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G.A. Kurina[1,2,*], M.A. Kalashnikova[3]

[1] *Voronezh State University, Voronezh, Russia;*
[2] *Federal Research Center «Computer Science and Control» of RAS, Moscow, Russia;*
[3] *Auxo, Voronezh, Russia*
(E-mail: kurina@math.vsu.ru, margarita.kalashnikova@mail.ru)

Minimizing sequences for a linear-quadratic control problem with three-tempo variables under weak nonlinear perturbations

The paper deals with the construction of minimizing sequences for the problem of minimizing a weakly nonlinearly perturbed quadratic performance index on trajectories of a weakly nonlinear system with three-tempo state variables. For this purpose, the so-called direct scheme for constructing an asymptotic solution is used, which consists in immediate substituting the postulated asymptotic expansion of the solution into the problem conditions and constructing a series of optimal control problems (in the case under consideration, linear-quadratic ones), the solutions of which are terms of the asymptotic expansion of the solution of the original nonlinear control problem. An estimate is obtained for the proximity of the optimal trajectory to the trajectory of the equation of state when some asymptotic approximation to the optimal control is used as a control. An example is given that illustrates in detail the proposed scheme for constructing minimizing sequences.

Keywords: three-tempo variables, nonlinear optimal control problems, asymptotic estimates, minimizing sequences.

Introduction

Mathematical models of many real processes contain multi-tempo fast variables. In review [1], there are 74 links to publications devoted to the study of such models.

Difficulties of using numerical methods for solving differential equations with quickly changing variables are well known. Therefore the employment of asymptotic methods is sometimes more preferable. The most popular method for asymptotic solving optimal control problems is constructing an asymptotic solution of problem following from control optimality conditions [2–4]. Another method, the so called direct scheme, consists of immediate substituting a postulated asymptotic solution into the problem condition and receiving a series of problems for finding asymptotic terms. The second approach allows to establish non-increasing of performance index values, if a next optimal control approximation is used, and gives the possibility to use standard programs for solving optimal control problems for finding asymptotics terms. For two-tempo systems, it is presented, for example, in [5, 6].

The direct scheme was applied in [7,8] for asymptotic solving an optimal control problem with weak nonlinear perturbations in a quadratic performance index and a linear state equation of the following form:

$$P_\varepsilon : \ J_\varepsilon(u) = \int_0^T (1/2(w(t,\varepsilon)'W(t)w(t,\varepsilon) + u(t,\varepsilon)'R(t)u(t,\varepsilon)) + \varepsilon F(w(t,\varepsilon), u(t,\varepsilon), t, \varepsilon))\, dt \to \min_u, \tag{1}$$

$$\mathcal{E}(\varepsilon)\frac{dw(t,\varepsilon)}{dt} = A(t)w(t,\varepsilon) + B(t)u(t,\varepsilon) + \varepsilon f(w(t,\varepsilon), u(t,\varepsilon), t, \varepsilon),\ t \in [0,T], \tag{2}$$

$$w(0,\varepsilon) = w^0. \tag{3}$$

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[*]Corresponding author.
E-mail: kurina@math.vsu.ru

Here ε is a non-negative small parameter, $T > 0$ is fixed, the prime means transposition; $w(t,\varepsilon) = (x(t,\varepsilon)', y(t,\varepsilon)', z(t,\varepsilon)')'$, $x(t,\varepsilon) \in \mathbb{R}^{n_1}$, $y(t,\varepsilon) \in \mathbb{R}^{n_2}$, $z(t,\varepsilon) \in \mathbb{R}^{n_3}$, $u(t,\varepsilon) \in \mathbb{R}^m$; $\mathcal{E}(\varepsilon) = diag(I_{n_1}, \varepsilon I_{n_2}, \varepsilon^2 I_{n_3})$, I_{n_i} is the identity matrix of order n_i, $f = (\overset{(1)}{f}{}', \overset{(2)}{f}{}', \overset{(3)}{f}{}')'$, $\overset{(i)}{f} \in \mathbb{R}^{n_i}$, $B = (\overset{(1)}{B}{}', \overset{(2)}{B}{}', \overset{(3)}{B}{}')'$, $\overset{(i)}{B} : \mathbb{R}^m \to \mathbb{R}^{n_i}$, $i = \overline{1,3}$; all functions in (1), (2) are sufficiently smooth with respect to their arguments; for all $t \in [0,T]$ matrices $W(t)$, $R(t)$ are symmetric, moreover, $W(t)$, $R(t)$ and $S(t) = B(t)R(t)^{-1}B(t)'$ are positive definite.

