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Extensions of some differential inequalities for fractional integro-differential equations via upper and lower solutions

This paper deals with some differential inequalities for generalized fractional integro-differential equations by using the technique of upper and lower solutions. The fractional differential operator is taken in Caputo's sense and the nonlinear term divided into two parts depends on the fractional integrals of an unknown function with two different fractional orders. The results are studied by employing a variety of coupled upper and lower solutions. These theorems have some potential for extending the iterative techniques to fractional order integro-differential equations and to coupled systems of integro-differential fractional equations to obtain the existence of solutions as well as approximate solutions for the considered problem.

Keywords: fractional differential equations, differential inequalities, upper and lower solutions, boundary value problem.

Introduction

Although fractional calculus has existed for as long as «conventional» calculus, it was not until recent decades that the study of fractional differential equations became popular. This is because fractional operators commonly offer better accurate models than those with integer derivatives. See [1,2] for the recent developments and further information. Among the different definitions for fractional order derivatives, the Caputo fractional derivative stands out and has been intensely utilized since it is best suited for describing many events and the initial conditions for fractional differential equations are the same form as that of ordinary differential equations with integer derivatives. Due to the fact that it is far more extensive than the theory of classical ordinary differential equations, the theory of fractional differential equations has drawn a lot of attention. Although there has been tremendous recent progress in the study of fractional differential equations, there is still a significant potential in this area. After reviewing the literature, we find a number of publications on basic arguments, such as existence, uniqueness and stability results for fractional differential equations. See [3–10] and the references therein.

Differential and integral inequalities are crucial in the qualitative study of differential and integral equations. They are used to investigate the concepts of existence, uniqueness, boundedness, stability, continuous dependence, and so on. The method of upper and lower solutions is a quite effective concept in the theory of nonlinear differential equations with initial or boundary conditions. Recently, these methods have been applied to fractional differential equations as well as differential inequalities [11–20]. We give some comparison results for several types of coupled upper and lower solutions for a given boundary value problems of fractional integro-differential equations. The results here can be viewed as expansions and generalizations of corresponding analogous results from the integer order case to the fractional order case.

The purpose of this paper is to refine some previously published results for a given boundary value problems of fractional integro-differential equations by employing the method of upper and lower solutions together with strict and non-strict inequalities.

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1 Mathematical Preliminaries

This section provides background knowledge on fractional calculus and fractional differential equations in order to improve understanding.

Definition 1. [1] Let $[a,b] \subset \mathbb{R}$, $Re(q) > 0$ and $f \in L_1[a,b]$. Then the left and right Riemann-Liouville fractional integrals I^q_{a+} and I^q_{b-} of order α are defined as

$$I^q_{a+} f(x) := \frac{1}{\Gamma(q)} \int\limits_a^x \frac{f(t)\,dt}{(x-t)^{1-q}},\ x \in (a,b]$$

and

$$I^q_{b-} f(x) := \frac{1}{\Gamma(q)} \int\limits_x^b \frac{f(t)\,dt}{(t-x)^{1-q}}, x \in [a,b)$$

respectively.

Definition 2. [1] Let $[a,b] \subset \mathbb{R}$, $Re(q) \in (0,1)$ and $f \in L_1[a,b]$. The left and right Caputo fractional derivatives of order q are given by

$${}^cD^q_{a+} f(x) := I^{1-q}_{a+} Df(x), \forall x \in (a,b]$$

and

$${}^cD^q_{b-} f(x) := -I^{1-q}_{b-} Df(x), \forall x \in [a,b)$$

respectively.

Let $F, G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$, $u \in C^1[J, \mathbb{R}]$, $J = [0,T]$. We consider the following fractional boundary value problem.

$${}^CD^{q_1} u(t) = F(t, u(t), I^{q_2} u(t)) + G(t, u(t), I^{q_3} u(t)), \quad g(u(0), u(T)) = 0, \tag{1}$$

where $0 < q_3 \le q_2 \le q_1 < 1$ and $g \in C[\mathbb{R}^2, \mathbb{R}]$. From now on, the fractional operator ${}^CD^q$ stands for the left Caputo fractional derivative as well as I^q represents the left Riemann Liouville fractional integral operator.

Definition 3. Let $\alpha, \beta \in C^1[J, \mathbb{R}]$. Then α and β are said to be
(i) natural lower and upper solutions of (1) respectively if

$$\begin{aligned} {}^CD^{q_1}\alpha(t) &\le F(t, \alpha(t), I^{q_2}\alpha(t)) + G(t, \alpha(t), I^{q_3}\alpha(t)), \quad g(\alpha(0), \alpha(T)) \le 0, &(2)\\ {}^CD^{q_1}\beta(t) &\ge F(t, \beta(t), I^{q_2}\beta(t)) + G(t, \beta(t), I^{q_3}\beta(t)), \ g(\beta(0), \beta(T)) \ge 0, &(3) \end{aligned}$$

