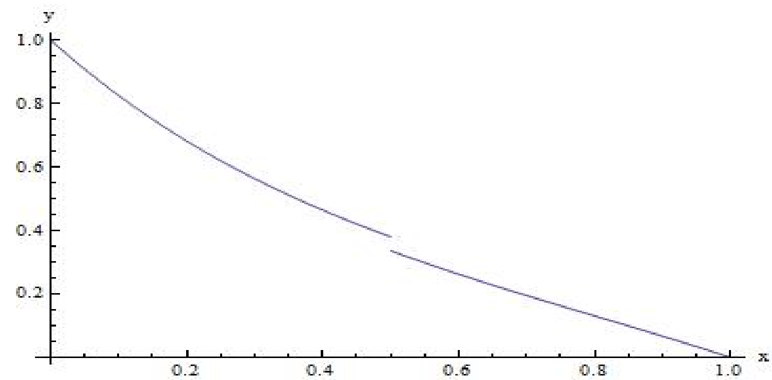

Figure 3. Approximate solution of the problem (9)–(12) for $c_1=3$, $c_2=4$, $c_3=5$, $c_4=6$.

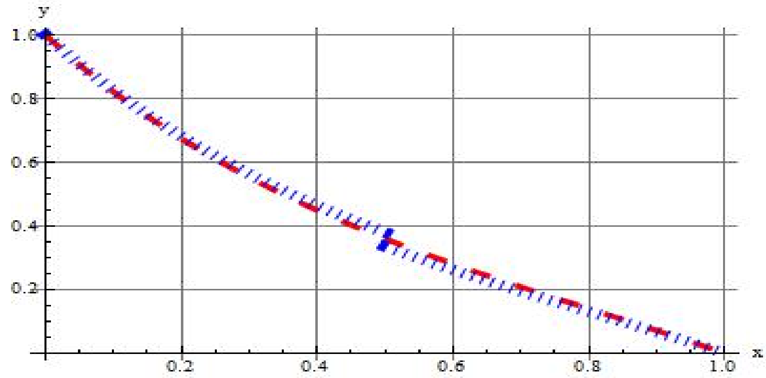

Figure 4. Comparison of the exact solution (red line) and the classical DTM-solution (blue line).

References

1 Snider, A.D. (1999). *Partial differential equations: sources and solutions.* Prentice Hall.

2 Olğar, H., Mukhtarov, O.Sh., & Aydemir, K. (2018). Some properties of eigenvalues and generalized eigenvectors of one boundary value problem. *Filomat, 32*(3), 911–920.

3 Mukhtarov, O.Sh., & Aydemir, K. (2021). Oscillation properties for non-classical Sturm-Liouville problems with additional transmission conditions. *Mathematical Modelling and Analysis, 26*(3), 432–443.

4 Paşaoğlu, B., & Tuna, H. (2018). Spectral expansion for the singular Dirac system with impulsive conditions. *Turkish Journal of Mathematics, 42*(5), 2527–2545.

5 Allahverdiev, B.P., Bairamov, E., & Ugurlu, E. (2013). Eigenparameter dependent Sturm–Liouville problems in boundary conditions with transmission conditions. *Journal of Mathematical Analysis and Applications, 401*(1), 388–396.

6 Aydemir, K., Olğar, H., Mukhtarov, O.S., & Muhtarov, F. (2018). Differential operator equations with interface conditions in modified direct sum spaces. *Filomat, 32*(3), 921–931.

7 Zhou, J.K. (1986). *Differential transformation and its application for electrical circuits.* Huazhong University Press, Wuhan.

8 Mukhtarov, O.Sh., Yücel, M., & Aydemir, K. (2020). Treatment a New Approximation Method and Its Justification for Sturm–Liouville Problems. *Complexity, 2020*, 8019460. https://doi.org/10.1155/2020/8019460

9 Mukhtarov, O.Sh., & Yücel, M. (2020). A study of the eigenfunctions of the singular Sturm–Liouville problem using the analytical method and the decomposition technique. *Mathematics, 8*(3), 415.

10 Gireesha, B.J., & Sowmya, G. (2022). Heat transfer analysis of an inclined porous fin using Differential Transform Method. *International Journal of Ambient Energy, 43*(1), 3189–3195.

11 Yücel, M., Muhtarov, F., & Mukhtarov, O. (2022). A New Transformation Method for Solving High-Order Boundary Value Problems. *Journal of New Theory, 40*, 90–100. https://doi.org/10.53570/jnt.1171760

12 Ashyralyev, A., & Arjmand, D. (2007). A note on the Taylor's decomposition on four points for a third-order differential equation. *Applied mathematics and computation, 188*(2), 1483–1490.

13 Ashyralyev, A., Arjmand, D., & Koksal, M. (2009). Taylor's decomposition on four points for solving third-order linear time-varying systems. *Journal of the Franklin Institute, 346*(7), 651–662.