The matrices A_{33} and $A_{22} - A_{23}A_{33}^{-1}A_{32}$ are assumed to be stable. Here and further A_{ij}, $i,j = \overline{1,3}$, mean matrices from a block representation of a matrix A with number of rows and columns n_1, n_2, n_3.

The rigorous justification of applying the direct scheme method to problem (1)–(2) is presented in [8]. The proof of estimates of the proximity between the exact solution and asymptotic one for the control, state trajectory and performance index value is also given. Moreover, this paper contains the proof of non-increasing performance index values when some new asymptotic approximations to the optimal control are used.

The construction of minimizing sequences is very important for approximate solving optimal control problems. Some facts concerning such sequences are given, for instance, in [9; 18, 22].

It should be noted that any illustrative examples are absent in [7, 8], though any example is very useful for understanding, in general, not simple algorithm of constructing minimizing sequences for problem (1)–(3). Such example is given in the present paper. A statement on estimate of the proximity between the optimal trajectory and a trajectory of system (2), (3), when some asymptotic approximation to the optimal control is used as control, is also proved here. In comparison with [8], some additional minimizing sequences are considered.

Some results of this paper were presented at the ICAAM 2022 [10].

This paper is organized as follows. For convenience, when considering an illustrative example, we present in the next section the algorithm of the direct scheme applied to problem (1)–(3) and give explicit formulas from [8] for linear-quadratic optimal control problems, solutions of which are asymptotic terms for a solution of problem (1)–(3). In section 2, we give some theorems on estimates from [8] and the proof of one theorem on an estimate of the proximity of the optimal trajectory to a trajectory of system (2), (3) under a special choice of the control. The last section is devoted to the detailed study of the first order approximation for an asymptotic solution of an illustrative example. A table containing values of the performance index for terms of constructed minimizing sequences is given.

1 *Formalism of direct scheme method with explicit forms of problems for finding asymptotics terms*

Following to the A.B. Vasil'eva's boundary function method [11], a solution of problem (1)–(3) is sought in the form

$$\vartheta(t,\varepsilon) = \overline{\vartheta}(t,\varepsilon) + \sum_{i=0}^{1}(\Pi_i\vartheta(\tau_i,\varepsilon) + Q_i\vartheta(\sigma_i,\varepsilon)), \tag{4}$$

where $\vartheta(t,\varepsilon) = (w(t,\varepsilon)', u(t,\varepsilon)')'$, $\tau_i = t/\varepsilon^{i+1}$, $\sigma_i = (t-T)/\varepsilon^{i+1}$, $i = 0,1$. Each term from (4) has an asymptotic expansion according to non-negative integer powers of the small parameter ε, i.e. $\overline{\vartheta}(t,\varepsilon) = \sum_{j\geq 0}\varepsilon^j\overline{\vartheta}_j(t)$, $\Pi_i\vartheta(\tau_i,\varepsilon) = \sum_{j\geq 0}\varepsilon^j\Pi_{ij}\vartheta(\tau_i)$, $Q_i\vartheta(\sigma_i,\varepsilon) = \sum_{j\geq 0}\varepsilon^j Q_{ij}\vartheta(\sigma_i)$. Here, $\overline{\vartheta}_j(t)$ are regular functions and $\Pi_{ij}\vartheta(\tau_i)$, $Q_{ij}\vartheta(\sigma_i)$ are boundary functions of exponential type in neighborhoods $t = 0$ and $t = T$ respectively.

For any sufficiently smooth function $G(w(t,\varepsilon), u(t,\varepsilon), t, \varepsilon)$ we will use the notation $G(\vartheta(t,\varepsilon), t, \varepsilon)$

and the asymptotic representation

$$G(\vartheta,t,\varepsilon)=\overline{G}(t,\varepsilon)+\sum_{i=0}^{1}(\Pi_iG(\tau_i,\varepsilon)+Q_iG(\sigma_i,\varepsilon)), \tag{5}$$