(ii) coupled lower and upper solutions of type I of (1) respectively if

$$\begin{aligned} {}^CD^{q_1}\alpha(t) &\le F(t, \alpha(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\beta(t)), \quad g(\alpha(0), \alpha(T)) \le 0, &(4)\\ {}^CD^{q_1}\beta(t) &\ge F(t, \beta(t), I^{q_2}\alpha(t)) + G(t, \beta(t), I^{q_3}\alpha(t)), \ g(\beta(0), \beta(T)) \ge 0, &(5) \end{aligned}$$

(iii) coupled lower and upper solutions of type II of (1) respectively if

$$\begin{aligned} {}^CD^{q_1}\alpha(t) &\le F(t, \beta(t), I^{q_2}\alpha(t)) + G(t, \beta(t), I^{q_3}\alpha(t)), \quad g(\alpha(0), \alpha(T)) \le 0,\\ {}^CD^{q_1}\beta(t) &\ge F(t, \alpha(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\beta(t)), \ g(\beta(0), \beta(T)) \ge 0, \end{aligned}$$

(iv) coupled lower and upper solutions of type III of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\beta(t), I^{q_2}\beta(t)) + G(t,\beta(t), I^{q_3}\beta(t)), \quad g(\alpha(0), \alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\alpha(t), I^{q_2}\alpha(t)) + G(t,\alpha(t), I^{q_3}\alpha(t)), \ g(\beta(0), \beta(T)) \geq 0, \end{aligned}$$

(v) coupled lower and upper solutions of type IV of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\alpha(t), I^{q_2}\alpha(t)) + G(t,\beta(t), I^{q_3}\beta(t)), \quad g(\alpha(0), \alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\beta(t), I^{q_2}\beta(t)) + G(t,\alpha(t), I^{q_3}\alpha(t)), \ g(\beta(0), \beta(T)) \geq 0, \end{aligned}$$

(vi) coupled lower and upper solutions of type V of (1) respectively if

$$\begin{aligned} {}^{C}D^{q_1}\alpha(t) &\leq F(t,\beta(t), I^{q_2}\beta(t)) + G(t,\alpha(t), I^{q_3}\alpha(t)), \quad g(\alpha(0), \alpha(T)) \leq 0, \\ {}^{C}D^{q_1}\beta(t) &\geq F(t,\alpha(t), I^{q_2}\alpha(t)) + G(t,\beta(t), I^{q_3}\beta(t)), \ g(\beta(0), \beta(T)) \geq 0. \end{aligned}$$

Lemma 1. [3] Let $m \in C^1[J,\mathbb{R}]$ and assume that $m(t_1) = 0$ for $t_1 \in (0,T]$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then we have ${}^{C}D^{q}m(t_1) \geq 0$.

The Laplace transform technique, as is well known, is a beneficial tool for solving initial value problems. Using this method, the stated problem is turned to an algebraic expression. The next lemma, which is about the inverse Laplace transform of the given function, is critical in this case.

Lemma 2. [21] Let $\alpha \geq \beta > 0$, $\alpha > \gamma$, $a, b \in \mathbb{R}$, $s^{\alpha-\beta} > |a|$ and $\left|s^{\alpha} + as^{\beta}\right| > |b|$. Then we get

$$\mathcal{L}^{-1}\left\{\frac{s^{\gamma}}{(s^{\alpha} + as^{\beta} + b)}\right\} = t^{\alpha-\gamma-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-b)^n(-a)^k\binom{n+k}{k}t^{k(\alpha-\beta)+n\alpha}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}.$$

We prove the following lemma in order to solve the given linear fractional initial value problem. It allows the corresponding result in [16] to be a specific case of this lemma.

Lemma 3. Assume that $\lambda \in C^1[J,\mathbb{R}]$, $0 < q_3 \leq q_2 \leq q_1 < 1$ and $L_1, M_1, M_2 \in \mathbb{R}$. The explicit solution of the following linear fractional integro-differential equation,

$$ {}^{C}D^{q_1}\lambda(t) = L_1\lambda(t) + M_1 I^{q_2}\lambda(t) + M_2 I^{q_3}\lambda(t), \quad \lambda(0) = \lambda_0 \tag{6}$$

is given by

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}t^{q_1(n+k+i)+nq_2+iq_3}}{\Gamma(q_1(n+k+i)+nq_2+iq_3+1)}\lambda_0$$

provided that $|s^{q_1+q_3}| > |M_2|$, $|s^{q_1} - M_2 s^{-q_3}| > |L_1|$ and $|s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}| > |M_1|$.