14 Karakoç, F., & Bereketoğlu, H. (2009). Solutions of delay differential equations by using differential transform method. *International Journal of Computer Mathematics, 86*(5), 914–923.

15 Sepasgozar, S., Faraji, M., & Valipour, P. (2017). Application of differential transformation method (DTM) for heat and mass transfer in a porous channel. *Propulsion and power Research, 6*(1), 41–48.

16 Usman, M., Hamid, M., Khan, U., Din, S.T.M., Iqbal, M.A., & Wang, W. (2018). Differential transform method for unsteady nanofluid flow and heat transfer. *Alexandria engineering journal, 57*(3), 1867–1875.

17 Wazwaz, A.M. (2002). The Numerical Solution of Special Fourth-Order Boundary Value Problems by the Modified Decomposition Method. *International journal of computer mathematics, 79*(3), 345–356.

18 Arslan, D. (2022). Approximate Solutions of the Fourth-Order Eigenvalue Problem. *Journal of Advanced Research in Natural and Applied Sciences, 8*(2), 214–221.

19 Ertürk, V.S., & Momani, S. (2007). Comparing numerical methods for solving fourth-order boundary value problems. *Applied Mathematics and Computation, 188*(2), 1963–1968.

20 Aljahdaly, N.H., & El-Tantawy, S.A. (2021). On the multistage differential transformation method for analyzing damping Duffing oscillator and its applications to plasma physics. *Mathematics, 9*(4), 432.

М. Южел[1], Ф.Ш. Мұхтаров[2], О.Ш. Мұхтаров[3]

[1] *Хитит Университеті, Чорум, Түркия;*
[2] *Әзірбайжан Ұлттық ғылым академиясының Математика және механика институты, Баку, Әзірбайжан;*
[3] *Токат Газиосманпаша Университеті, Токат, Түркия*

Тарату шарттарын ескере отырып, Вебердің екіинтервалды теңдеуін шешуге арналған жалпыланған дифференциалдық түрлендіру әдісі

Зерттеудің негізгі мақсаты – классикалық дифференциалдық түрлендіру әдісін жаңа шеттік есептердің шешуге бейімдеу. Бұл әдістің артықшылығы оның қарапайымдылығында, өйткені шешілетін дифференциалдық теңдеуді іріктеу, ауытқу немесе сызықтық ету қажет емес. Осы сызықтық және бейсызықты дифференциалдық теңдеулер үшін қатарлар түрінде шешімдер алудың тиімді әдісі және берілген функцияның жоғары туындыларының мәндерін есептеуді қажет ететін Тейлор қатарларының классикалық әдісінен ерекшеленеді. Дифференциалды түрлендіру әдісі бір интервалды есептерді шешуге арналғаны белгілі және оны көп интервалды есептерге қалай қолдану керектігі белгісіз. Осы мақалада біз екі интервалды дифференциалдық теңдеулер үшін шеттік есептерді шешу үшін классикалық дифференциалдық түрлендіру әдісін бейімдедік. Ұсынылған жаңа әдістемені негіздеу үшін сол және оң жақты шешімдер екі қосымша берілу шарттарымен байланысты болатын ортақ ұшы бар екі қиылыспайтын кесінділер бойынша берілген Вебер теңдеуінің шеттік есебі шешілді.

Кілт сөздер: екіинтервалды есептер, дифференциалды түрлендіру әдісі, Вебер теңдеуі, тарату шарттары.

М. Южел[1], Ф.Ш. Мухтаров[2], О.Ш. Мухтаров[3]

[1] *Университет Хитит, Чорум, Турция;*
[2] *Институт математики и механики Национальной академии наук Азербайджана, Баку, Азербайджан;*
[3] *Университет Токат Газиосманпаша, Токат, Турция*

Метод обобщенного дифференциального преобразования для решения двухинтервального уравнения Вебера с учетом условий передачи

Основная цель данного исследования состоит в том, чтобы адаптировать классический дифференциальный метод преобразования для решения новых типов краевых задач. Преимущество этого метода заключается в его простоте, так как нет необходимости в дискретизации, возмущении или линеаризации решаемого дифференциального уравнения. Это эффективный метод получения решений в виде рядов как для линейных, так и нелинейных дифференциальных уравнений, и он отличается от классического метода рядов Тейлора, который требует вычисления значений высших производных заданной функции. Известно, что метод дифференциального преобразования предназначен для решения одноинтервальных задач и не ясно, как его применять к многоинтервальным задачам. В настоящей статье мы адаптировали классический метод дифференциального преобразования для решения краевых задач для двухинтервальных дифференциальных уравнений. Для обоснования предложенной новой методики решалась краевая задача для уравнения Вебера, заданного на двух непересекающихся отрезках с общим концом, на которых левое и правое решения были связаны двумя дополнительными условиями передачи.

Ключевые слова: двухинтервальные задачи, метод дифференциального преобразования, уравнение Вебера, условия передачи.