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Numerical method to solution of generalized model Buckley-Leverett in a class of discontinuous functions

A new numerical method is proposed for solving the generalized Buckley-Leverett problem, which describes the movement of two-phase mixtures of Bazhenov bed sediments in a class of discontinuous functions. To this end, we introduce an auxiliary problem that has advantages over the main problem, and using these advantages, an original finite difference method to solve of the auxiliary problem is developed. Using the suggested auxiliary problem, a solution which expresses exactly all physical characteristics of the problem is obtained.

Keywords: generalized Buckley-Leverett problem, auxiliary problem, finite differences method in a class of discontinuous functions.

Introduction

We consider the following problem in the upper half of the Euclidean $R^2_+(x,t)$ space

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial \varphi(u(x,t))}{\partial x} - \psi\big(u(x,t)\big) = 0, \tag{1}$$

$$u(x,0) = u_0(x), x \geq 0, \tag{2}$$

$$u(0,t) = u_1(x), t \geq 0, \tag{3}$$

where $\varphi(u)$ and $\psi(u)$ are known functions according to argument u and have the following properties:

- $\varphi(u)$, $\psi(u)$ and $\varphi'(u)$, $\psi'(u)$ are continuous functions, and they are bounded for bounded u, and $\varphi''(u)$ does not change its sign,
- $\varphi(u) \geq 0$ and $\varphi'(u) \geq 0$ for $u \geq 0$, and the argument u has values such that the function $\psi(u)$ becomes zero at these points,
- $\psi'(u)$ is bounded function for $u \geq 0$.

Here, $u_0(x)$ and $u_1(x)$ are given functions satisfying the condition $u_0(0) \neq u_1(0)$.

In the case of $\psi\big(u(x,t)\big) \equiv 0$, the problem (1)–(3) is used to solve many problems in hydrodynamics, including the qualitative characteristics of the mechanism of compression of oil with gas or water in a porous medium, which is called the Buckley-Leverett model in the literature [1]. It has been proven that when the initial and boundary functions are incompatible (for the initial-boundary problem) or the initial profile has a decreasing part with respect to the spatial coordinate (for the initial value problem), the jump points, locations of which are not known beforehand, occur in the solution of the problem [2–9]. In other words, there is no classical solution for the problem under consideration, and the question of the uniqueness of the solution remains open. For this purpose, criteria for the uniqueness

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of the solution and robustness of the jump are proposed in [4, 6, 8, 10, 11]. In the theory of hyperbolic equations, the stable jump motion in a problem is called a discontinuity disintegration problem and has been widely studied in the literature, as specified in [2, 3, 5, 8, 12, 13], etc.

There are conservative finite volume methods of practical importance, which are based on dividing the spatial domain into intervals (also called «finite volumes» or grid cells), and establishing certain approximations to the integral of the flow over each of these volumes in [10, 11, 14–18], etc. Also, Godunov-type finite difference algorithms were developed considering the properties of the analytical solution in [13].

In [19], a method in the class of discontinuous functions was proposed to find an analytical solution of the problem (1)–(3), and using this method, a finite difference algorithm was established that accurately expresses all the properties of the physical processes of the problem in [20].

In the case of $\psi\big(u(x,t)\big) \neq 0$, the problem (1)–(3) is called generalized Buckley-Leverett problem in a physical sense and differs from the classic Buckley-Leverett problem in that the trajectory of the jump does not coincide with the characteristics, and the discontinuity jump approaches zero as time values increase [21].

In [21], the dynamics of chemical and physico-chemical changes in a multi-phase and multi-component oilfield after exposure to thermogas was investigated by means of a mathematical model, where the process of injecting hot water into the reservoir containing hydrocarbons was specifically discussed. Usually, this type of impact method is applied to an oilfield (Bazhenov-type deposit) including kerogen containing oil in a bound state. Such deposits have a layered structure in which oil is located in the pores as well. Permeable non-productive strata alternate with productive impermeable strata. Mathematical modeling of deposits with such a structure in the non-isothermal mode becomes even more complicated.