Proof. If we apply the Laplace transform on both side of the equation (6), we find the following relations

$$\begin{aligned}
\mathcal{L}\left\{{}^{C}D^{q_1}\lambda(t)\right\} &= L_1\mathcal{L}\{\lambda(t)\} + M_1\mathcal{L}\{I^{q_2}\lambda(t)\} + M_2\mathcal{L}\{I^{q_3}\lambda(t)\} \\
\frac{s\lambda(s)-\lambda_0}{s^{1-q_1}} &= L_1\lambda(s) + M_1\frac{\lambda(s)}{s^{q_2}} + M_2\frac{\lambda(s)}{s^{q_3}} \\
s^{q_1}\lambda(s) - s^{q_1-1}\lambda_0 &= L_1\lambda(s) + M_1\lambda(s)s^{-q_2} + M_2\lambda(s)s^{-q_3} \\
\lambda(s) &= \frac{s^{q_1+q_2-1}}{s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2} - M_1}\lambda_0 \\
&= \frac{s^{q_1+q_2-1}}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}\right)\left(1 - \frac{M_1}{s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}}\right)}\lambda_0 \\
&= \frac{s^{q_1+q_2-1}}{s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}\right)^n}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}\right)^{n+1}}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3}\right)^{n+1}\left(1 - \frac{L_1 s^{q_2}}{s^{q_1+q_2} - M_2 s^{q_2-q_3}}\right)^{n+1}}\lambda_0 \\
&= s^{q_1+q_2-1}\sum_{n=0}^{\infty}\frac{(M_1)^n}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3}\right)^{n+1}}\sum_{k=0}^{\infty}\frac{(L_1)^k\left(s^{q_2}\right)^k\binom{n+k}{k}}{\left(s^{q_1+q_2} - M_2 s^{q_2-q_3}\right)^k}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2} - M_2 s^{q_2-q_3}\right)^{n+k+1}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2}\right)^{n+k+1}\left(1 - \frac{M_2 s^{q_2-q_3}}{s^{q_1+q_2}}\right)^{n+k+1}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(M_1)^n(L_1)^k\binom{n+k}{k}}{s^{-q_1-q_2(k+1)+1}\left(s^{q_1+q_2}\right)^{n+k+1}}\sum_{i=0}^{\infty}(M_2)^i\binom{n+k+i}{i}s^{(-q_3-q_1)i}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}}{s^{-q_1-q_2(k+1)+1+(q_1+q_2)(n+k+1)+i(q_1+q_3)}}\lambda_0 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}}{s^{q_1(n+k+i)+nq_2+iq_3}}\lambda_0
\end{aligned}$$

provided that $|s^{q_1+q_3}| > |M_2|$, $|s^{q_1} - M_2 s^{-q_3}| > |L_1|$ and $|s^{q_1+q_2} - M_2 s^{q_2-q_3} - L_1 s^{q_2}| > |M_1|$. At this stage, we arrive at by implementing the inverse Laplace transform

$$\begin{aligned}
\mathcal{L}^{-1}\{\lambda(s)\} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}\mathcal{L}^{-1}\left\{\frac{1}{s^{q_1(n+k+i)+nq_2+iq_3}}\right\}\lambda_0 \\
\lambda(t) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(M_1)^n(L_1)^k(M_2)^i\binom{n+k}{k}\binom{n+k+i}{i}t^{q_1(n+k+i)+nq_2+iq_3}}{\Gamma\left(q_1(n+k+i)+nq_2+iq_3+1\right)}\lambda_0.
\end{aligned}$$

2 Formulas and theorems

Depending on the selection of upper and lower solutions of (1), we will assume the suitable conditions to establish some differential inequalities.

Theorem 1. Let α and β be natural lower and upper solutions of (1). $F(t,u,v)$ and $G(t,u,v)$ is non-decreasing in v and following Lipschitz-like conditions are also satisfied for $L_1, L_2, M_1, M_2 > 0$

$$\begin{aligned} F\left(t,u_1\left(t\right),v_1\left(t\right)\right)-F\left(t,u_2\left(t\right),v_2\left(t\right)\right) &\le L_1\left(u_1-u_2\right)+M_1\left(v_1-v_2\right), \quad (7)\\ G\left(t,u_1\left(t\right),\bar{v}_1\left(t\right)\right)-G\left(t,u_2\left(t\right),\bar{v}_2\left(t\right)\right) &\le L_2\left(u_1-u_2\right)+M_2\left(\bar{v}_1-\bar{v}_2\right) \quad (8)\end{aligned}$$

whenever $u_1 \ge u_2$. Then we have $\alpha\left(t\right) \le \beta\left(t\right)$ provided $\alpha\left(0\right) \le \beta\left(0\right)$.

Proof. In order to make it compatible with the problem (1), the functions $v_i, \bar{v}_i$ must be evaluated as follows $v_i = I^{q_2}u_i$ and $\bar{v}_i = I^{q_3}u_i$, $i = 1,2$. Clearly, $u_1 \le u_2$ implies that $v_1 \le v_2$ and $\bar{v}_1 \le \bar{v}_2$.