$\overline{G}(t,\varepsilon)=G(\overline{\vartheta}(t,\varepsilon),t,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\overline{G}_j(t)$, $\Pi_0G(\tau_0,\varepsilon)=G(\overline{\vartheta}(\varepsilon\tau_0,\varepsilon)+\Pi_0\vartheta(\tau_0,\varepsilon),\varepsilon\tau_0,\varepsilon)-$ $-G(\overline{\vartheta}(\varepsilon\tau_0,\varepsilon),\varepsilon\tau_0,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\Pi_{0j}G(\tau_0)$, $\Pi_1G(\tau_1,\varepsilon)=G(\overline{\vartheta}(\varepsilon^2\tau_1,\varepsilon)+\Pi_0\vartheta(\varepsilon\tau_1,\varepsilon)+$ $+\Pi_1\vartheta(\tau_1,\varepsilon),\varepsilon^2\tau_1,\varepsilon)-G(\overline{\vartheta}(\varepsilon^2\tau_1,\varepsilon)+\Pi_0\vartheta(\varepsilon\tau_1,\varepsilon),\varepsilon^2\tau_1,\varepsilon)=\sum_{j\geq 0}\varepsilon^j\Pi_{1j}G(\tau_1)$, $Q_0G(\sigma_0,\varepsilon)=G(\overline{\vartheta}(T+$ $\varepsilon\sigma_0,\varepsilon)+Q_0\vartheta(\sigma_0,\varepsilon),T+\varepsilon\sigma_0,\varepsilon)-G(\overline{\vartheta}(T+\varepsilon\sigma_0,\varepsilon),T+\varepsilon\sigma_0,\varepsilon)=\sum_{j\geq 0}\varepsilon^jQ_{0j}G(\sigma_0)$, $Q_1G(\sigma_1,\varepsilon)=$ $G(\overline{\vartheta}(T+\varepsilon^2\sigma_1,\varepsilon)+Q_0\vartheta(\varepsilon\sigma_1,\varepsilon)+Q_1\vartheta(\sigma_1,\varepsilon),T+\varepsilon^2\sigma_1,\varepsilon)-G(\overline{\vartheta}(T+\varepsilon^2\sigma_1,\varepsilon)+Q_0\vartheta(\varepsilon\sigma_1,\varepsilon),T+\varepsilon^2\sigma_1,\varepsilon)=$ $\sum_{j\geq 0}\varepsilon^jQ_{1j}G(\sigma_1)$.

The first step of the algorithm of the direct scheme method consists of the substitution of expansion (4) into problem condition (1)–(3) taking into account (5). Equating in the transformed expressions for (2),(3) terms of the same powers of ε, separately depending on t, τ_i, σ_i, $i=0,1$, we obtain relations for defining asymptotics terms. Whence, in particular, it follows that

$$E_1\Pi_{00}w(\tau_0)=0,\ E_1\Pi_{10}w(\tau_1)=E_1\Pi_{11}w(\tau_1)=0,\ E_1Q_{00}w(\sigma_0)=0,$$
$$E_1Q_{10}w(\sigma_1)=E_1Q_{11}w(\sigma_1)=0,\ E_2\Pi_{10}w(\tau_1)=0,\ E_2Q_{10}w(\sigma_1)=0.$$

With the help of passing in the integrals from the expressions depending on τ_i, σ_i, $i=0,1$, to integrals over the corresponding intervals $[0,+\infty)$ and $(-\infty,0]$, in the transformed integrand from (1) the functional $J_\varepsilon(u)$ is written in the form

$$J_\varepsilon(u)=\sum_{j\geq 0}\varepsilon^jJ_j. \tag{6}$$

Analyzing the structure of coefficients J_j with even and odd indices separately, five linear-quadratic optimal control problems $\overline{P}_j$, $\Pi_{ij}P$, $Q_{ij}P$, $i=0,1$, solutions of which are terms of asymptotic solution of problem (1)–(3), are formulated in [8]. Further, the explicit formulas for these problems will be given.

Let us introduce the following notation:

$$E_1=diag(I_{n_1},0,0),\ E_2=diag(0,I_{n_2},0),\ E_3=diag(0,0,I_{n_3}),$$
$$\phi(\vartheta,t,\varepsilon)=A(t)w(t,\varepsilon)+B(t)u(t,\varepsilon)+\varepsilon f(w(t,\varepsilon),u(t,\varepsilon),t,\varepsilon),$$
$$\rho(\vartheta,\psi,t,\varepsilon)=W(t)w(t,\varepsilon)-A(t)'\psi(t,\varepsilon)+\varepsilon(F_w(\vartheta,t,\varepsilon)'-f_w(\vartheta,t,\varepsilon)'\psi(t,\varepsilon)),$$
$$\chi(\vartheta,\psi,t,\varepsilon)=R(t)u(t,\varepsilon)-B(t)'\psi(t,\varepsilon)+\varepsilon(F_u(\vartheta,t,\varepsilon)'-f_u(\vartheta,t,\varepsilon)'\psi(t,\varepsilon)).$$