The purpose of the thermal impact mechanism is to inject a certain amount of hot fluid such as hot water into the reservoir in order to increase the reservoir temperature, and then to displace oil by water at the common contact interface. This can also release trapped oil and isolated pores. During treatment with hot water injected into the reservoir, some additional amount of oil is released into the pore volume, which affects the regime of the displacement of oil by water. Ultimately, it leads to an increase in the flow rate of light crude oil trapped in the reservoir.

Since the filtration process is slow, the deformation of the bed can be neglected. On the other hand, the movement happens so quickly that it is possible to ignore the conductive heat transfer as the essential mechanism of heat transfer is convection.

More interesting problems arise in investigating the role of spatial structures in the creation and evolution of living organisms in molecular biology [22]. When studying this type of problem, $\varphi(u(x,t))$ in Eq. (1) represents the convective flow function of the reaction component, and $\psi(u(x,t))$ represents the kinetics of the reaction. If $\partial\varphi(u(x,t))/\partial x$ is a strictly non-linear function, then jumps occur in the solution of the problem, in which case such solutions are understood as weak solutions.

In this study, we consider problems with source terms that do not include delta functions that typically converge to zero throughout most of the region, ignoring the existence of very thin reaction zones that occur dynamically as part of the solution. Such source terms are often expressed in delta functions, but their positions and strengths are often not known in advance.

In this article, in order to show what behaviors are expected from the process, problem (1)–(3) is handled only mathematically, with respect to wave propagation, without considering the mechanism of any chemical reaction. In general, soft solutions found by the characteristics method do not enable us to explore the dynamics from the beginning of the process to the end.

As it is known that, the solutions of problem (1)–(3) has the points of discontinuities locations of which are unknown beforehand. Existence of the points of discontinuities in the solutions involves difficulties in applying the classical numerical methods to that equations [19]. The necessity to work with discontinuous functions and to find a solution that can accurately express the dynamics of the process require the creation of sensitive numerical methods in the class of discontinuous functions.

1 Finding the analytical solution

For the sake of simplicity, we will first consider the Cauchy problem for Eq. (1). We can easily get the solution of problem (1), (2) by the method of characteristics. For this, if we search for the solution of problem (1), (2) in the closed form $V(t,x,u) = 0$, we reach the following quasi linear equation in accordance with the V function

$$\frac{\partial V}{\partial t} + \varphi'(u)\frac{\partial V}{\partial x} + \psi(u)\frac{\partial V}{\partial u} = 0. \tag{4}$$

The system of characteristic equations for (4) is

$$\frac{dt}{1} = \frac{dx}{\varphi'(u)} = \frac{du}{\psi(u)}.$$

From here, the following system of equations is obtained

$$\begin{cases} \frac{dx}{dt} = \varphi'(u), \\ \frac{du}{dt} = \psi(u). \end{cases} \tag{5}$$

The first intermediate integrals for the system (5) are

$$c_1 = x - \int \frac{\varphi'(u)du}{\psi(u)}, \quad c_2 = t - \int \frac{du}{\psi(u)}. \tag{6}$$

According to the general theory, for an arbitrary continuously differentiable function F, the general solution of problem (1), (2) is written as $F(c_1, c_2) = 0$ or

$$x - \int \psi^{-1}(u)\varphi'(u)du = f\Big(t - \int \psi^{-1}(u)du\Big), \tag{7}$$

where the function f is any continuously differentiable function. Expression in the form of (7) is called soft solution.