We now set $\alpha_\epsilon\left(t\right) = \alpha\left(t\right) - \epsilon\lambda\left(t\right)$ for arbitrary small number $\epsilon > 0$, where

$$\lambda\left(t\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{\left(L\right)^k\left(M_1\right)^n\left(N_1\right)^i\binom{n+k}{k}\binom{n+k+i}{i}t^{q_1\left(n+k+i\right)+nq_2+iq_3}}{\Gamma\left(q_1\left(n+k+i\right)+nq_2+iq_3+1\right)}$$

is unique positive solution of the equation

$$^{C}D^{q_1}\lambda\left(t\right)=L\lambda\left(t\right)+M_1I^{q_2}\lambda\left(t\right)+M_2I^{q_3}\lambda\left(t\right), \quad \lambda\left(0\right)=1, \tag{9}$$

where L is a positive number such that $L > L_1 + L_2$. Notice that $\alpha_\epsilon\left(0\right) = \alpha\left(0\right) - \epsilon\lambda\left(0\right) < \alpha\left(0\right)$, $\alpha_\epsilon\left(t\right) < \alpha\left(t\right)$ for $0 \le t \le T$. If we differentiate $\alpha_\epsilon\left(t\right)$ in terms of Caputo's sense, and using (2) we get

$$\begin{aligned} ^{C}D^{q_1}\alpha_\epsilon\left(t\right) &= {}^{C}D^{q_1}\alpha\left(t\right)-\epsilon{}^{C}D^{q_1}\lambda\left(t\right)\\ &\le F\left(t,\alpha\left(t\right),I^{q_2}\alpha\left(t\right)\right)+G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\\ &\quad -L\epsilon\lambda\left(t\right)-M_1\epsilon I^{q_2}\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right).\end{aligned}$$

We observe that $\alpha_\epsilon\left(t\right) < \alpha\left(t\right)$ on J yields $I^{q_2}\alpha_\epsilon\left(t\right) < I^{q_2}\alpha\left(t\right)$ and $I^{q_3}\alpha_\epsilon\left(t\right) < I^{q_3}\alpha\left(t\right)$ on J by the definition of R-L fractional integral. We then employ the Lipschitz-like inequalities in (7) and (8) to obtain

$$\begin{aligned} ^{C}D^{q_1}\alpha_\epsilon\left(t\right) &\le F\left(t,\alpha\left(t\right),I^{q_2}\alpha\left(t\right)\right)-F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\\ &\quad -G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-L\epsilon\lambda\left(t\right)-M_1\epsilon I^{q_2}\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\quad +F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)\\ &\le L_1\epsilon\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right)+M_1\epsilon I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right)+L_2\epsilon\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right)\\ &\quad +M_2\epsilon I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right)-L\epsilon\lambda\left(t\right)-M_1\epsilon I^{q_2}\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\quad +F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)\\ &= F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)+\epsilon\lambda\left(t\right)\left(L_1+L_2-L\right)\\ &< F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right).\end{aligned}$$

We intend to demonstrate $\alpha_\epsilon\left(t\right) < \beta\left(t\right)$ for $t \in [0,T]$, which concludes the proof by letting $\epsilon \longrightarrow 0$. Suppose that $\alpha_\epsilon\left(t\right) < \beta\left(t\right)$ on $t \in [0,T]$ is false. Then the set $A = \{t : t \in [0,T], \alpha_\epsilon\left(t\right) \ge \beta\left(t\right)\}$ is nonempty. Let t_* be the greatest lower bound of A, then $\alpha_\epsilon\left(t_*\right) = \beta\left(t_*\right)$ and $\alpha_\epsilon\left(t\right) < \beta\left(t\right)$ for $0 \le t < t_*$.

By generating $m\left(t\right) = \alpha_\epsilon\left(t\right) - \beta\left(t\right)$, it is written that $m\left(t\right) \le 0$ for $0 \le t < t_*$ and $m\left(t_*\right) = 0$. Because of Lemma 1, it leads to $^{C}D^{q_1}m\left(t_*\right) \ge 0$.

Since $\alpha_\epsilon(s) \leq \beta(s)$ for $0 \leq s \leq t_*$, we immediately get

$$\begin{aligned} I^{q_2}\alpha_\epsilon(t_*) &= \frac{1}{\Gamma(q_2)}\int\limits_0^{t_*}(t_*-s)^{1-q_2}\alpha_\epsilon(s)\,ds \\ &\leq \frac{1}{\Gamma(q_2)}\int\limits_0^{t_*}(t_*-s)^{1-q_2}\beta(s)\,ds \\ &= I^{q_2}\beta(t_*). \end{aligned}$$

A similar discussion offers $I^{q_3}\alpha_\epsilon(t_*) \leq I^{q_3}\beta(t_*)$. By recalling the non-decreasing features of F and G, we follow that

$$\begin{aligned} F(t_*,\alpha_\epsilon(t_*),I^{q_2}\alpha_\epsilon(t_*))+G(t_*,\alpha_\epsilon(t_*),I^{q_3}\alpha_\epsilon(t_*)) &> {}^C D^{q_1}\alpha_\epsilon(t_*) \\ &\geq {}^C D^{q_1}\beta(t_*) \\ &\geq F(t_*,\beta(t_*),I^{q_2}\beta(t_*))+G(t_*,\beta(t_*),I^{q_3}\beta(t_*)) \\ &\geq F(t_*,\beta(t_*),I^{q_2}\alpha_\epsilon(t_*))+G(t_*,\beta(t_*),I^{q_3}\alpha_\epsilon(t_*)) \end{aligned}$$

giving rise to a contradiction because of the fact that $\alpha_\epsilon(t_*) = \beta(t_*)$. Then the inequality

$$\alpha_\epsilon(t) < \beta(t), \forall t \in J$$

holds, which proves $\alpha(t) \leq \beta(t)$ on J.