The coefficient with ε^j in an expansion of a function $\omega=\omega(\varepsilon)$ in a series in powers of ε will be denoted by w_j or $[w]_j$. The k-th partial sum of a series will denoted by upper wave and the low index k, i.e. $\widetilde{\omega}_k=\sum_{j=0}^{k}\varepsilon^j\omega_j$. The hat and the low index k in a function notation will be mean that the function is calculated with the functional argument equal to the k-th partial sum of the corresponding expansion, e.g., $\widehat{\overline{f}}_k(t,\varepsilon)=f(\widetilde{\overline{\vartheta}}_k(t,\varepsilon),t,\varepsilon)$. Functions with negative indices will be considered equal to zero.

In the following expressions with ρ and χ in the performance indices of the formulated linear-quadratic optimal control problems we take $\psi(t,\varepsilon)=\sum_{j=0}^{\infty}\varepsilon^j(\overline{\psi}_j(t)+(\varepsilon E_1+E_2+E_3)(\Pi_{0j}\psi(\tau_0)+$ $Q_{0j}\psi(\sigma_0))+(\varepsilon^2E_1+\varepsilon E_2+E_3)(\Pi_{1j}\psi(\tau_1)+Q_{1j}\psi(\sigma_1)))$, where $\overline{\psi}_j$, $\Pi_{ij}\psi(\tau_i)$, $Q_{ij}\psi(\sigma_i)$, $i=0,1$, are costate variables in problems $\overline{P}_j$, $\Pi_{ij}P$, $Q_{ij}P$, $i=0,1$, respectively.

Regular functions $\overline{\vartheta}_j(t)$, $t\in[0,T]$, are solutions of the following problems

$$\overline{P}_j:\ \overline{J}_j(\overline{u}_j)=\overline{w}_j(T)'E_1(Q_{0(j-1)}\psi(0)+Q_{1(j-2)}\psi(0))+\int\limits_0^T(\overline{w}_j(t)'(\frac{1}{2}W(t)\overline{w}_j(t)+$$
$$+[\widehat{\overline{\rho}}_{j-1}(t,\varepsilon)]_j-E_2\frac{d\overline{\psi}_{j-1}(t)}{dt}-E_3\frac{d\overline{\psi}_{j-2}(t)}{dt})+\overline{u}_j(t)'(\frac{1}{2}R(t)\overline{u}_j(t)+[\widehat{\overline{\chi}}_{j-1}(t,\varepsilon)]_j))\,dt\to\min_{\overline{u}_j}, \tag{7}$$

$$E_1\frac{d\overline{w}_j(t)}{dt}+E_2\frac{d\overline{w}_{j-1}(t)}{dt}+E_3\frac{d\overline{w}_{j-2}(t)}{dt}=[\overline{\phi}(t,\varepsilon)]_j, \tag{8}$$

$$E_1\overline{w}_j(0)=E_1w^0,\ j=0,\ E_1(\overline{w}_j(0)+\Pi_{0j}w(0))=0,\ j=1,$$
$$E_1(\overline{w}_j(0)+\Pi_{0j}w(0)+\Pi_{1j}w(0))=0,\ j\ge 2. \tag{9}$$

The boundary functions $\Pi_{0j}\vartheta(\tau_0)$, $\tau_0\in[0,+\infty)$, are solutions of the problems

$$\Pi_{0j}P:\ \Pi_{0j}J(\Pi_{0j}u)=\int\limits_0^{+\infty}(\Pi_{0j}w(\tau_0)'(\frac{1}{2}W(0)\Pi_{0j}w(\tau_0)+[\widehat{\Pi}_{0(j-1)}\rho(\tau_0,\varepsilon)]_j-E_3\frac{d\Pi_{0(j-1)}\psi(\tau_0)}{d\tau_0})+$$
$$+\Pi_{0j}u(\tau_0)'(\frac{1}{2}R(0)\Pi_{0j}u(\tau_0)+[\widehat{\Pi}_{0(j-1)}\chi(\tau_0,\varepsilon)]_j))\,d\tau_0\to\min_{\Pi_{0j}u}, \tag{10}$$