To check the effectiveness of the proposed method, and to find a clear expression of the analytical solution, instead of Eq. (1), the following equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - u(1-u) = 0 \tag{8}$$

is considered in the special case of $\varphi(u) = \frac{u^2}{2}$ and $\psi(u) = u(1-u)$. In this case, Eq. (8) is called the Burgers equation with a robust source in hydrodynamics. For Eq. (8), the expression in (6) and (7) take the following form

$$c_1 = x + \ln(1-u), \quad c_2 = \frac{e^t}{u} - e^t$$

and

$$x + \ln(1-u) = f\Big(\frac{e^t}{u} - e^t\Big)$$

respectively. Here, f is any continuously differentiable function. Considering the initial condition (2), the soft solution of problem (1), (2) is obtained as

$$u(x,t) = e^t(1 - u + ue^{-t})u_0\big(\ln e^x(1 - u + ue^{-t})\big). \tag{9}$$

By a simple calculation, it is verified that the function $u(x,t)$ given by (9) is a soft solution of problem (1), (2). In a special case if it is assumed $u_0(x) = e^{-x}$, then expression (9) takes the following form

$$u(x,t) = e^{-x+t}.$$

Also, it is easily shown that function (9) satisfies Eq. (8). In other words, function (9) is a soft solution of Eq. (8).

When the second equation of system (5) is considered

$$\frac{du}{dt} = u(1-u), \tag{10}$$

it is seen that the constant functions $u = 0$ and $u = 1$ are equilibrium solutions. This equation, called the logistic equation, describes the growth of population and is also applied to the growth of bacteria, fruit flies and flour beetles, etc. [22]. It can be shown that $u = 0$ is an unstable equilibrium point, and $u = 1$ is a stable one. The initial condition $u(x,0)$ just indicates a non-regular spatial distribution.

Solution of Eq. (10) is

$$u(\xi) = \frac{1}{2}\Big(1 + \tanh\frac{\xi}{2D}\Big), \quad \xi = x - Dt$$

and this becomes a piece-wise continuous function from zero to one rapidly, regardless of the initial profile. To see the subsequent evolution of the solution, it is sufficient to consider the Riemann problem with a jump from 0 to 1 or from 1 to 0. In the Burgers equation without source function, the jump line moves with velocity $D = 1/2$ for $u_{left} = 1$ and $u_{right} = 0$. Then the source term is identically equal to zero, and therefore, has no effect on the solution of the problem. An interesting situation occurs if $u_{left} = 0$ and $u_{right} = 1$, in which case the Burgers equation converts the jump in the initial profile into a rarefaction wave.

2 Finite differences in a class of continuous functions

Firstly, let's divide the interval $[a,b]$ into n equal parts by means of the points x_j, $(j = 0,1,2,...,n)$ and by setting $h_x = (b-a)/n$, $(i = 0,1,2,...,n)$ that is, $x_j = a + jh_x$. In a similar way, let's divide the interval $[0,T)$ into time layers by means of points $t_k = kh_\tau$, $(k = 0,1,2,...)$, where $h_\tau > 0$. Here a, b, and T are given real numbers. Thus, we have constructed two one-dimensional grids over the intervals $[a,b]$ and $[0,T)$, respectively

$$\omega_{h_x} = \{x_j = a + jh_x, \quad h_x = (b-a)/n, \quad (j = 0,1,2,...,n)\},$$

$$\omega_{h_\tau} = \{t_k = kh_\tau, \quad h_\tau > 0, \quad (k = 0,1,2,...)\}.$$

Eventually, we cover the region by a uniform grid $\Omega_{h_x h_\tau} = \omega_{h_x} \times \omega_{h_\tau}$.

The need to work with discontinuous functions and find a solution that can accurately express the dynamics of the process requires the creation of a sensitive numerical method in the class of discontinuous functions. Now, we can study the techniques in discretizing the differential problem.

To find the numerical solution of problem (8), (2), let us include the following operator

$$A(\boldsymbol{.}) = \frac{\partial(\boldsymbol{.})}{\partial x}.$$

It is clear that this operator has an inverse denoting by $A^{-1}(\boldsymbol{.})$, which differs from it by a constant. Applying the operator $A^{-1}(\boldsymbol{.})$ to both sides of the Eq. (3) we get

$$A^{-1}\left(\frac{\partial u}{\partial t}\right) + A^{-1}\left(\frac{1}{2}\frac{\partial u^2}{\partial x}\right) - A^{-1}\big(u(1-u)\big) = A^{-1}(0).$$