Corollary 1. This result includes the Theorem 2 in [11] as a special case when $F \equiv 0$ and $q_1 = q_2$ or $G \equiv 0$ and $q_1 = q_3$.

Theorem 2. Let α and β be coupled lower and upper solutions of type I of (1). $F(t,u,v)$ and $G(t,u,v)$ is both non-increasing in v and they hold the following inequalities for $u_1 \geq u_2, v_1 \geq v_2$ and L_1, L_2, M_1, M_2 positive constants such that

$$F(t,u_1(t),v(t)) - F(t,u_2(t),v(t)) \leq L_1(u_1-u_2), \tag{10}$$
$$G(t,u_1(t),\bar{v}(t)) - G(t,u_2(t),\bar{v}(t)) \leq L_2(u_1-u_2), \tag{11}$$
$$F(t,u(t),v_1(t)) - F(t,u(t),v_2(t)) \geq -M_1(v_1-v_2), \tag{12}$$
$$G(t,u(t),\bar{v}_1(t)) - G(t,u(t),\bar{v}_2(t)) \geq -M_2(\bar{v}_1-\bar{v}_2). \tag{13}$$

If $\alpha(0) \leq \beta(0)$, then it yields that $\alpha(t) \leq \beta(t)$ on J.

Proof. We begin by constructing $\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ and $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ for $\epsilon > 0$. The function $\lambda(t)$ is also supposed to be unique positive solution of (9) with $L_1 + L_2 > L > 0$. It is clear that $\beta_\epsilon(0) = \beta(0) + \epsilon\lambda(0) > \beta(0)$ and $\alpha_\epsilon(0) = \alpha(0) - \epsilon\lambda(0) < \alpha(0)$ that imply $\alpha_\epsilon(0) < \beta_\epsilon(0)$ and for $0 \leq t \leq T$, we get $\beta_\epsilon(t) > \beta(t)$ and $\alpha_\epsilon(t) < \alpha(t)$.

Differentiating both sides of $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ leads to

$$\begin{aligned} {}^C D^{q_1}\beta_\epsilon(t) &= {}^C D^{q_1}\beta(t) + \epsilon^C D^{q_1}\lambda(t) \\ &\geq F(t,\beta(t),I^{q_2}\alpha(t)) + G(t,\beta(t),I^{q_3}\alpha(t)) \\ &\quad + L\epsilon\lambda(t) + M_1\epsilon I^{q_2}\lambda(t) + M_2\epsilon I^{q_3}\lambda(t). \end{aligned}$$

Since $\beta_\epsilon(t) > \beta(t)$, we can utilize the inequalities (10) and (11) in hypothesis to get

$$\begin{aligned} F(t,\beta_\epsilon(t),I^{q_2}\alpha(t)) - F(t,\beta(t),I^{q_2}\alpha(t)) &\leq L_1(\beta_\epsilon(t)-\beta(t)), \\ F(t,\beta(t),I^{q_2}\alpha(t)) &\geq F(t,\beta_\epsilon(t),I^{q_2}\alpha(t)) - L_1(\beta_\epsilon(t)-\beta(t)) \end{aligned}$$

and

$$\begin{aligned} G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\beta\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\leq L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right), \\ G\left(t,\beta\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\geq G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right). \end{aligned}$$

Putting these results into the foregoing inequality, we write

$$\begin{aligned} {}^{C}D^{q_1}\beta_\epsilon\left(t\right) &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)-L_1\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right) \\ &\quad +G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-L_2\left(\beta_\epsilon\left(t\right)-\beta\left(t\right)\right) \\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &= F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)+(L-L_1-L_2)\epsilon\lambda\left(t\right) \\ &\quad +M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right). \end{aligned}$$

Since the fact that $\alpha\left(t\right)>\alpha_\epsilon\left(t\right)$, we can have $I^{q_2}\alpha\left(t\right)>I^{q_2}\alpha_\epsilon\left(t\right)$ and $I^{q_3}\alpha\left(t\right)>I^{q_3}\alpha_\epsilon\left(t\right)$. Therefore, the following inequalities can be found by considering inequalities (12) and (13)

$$\begin{aligned} F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right)-F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right) &\geq -M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right), \\ F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha\left(t\right)\right) &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)-M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \end{aligned}$$

and

$$\begin{aligned} G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right) &\geq -M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right), \\ G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha\left(t\right)\right) &\geq G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right). \end{aligned}$$

Combining these results with previous inequality, we arrive at

$$\begin{aligned} {}^{C}D^{q_1}\beta_\epsilon\left(t\right) &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)-M_1I^{q_2}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \\ &\quad +G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-M_2I^{q_3}\left(\alpha\left(t\right)-\alpha_\epsilon\left(t\right)\right) \\ &\quad +M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right) \\ &= F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right). \end{aligned}$$

We intend to demonstrate $\alpha\left(t\right)<\beta_\epsilon\left(t\right)$ on J. If we use the similar technique as before we first assume that assertion is false which gives a contradiction itself. Therefore, when $\epsilon\longrightarrow 0$ gives the desired result.