$$(E_1+E_2)\frac{d\Pi_{0j}w(\tau_0)}{d\tau_0}+E_3\frac{d\Pi_{0(j-1)}w(\tau_0)}{d\tau_0}=E_1[\Pi_0\phi(\tau_0,\varepsilon)]_{j-1}+(E_2+E_3)[\Pi_0\phi(\tau_0,\varepsilon)]_j, \tag{11}$$

$$\Pi_{0j}x(+\infty)=0,\ E_2(\overline{w}_j(0)+\Pi_{0j}w(0))=E_2w^0,\ j=0,$$
$$E_2(\overline{w}_j(0)+\Pi_{0j}w(0)+\Pi_{1j}w(0))=0,\ j\ge 1. \tag{12}$$

The boundary functions $Q_{0j}\vartheta(\sigma_0)$, $\sigma_0\in(-\infty,0]$, are solutions of the problems

$$Q_{0j}P:\ Q_{0j}J(Q_{0j}u)=Q_{0j}w(0)'E_2(\overline{\psi}_j(T)+Q_{1(j-1)}\psi(0))+$$
$$+\int\limits_{-\infty}^{0}(Q_{0j}w(\sigma_0)'(\frac{1}{2}W(T)Q_{0j}w(\sigma_0)+[\widehat{Q}_{0(j-1)}\rho(\sigma_0,\varepsilon)]_j-E_3\frac{dQ_{0(j-1)}\psi(\sigma_0)}{d\sigma_0})+$$
$$+Q_{0j}u(\sigma_0)'(\frac{1}{2}R(T)Q_{0j}u(\sigma_0)+[\widehat{Q}_{0(j-1)}\chi(\sigma_0,\varepsilon)]_j))\,d\sigma_0\to\min_{Q_{0j}u}, \tag{13}$$

$$(E_1+E_2)\frac{dQ_{0j}w(\sigma_0)}{d\sigma_0}+E_3\frac{dQ_{0(j-1)}w(\sigma_0)}{d\sigma_0}=E_1[Q_0\phi(\sigma_0,\varepsilon)]_{j-1}+(E_2+E_3)[Q_0\phi(\sigma_0,\varepsilon)]_j, \tag{14}$$

$$(E_1+E_2)Q_{0j}w(-\infty)=0. \tag{15}$$

The boundary functions $\Pi_{1j}\vartheta(\tau_1)$, $\tau_1\in[0,+\infty)$ are solutions of the problems

$$\Pi_{1j}P:\ \Pi_{1j}J(\Pi_{1j}u)=\int\limits_0^{+\infty}(\Pi_{1j}w(\tau_1)'(\frac{1}{2}W(0)\Pi_{1j}w(\tau_1)+[\widehat{\Pi}_{1(j-1)}\rho(\tau_1,\varepsilon)]_j)+$$
$$+\Pi_{1j}u(\tau_1)'(\frac{1}{2}R(0)\Pi_{1j}u(\tau_1)+[\widehat{\Pi}_{1(j-1)}\chi(\tau_1,\varepsilon)]_j))\,d\tau_1\to\min_{\Pi_{1j}u}, \tag{16}$$

$$\frac{d\Pi_{1j}w(\tau_1)}{d\tau_1}=E_1[\Pi_1\phi(\tau_1,\varepsilon)]_{j-2}+E_2[\Pi_1\phi(\tau_1,\varepsilon)]_{j-1}+E_3[\Pi_1\phi(\tau_1,\varepsilon)]_j, \tag{17}$$

$$(E_1+E_2)\Pi_{1j}w(+\infty)=0,\ E_3(\overline{w}_j(0)+\Pi_{0j}w(0)+\Pi_{1j}w(0))=\begin{cases}E_3w^0,\ j=0,\\ 0,\ j\geq 1.\end{cases} \tag{18}$$