Let $A^{-1}(0) = h(t)$, from here we have $Ah = 0$. The last equation is written as

$$\frac{\partial A^{-1}u}{\partial t} + \frac{u^2}{2} - A^{-1}\big(u(1-u)\big) = h(t), \tag{11}$$

where $h(t) \in A^{-1}(0) = \ker A = \{h(t) \in C[0,\infty) \ : \ Ah = 0\}$ is any function. We introduce the following transformation

$$A^{-1}u + h(t) = v(x,t). \tag{12}$$

From (12) we obtain

$$u(x,t) = A\big(v(x,t)\big). \tag{13}$$

Substituting the relations (12) and (13) in Eq. (11), we get

$$\frac{\partial v(x,t)}{\partial t} + \frac{1}{2}\big(u(x,t)\big)^2 - \alpha\int_{-\infty}^{x} u(\xi,t)\big(1-u(\xi,t)\big)d\xi = 0. \tag{14}$$

The initial condition for Eq. (14) is

$$v(x,0) = v_0(x). \tag{15}$$

Here the function $v_0(x)$ is any continuously differentiable solution of equation $A\big(v(x,0)\big) = u(x,0)$, which is

$$\frac{dv_0(x)}{dx} = u_0(x). \tag{16}$$

We will call the problem (14)–(15) as an auxiliary problem. In accordance with [19] and [23] we consider this special auxiliary problem in order to determine the weak solution of the problem (8), (2).

The auxiliary problem has the following advantages:

- The differentiability property of the function $v(x,t)$ is of a higher order than the differentiability property of the function $u(x,t)$,
- The function $u(x,t)$ may be a discontinuous function, as long as it is an integrable one,
- Algorithms built to calculate the function $u(x,t)$ do not require the derivative of $u(x,t)$ with respect to any variables.

Theorem 1. If the function $v(x,t)$ is a solution of the auxiliary problem (14), (15), then the function $u(x,t) = A\big(v(x,t)\big)$ is a weak solution of the main problem (8), (2).

Proof. To prove the theorem, it is sufficient to apply the operator A directly to the Eq. (14) and consider the expression in (13).

The construction of finite difference algorithms

We will apply two finite difference schemes for Eq. (14) using explicit and implicit schemes.

Explicit scheme : Firstly, let us discretize Eq. (14) as follows

$$\frac{V_{i,k+1} - V_{i,k}}{\tau} + \frac{1}{2}U_{i,k}^2 - \alpha h\sum\nolimits_{j=1}^{i} U_{j,k}(1-U_{j,k}) = 0$$

or

$$V_{i,k+1} = V_{i,k} - \tau U_{i,k}\Big[U_{i,k}\Big(\frac{1}{2} + \alpha h\Big)\Big] + \tau\alpha h\sum\nolimits_{j=1}^{i-1} U_{j,k}(1-U_{j,k}), \quad j = 1,2,...,n-1; \quad k = 0,1,2,... \tag{17}$$

where h and τ are steps of the grid for x and t variables, respectively.

The initial condition for (17) is

$$V_{j,0} = v_0(x_j), \quad j = 0, 1, 2, ..., n.$$

Here, $v_0(x)$ is the grid function corresponding to the continuous function found from Eq. (16). The validity of the following equality can be easily shown as

$$U_{i,k+1} = \frac{V_{i,k+1} - V_{i-1,k+1}}{h}.$$

Implicit scheme : Now let's write an implicit scheme for problem (14), (15). For this, let's write equation (14) as a finite difference equivalent as follows

$$V_{i,k+1} = V_{i,k} - \tau U_{i,k+1}\Big[U_{i,k+1}\Big(\frac{1}{2} + \alpha h\Big) + \alpha h\Big] + \tau\alpha h\sum_{j=1}^{i-1} U_{j,k+1}(1 - U_{j,k+1}),$$

$$j = 1, 2, ..., n; k = 0, 1, 2, ...$$

We can obtain the solution of the last system of nonlinear algebraic equations by applying Newton's method.