Remark 1. Notice that if $F(t,u,v)$ and $G(t,u,v)$ are non-decreasing in v for each (t,u) whenever $\alpha\leq\beta$, then natural lower and upper solutions given by (2) and (3) imply the coupled lower and upper solutions of type I given by (4) and (5). Conversely, if $F(t,u,v)$ and $G(t,u,v)$ are non-increasing in v for each (t,u) whenever $\alpha\leq\beta$, then coupled lower and upper solutions of type I reduce to natural lower and upper solutions respectively.

Theorem 3. Let α and β be coupled lower and upper solutions of type II of (1) as well as $F(t,u,v)$ and $G(t,u,v)$ is non-decreasing in v. We also assume that

$$\begin{aligned} F\left(t,u_1\left(t\right),v\left(t\right)\right)-F\left(t,u_2\left(t\right),v\left(t\right)\right) &\geq -L_1\left(u_1-u_2\right) \\ G\left(t,u_1\left(t\right),\bar{v}\left(t\right)\right)-G\left(t,u_2\left(t\right),\bar{v}\left(t\right)\right) &\geq -L_2\left(u_1-u_2\right) \\ F\left(t,u\left(t\right),v_1\left(t\right)\right)-F\left(t,u\left(t\right),v_2\left(t\right)\right) &\leq M_1\left(v_1-v_2\right) \\ G\left(t,u\left(t\right),\bar{v}_1\left(t\right)\right)-G\left(t,u\left(t\right),\bar{v}_2\left(t\right)\right) &\leq M_2\left(\bar{v}_1-\bar{v}_2\right) \end{aligned}$$

whenever $u_1\geq u_2, v_1\geq v_2$, where $L_1,L_2>0, M_1,M_2\geq 0$. Then $\alpha\left(0\right)\leq\beta\left(0\right)$ implies that $\alpha\left(t\right)\leq\beta\left(t\right)$ on J.

Proof. For the proof, we recall the previous definitions of functions $\alpha_\epsilon(t), \beta_\epsilon(t)$ on J such that for $\epsilon > 0$

$$\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t), \beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t).$$

The function $\lambda(t)$ is the unique positive solution of (9) with $L_1 + L_2 < L$. We can achieve the desired conclusion by using a similar process as described above.

Theorem 4. Let α and β be coupled lower and upper solutions of type III of (1) as well as both $F(t,u,v)$ and $G(t,u,v)$ is non-increasing in v. We also assume that

$$F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) \geq -L_1(u_1 - u_2) - M_1(v_1 - v_2),$$

$$G(t, u_1(t), \bar{v}_1(t)) - G(t, u_2(t), \bar{v}_2(t)) \geq -L_2(u_1 - u_2) - M_2(\bar{v}_1 - \bar{v}_2)$$

whenever $u_1 \geq u_2, v_1 \geq v_2$ and $L_1, L_2 > 0, M_1,\ M_2 \geq 0$. Then $\alpha(t) \leq \beta(t)$ on J provided that $\alpha(0) \leq \beta(0)$.

Proof. By using analogous considerations as mentioned previously, we can gain the conclusion of theorem directly. For space-saving, we omit the details here.

Corollary 2. If we take $G \equiv 0$ in the problem (1), then the results in Theorems 1–4 are reduced to the results in [16].

Remark 2. It is worthwhile to note that if $\alpha \leq \beta$ on J, then the monotonicity assumption of F and G in Theorem 3 combined with allowing α, β to be the coupled lower and upper solutions of type II respectively is equivalent to the case in which the monotonicity assumption of F and G in Theorem 4 combined with α, β being the coupled lower and upper solutions of type III respectively.

Theorem 5. Let α and β be coupled lower and upper solutions of type IV of (1). $F(t,u,v)$ is non-decreasing in v while $G(t,u,v)$ is non-increasing in v. Assume further that following inequalities are satisfied:

$$\begin{aligned} F(t, u_1(t), v_1(t)) - F(t, u_2(t), v_2(t)) &\leq L_1(u_1 - u_2) + M_1(v_1 - v_2), & (14)\\ G(t, u_1(t), \bar{v}_1(t)) - G(t, u_2(t), \bar{v}_2(t)) &\geq -L_2(u_1 - u_2) - M_2(\bar{v}_1 - \bar{v}_2), & (15)\end{aligned}$$

where $L_1, L_2 > 0, M_1, M_2 \geq 0$, whenever $u_1 \geq u_2, v_1 \geq v_2$. Then $\alpha(0) \leq \beta(0)$ implies that $\alpha(t) \leq \beta(t)$ on J.