The boundary functions $Q_{1j}\vartheta(\sigma_1)$, $\sigma_1\in(-\infty,0]$ are solutions of the problems

$$Q_{1j}P:\ Q_{1j}J(Q_{1j}u)=Q_{1j}w(0)'E_3(\overline{\psi}_j(T)+Q_{0j}\psi(0))+\int\limits_{-\infty}^{0}(Q_{1j}w(\sigma_1)'(\frac{1}{2}W(T)Q_{1j}w(\sigma_1)+$$
$$+[\widehat{Q}_{1(j-1)}\rho(\sigma_1,\varepsilon)]_j)+Q_{1j}u(\sigma_1)'(\frac{1}{2}R(T)Q_{1j}u(\sigma_1)+[\widehat{Q}_{1(j-1)}\chi(\sigma_1,\varepsilon)]_j))\,d\sigma_1\to\min_{Q_{1j}u}, \tag{19}$$

$$\frac{dQ_{1j}w(\sigma_1)}{d\sigma_1}=E_1[Q_1\phi(\sigma_1,\varepsilon)]_{j-2}+E_2[Q_1\phi(\sigma_1,\varepsilon)]_{j-1}+E_3[Q_1\phi(\sigma_1,\varepsilon)]_j, \tag{20}$$

$$Q_{1j}w(-\infty)=0. \tag{21}$$

2 Asymptotic estimates

Let eigenvalues of the matrix $\begin{pmatrix}A_{33} & S_{33}\\ W_{33} & -A_{33}'\end{pmatrix}$ are different for all $t\in[0,T]$ (condition I from [8]) and the same condition is satisfied for the matrix
$\begin{pmatrix}A_{22} & S_{22}\\ W_{22} & -A_{22}'\end{pmatrix}-\begin{pmatrix}A_{23} & S_{23}\\ W_{23} & -A_{32}'\end{pmatrix}\begin{pmatrix}A_{33} & S_{33}\\ W_{33} & -A_{33}'\end{pmatrix}^{-1}\begin{pmatrix}A_{32} & S_{23}'\\ W_{23}' & -A_{23}'\end{pmatrix}$ (condition II from [8]).

Under these conditions, the following Theorems 1–3 have been proved in [8].

Theorem 1. Solution $\vartheta_*(t,\varepsilon)$ of problem P_ε for sufficiently small $\varepsilon>0$, $t\in[0,T]$, satisfies the inequality

$$\|\vartheta_*(t,\varepsilon)-\widetilde{\vartheta}_n(t,\varepsilon)\|\leq c\varepsilon^{n+1}.$$

Here and further c is a positive constant independent of t, ε.

Note a slip of the presentation of Theorem 1 in [8], namely, the condition II for this theorem has been formulated inside of the theorem proof.

Theorem 2. For sufficiently small $\varepsilon>0$, the following inequality for the performance index is valid

$$J_\varepsilon(\widetilde{u}_n)-J_\varepsilon(u_*)\leq c\varepsilon^{2n+2}.$$

Theorem 3. For sufficiently small $\varepsilon>0$, the following inequalities are valid

$$J_\varepsilon(\widetilde{u}_{*(n-1)})\geqslant J_\varepsilon(\widetilde{u}_{*(n-1)}+\varepsilon^n\overline{u}_{*n})\geqslant$$
$$\geqslant J_\varepsilon(\widetilde{u}_{*(n-1)}+\varepsilon^n(\overline{u}_{*n}+\Pi_{0n}u_*+Q_{0n}u_*))\geqslant J_\varepsilon(\widetilde{u}_{*n}),\ n\geqslant 1.$$

If an addition to $\widetilde{u}_{*(n-1)}$ is non-zero, then the corresponding inequality is strict.

Here the notation $\widetilde{u}_*$, $j=n,n-1$ is used for the j-th order approximation for the optimal control u_*.

Denote by $\widetilde{w}=\widetilde{w}(t,\varepsilon)$ a solution of problem (2)–(3) at $u=\widetilde{u}_{*n}$ and $\delta w=\delta w(t,\varepsilon)=w_*(t,\varepsilon)-\widetilde{w}(t,\varepsilon)$, $\delta u=\delta u(t,\varepsilon)=u_*(t,\varepsilon)-\widetilde{u}_{*n}(t,\varepsilon)$.

Under proving Theorem 2 in [8], the estimate for $\delta w(t,\varepsilon)$ has been used without the rigorous proof. The proof of it will be given below, i.e. we will prove the following.

Theorem 4. For sufficiently small $\varepsilon>0$, the inequality

$$\|\delta w(t,\varepsilon)\|\leqslant c\varepsilon^{n+1} \tag{22}$$

is fulfilled.