Computer tests

In order to compare the solutions found by the finite difference algorithm we proposed solutions in the literature, as $u_0(x)$ function e^{-x} is accepted. The calculation results are shown in Figures 1-3 accordingly.

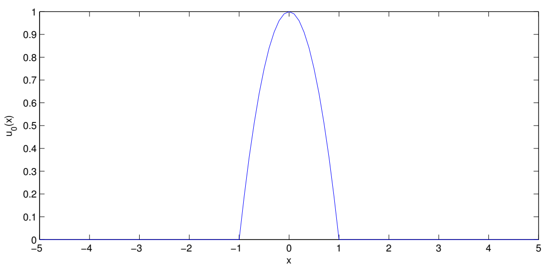

Figure 1. The source function.

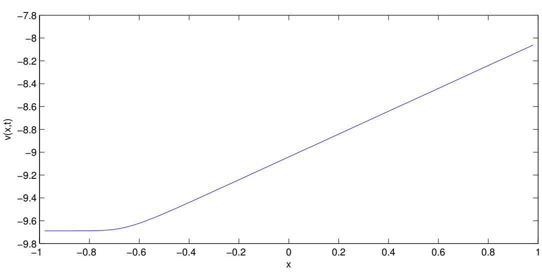

Figure 2. Graph of the solution of problem (14), (15) at $T = 1.5$.

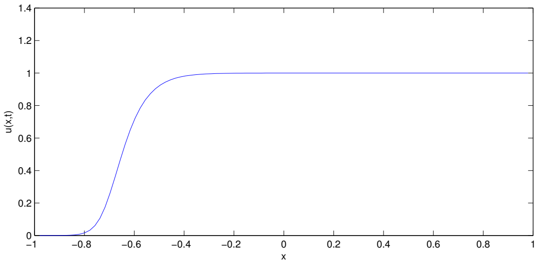

Figure 3. Graph of the function $u(x,t) = A(v(x,t))$.

3 Conclusion

The results obtained in this paper can be listed as follows:

- An original method in the class of discontinuous functions is proposed to find the numerical solution of the Cauchy problem for the first order nonlinear partial differential equation with a nonlinear source function.

- The special auxiliary problem of which the differentiable properties of the solution one order higher than the differentiable properties of the main problem is introduced.

- Using the advantages of the auxiliary problem the efficient numerical algorithms are suggested in a class of discontinuous functions. The obtained solutions express the all physical properties accurately.

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Үзілісті функциялар класындағы Бакли-Левереттің жалпыланған моделін шешудің сандық әдісі

Бакли-Левереттің жалпыланған есебін шешудің жаңа сандық әдісі ұсынылды, ол үзілісті функциялар класындағы Бажен қабатының екі фазалы қоспаларының қозғалысын сипаттайды. Ол үшін негізгі есептен артықшылығы бар көмекші есеп енгізілді және осы басымдықтардың көмегімен көмекші есепті шешу үшін ақырлы айырымдық түпнұсқа әдісі әзірленді. Ұсынылған көмекші есептің көмегімен есептің барлық физикалық сипаттамаларын дәл көрсететін шешім алынды.

Кілт сөздер: Бакли-Левереттің жалпыланған есебі, көмекші есеп, үзіліс функциялары класындағы ақырлы айырымдық әдіс.

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Численный метод решения обобщенной модели Бакли-Леверетта в классе разрывных функций

Предложен новый численный метод решения обобщенной задачи Бакли-Леверетта, описывающий движение двухфазных смесей отложений баженовской толщи в классе разрывных функций. Для этого введена вспомогательная задача, имеющая преимущества перед основной, и с помощью данных приоритетов разработана оригинальный метод конечных разностей для решения вспомогательной задачи. С помощью предложенной вспомогательной задачи получено решение, точно выражающее все физические характеристики задачи.

Ключевые слова: обобщенная задача Бакли-Леверетта, вспомогательная задача, метод конечных разностей в классе разрывных функций.

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