Proof. We begin by constructing $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ and $\alpha_\epsilon(t) = \alpha(t) - \epsilon\lambda(t)$ for $\epsilon > 0$. The function $\lambda(t)$ is also supposed to be unique positive solution of (9) such that $L > L_1 + L_2$. It is clear that $\beta_\epsilon(0) = \beta(0) + \epsilon\lambda(0) > \beta(0)$ and $\alpha_\epsilon(0) = \alpha(0) - \epsilon\lambda(0) < \alpha(0)$ imply $\alpha_\epsilon(0) < \beta_\epsilon(0)$. In addition to that for $0 \leq t \leq T$ we get $\beta_\epsilon(t) > \beta(t)$ and $\alpha_\epsilon(t) < \alpha(t)$.

Differentiating both sides of $\beta_\epsilon(t) = \beta(t) + \epsilon\lambda(t)$ leads to

$$\begin{aligned} {}^C D^{q_1}\beta_\epsilon(t) &= {}^C D^{q_1}\beta(t) + {}^C D^{q_1}\epsilon\lambda(t)\\ &\geq F(t, \beta(t), I^{q_2}\beta(t)) + G(t, \alpha(t), I^{q_3}\alpha(t))\\ &\quad + L\epsilon\lambda(t) + M_1\epsilon I^{q_2}\lambda(t) + M_2\epsilon I^{q_3}\lambda(t). \qquad (16)\end{aligned}$$

Since $\beta_\epsilon(t) > \beta(t)$ for $0 \leq t \leq T$, we can employ the inequality (14) and (15) then it yields

$$F(t, \beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) - F(t, \beta(t), I^{q_2}\beta(t)) \leq L_1(\beta_\epsilon - \beta) + M_1 I^{q_2}(\beta_\epsilon - \beta),$$

$$F(t, \beta(t), I^{q_2}\beta(t)) \geq F(t, \beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) - L_1\epsilon\lambda(t) - M_1\epsilon I^{q_2}\lambda(t) \qquad (17)$$

and

$$G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)-G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)\geq -L_2\left(\alpha-\alpha_\epsilon\right)-M_2I^{q_3}\left(\alpha-\alpha_\epsilon\right),$$
$$G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\geq G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-L_2\epsilon\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right). \tag{18}$$

If we substitute (17) and (18) into (16), we get

$$\begin{aligned}{}^{C}D^{q_1}\beta_\epsilon\left(t\right) &\geq F\left(t,\beta\left(t\right),I^{q_2}\beta\left(t\right)\right)+G\left(t,\alpha\left(t\right),I^{q_3}\alpha\left(t\right)\right)\\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\geq F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\beta_\epsilon\left(t\right)\right)-L_1\epsilon\lambda\left(t\right)-M_1\epsilon I^{q_2}\lambda\left(t\right)\\ &\quad +G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right)-L_2\epsilon\lambda\left(t\right)-M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &\quad +L\epsilon\lambda\left(t\right)+M_1\epsilon I^{q_2}\lambda\left(t\right)+M_2\epsilon I^{q_3}\lambda\left(t\right)\\ &> F\left(t,\beta_\epsilon\left(t\right),I^{q_2}\beta_\epsilon\left(t\right)\right)+G\left(t,\alpha_\epsilon\left(t\right),I^{q_3}\alpha_\epsilon\left(t\right)\right).\end{aligned}$$

A similar procedure can be applied to $\alpha_\epsilon\left(t\right)=\alpha\left(t\right)-\epsilon\lambda\left(t\right)$ to achieve the following result

$${}^{C}D^{q_1}\alpha_\epsilon\left(t\right)<F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\beta_\epsilon\left(t\right)\right)$$

on $[0,T]$.

We next prove that $\alpha_\epsilon\left(t\right)<\beta_\epsilon\left(t\right)$ on $[0,T]$. Contrary to this claim, we presume for a moment that the inequality is not true and, setting $m\left(t\right)=\alpha_\epsilon\left(t\right)-\beta_\epsilon\left(t\right)$ there would exist a point t_* such that $m\left(t_*\right)=0$ and $m\left(t\right)\leq 0$ for $0\leq t<t_*$. We get at once ${}^{C}D^{q_1}m\left(t_*\right)\geq 0$ by Lemma 1. Obviously, it causes a contradiction. Then, it has to be

$$\alpha_\epsilon\left(t\right)<\beta_\epsilon\left(t\right)$$

on J. Finally, letting $\epsilon\longrightarrow 0$, we reach at

$$\begin{aligned}\lim_{\epsilon\longrightarrow 0}\left(\alpha\left(t\right)-\epsilon\lambda\left(t\right)\right) &\leq \lim_{\epsilon\longrightarrow 0}\left(\beta\left(t\right)+\epsilon\lambda\left(t\right)\right),\\ \alpha\left(t\right) &\leq \beta\left(t\right),\end{aligned}$$

for $t\in J$, ending the proof.

Corollary 3. This result is evaluated as the generalization of Theorem 2.10 in [17] to fractional orders by simple modifications.

Theorem 6. Let α and β be coupled lower and upper solutions of type V of (1). $F(t,u,v)$ is non-increasing and $G(t,u,v)$ is non-decreasing in v. Additionally, following inequalities hold:

$$F\left(t,u_1\left(t\right),v_1\left(t\right)\right)-F\left(t,u_2\left(t\right),v_2\left(t\right)\right) \geq -L_1\left(u_1-u_2\right)-M_1\left(v_1-v_2\right), \tag{19}$$
$$G\left(t,u_1\left(t\right),\bar{v}_1\left(t\right)\right)-G\left(t,u_2\left(t\right),\bar{v}_2\left(t\right)\right) \leq L_2\left(u_1-u_2\right)+M_2\left(\bar{v}_1-\bar{v}_2\right), \tag{20}$$

where $L_1,L_2,M_1,M_2>0$, whenever $u_1\geq u_2,v_1\geq v_2$. Then $\alpha\left(0\right)\leq\beta\left(0\right)$ implies that $\alpha\left(t\right)\leq\beta\left(t\right)$ on J.

Proof. In that case, for some $\epsilon>0$, we compose $\beta_\epsilon\left(t\right)=\beta\left(t\right)+\epsilon\lambda\left(t\right)$ and $\alpha_\epsilon\left(t\right)=\alpha\left(t\right)-\epsilon\lambda\left(t\right)$ where the function $\lambda\left(t\right)$ is taken as the nonnegative unique solution of the following linear equation

$${}^{C}D^{q_1}\lambda\left(t\right)=L\lambda\left(t\right)+M_1I^{q_2}\lambda\left(t\right)+M_2I^{q_3}\lambda\left(t\right),\quad \lambda\left(0\right)=1.$$

Taking derivatives in Caputo's sense on both sides of constructed functions and using (19) and (20), we have the following strict inequalities

$${}^{C}D^{q_1}\beta_\epsilon\left(t\right)>F\left(t,\alpha_\epsilon\left(t\right),I^{q_2}\alpha_\epsilon\left(t\right)\right)+G\left(t,\beta_\epsilon\left(t\right),I^{q_3}\beta_\epsilon\left(t\right)\right)$$

and

$$^{C}D^{q_1}\alpha_\epsilon(t) < F(t, \beta_\epsilon(t), I^{q_2}\beta_\epsilon(t)) + G(t, \alpha_\epsilon(t), I^{q_3}\alpha_\epsilon(t)).$$

At this stage we apply proof by contradiction with the help of Lemma 1 to show $\alpha_\epsilon(t) < \beta_\epsilon(t)$ on J. As a final step, performing $\epsilon \longrightarrow 0$, we get the desired result

$$\alpha(t) \leq \beta(t),$$

for $t \in J$, which completes the proof.

3 Conclusion

Using the method of upper and lower solutions, this research discusses some differential inequalities for generalized fractional integro-differential equations. Multiple coupled upper and lower solutions are used to examine the results. These theorems provide some possibilities for stretching iterative techniques to fractional order integro-differential equations and coupled systems of integro-differential fractional equations in order to determine the existence of solutions as well as approximations for the problem under consideration.

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Жоғарғы және төменгі шешімдер арқылы бөлшек ретті интегралды дифференциалдық теңдеулер үшін кейбір дифференциалдық теңсіздіктердің кеңеюі

Мақалада жоғарғы және төменгі шешімдер техникасын қолдана отырып, бөлшек ретті жалпыланған интегралды-дифференциалдық теңдеулер үшін кейбір дифференциалдық теңсіздіктер қарастырылған. Бөлшек дифференциалдық оператор Капуто мағынасында түсініледі, ал екіге бөлінген сызықтық емес термин екі түрлі бөлшек реті бар белгісіз функцияның бөлшек интегралдарына тәуелді. Нәтижелер әртүрлі байланысты жоғарғы және төменгі шешімдерді қолдану арқылы зерттелген. Бұл теоремалар қайталанатын әдістерді бөлшек ретті интегралды-дифференциалдық теңдеулерге және шешімдердің болуын, сондай-ақ қарастырылып отырған мәселе үшін жуықталған шешімдерді алу үшін бөлшек ретті интегралды-дифференциалдық теңдеулердің байланысты жүйелеріне тарату үшін белгілі бір әлеуетке ие.

Кілт сөздер: бөлшек дифференциалдық теңдеулер, дифференциалдық теңсіздіктер, жоғарғы және төменгі шешімдер, шеткі есеп.

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Расширения некоторых дифференциальных неравенств для интегро-дифференциальных уравнений дробного порядка через верхние и нижние решения

В статье рассмотрены некоторые дифференциальные неравенства для обобщенных интегро-дифференциальных уравнений дробного порядка с использованием техники верхних и нижних решений. Дробно-дифференциальный оператор понимается в смысле Капуто, а нелинейный член, разделенный на две части, зависит от дробных интегралов неизвестной функции с двумя различными дробными порядками. Результаты изучены с использованием различных связанных верхних и нижних решений. Эти теоремы имеют некоторый потенциал для распространения итерационных методов на интегро-дифференциальные уравнения дробного порядка и на связанные системы интегро-дифференциальных уравнений дробного порядка для получения существования решений, а также приближенных решений для рассматриваемой задачи.

Ключевые слова: дробные дифференциальные уравнения, дифференциальные неравенства, верхние и нижние решения, краевая задача.