Proof. It follows from (2), (3), that δw satisfies the system

$$\mathcal{E}(\varepsilon)\frac{d\delta w(t,\varepsilon)}{dt} = A(t)\delta w(t,\varepsilon) + B(t)\delta u(t,\varepsilon) + \varepsilon(f(w_*(t,\varepsilon), u_*(t,\varepsilon), t, \varepsilon) - $$
$$-f(w_*(t,\varepsilon) - \delta w(t,\varepsilon), u_*(t,\varepsilon) - \delta u(t,\varepsilon), t, \varepsilon)),$$
$$\delta w(0,\varepsilon) = 0.$$

In view of Theorem 1

$$\|\delta u(t,\varepsilon)\| \leqslant c\varepsilon^{n+1}. \tag{23}$$

Write out the problem for $\delta w = (\delta x', \delta y', \delta z')'$ in the form

$$\frac{d\delta x}{dt} = A_{11}(t)\delta x + A_{12}(t)\delta y + A_{13}(t)\delta z + \overset{(1)}{B}(t)\delta u + \varepsilon \overset{(1)}{g}(\delta w, \delta u, t, \varepsilon),\ \delta x(0,\varepsilon) = 0, \tag{24}$$

$$\varepsilon\frac{d\delta y}{dt} = A_{21}(t)\delta x + A_{22}(t)\delta y + A_{23}(t)\delta z + \overset{(2)}{B}(t)\delta u + \varepsilon \overset{(2)}{g}(\delta w, \delta u, t, \varepsilon),\ \delta y(0,\varepsilon) = 0, \tag{25}$$

$$\varepsilon^2\frac{d\delta z}{dt} = A_{31}(t)\delta x + A_{32}(t)\delta y + A_{33}(t)\delta z + \overset{(3)}{B}(t)\delta u + \varepsilon \overset{(3)}{g}(\delta w, \delta u, t, \varepsilon),\ \delta z(0,\varepsilon) = 0, \tag{26}$$

where $\overset{(i)}{g}(\delta w, \delta u, t, \varepsilon) = \varepsilon(\overset{(i)}{f}(w_*, u_*, t, \varepsilon) - \overset{(i)}{f}(w_* - \delta w, u_* - \delta u, t, \varepsilon))$, $i = \overline{1,3}$.

For brevity, the arguments t, ε are dropped in some of the last relations.

Using the fundamental matrix $\overset{(3)}{U}(t,s,\varepsilon)$ of the system

$$\varepsilon^2\frac{dZ}{dt} = A_{33}(t)Z, \tag{27}$$

we write out the problem (26) in the integral form

$$\delta z(t,\varepsilon) = \frac{1}{\varepsilon^2}\int_0^t \overset{(3)}{U}(t,s,\varepsilon)(A_{31}(s)\delta x(s,\varepsilon) + A_{32}(s)\delta y(s,\varepsilon) + \overset{(3)}{B}(s)\delta u(s,\varepsilon) + $$
$$+\varepsilon\overset{(3)}{g}(\delta w(s,\varepsilon), \delta u(s,\varepsilon), s, \varepsilon))\,ds. \tag{28}$$

Due to stability of the matrix $A_{33}(t)$ the matrix $\overset{(3)}{U}(t,s,\varepsilon)$ has the estimate [10; 69]

$$\|\overset{(3)}{U}(t,s,\varepsilon)\| \leqslant c\exp(-\frac{æ(t-s)}{\varepsilon^2}), \tag{29}$$

where $0 \leqslant s \leqslant t \leqslant T$ and here and further æ is a positive constant independent of t, ε.

In the following, we will denote functions, appearing under transformations of problems (24)–(26) and satisfying the next two conditions 1) and 2) by $\overset{(i)}{h}(\delta w, t, \varepsilon)$, $i = \overline{1,3}$. Specific forms of these functions are omitted since they are insignificant for the proof.

1) For any $q > 0$, there exist such constants $\Delta = \Delta(q)$ and $\varepsilon_0 = \varepsilon_0(q)$ that, for $\|\delta w_i\|_{C_{[0,T]}} \leqslant \Delta$, $i = \overline{1,2}$, $0 < \varepsilon \leqslant \varepsilon_0$

$$\|\overset{(i)}{h}(\delta w_1, t, \varepsilon) - \overset{(i)}{h}(\delta w_2, t, \varepsilon)\| \leqslant q\|\delta w_1 - \delta w_2\|_{C_{[0,T]}},$$

2) $\|\overset{(i)}{h}(0,t,\varepsilon)\| \leqslant c\varepsilon^{n+1}$.

In view of (27), taking into account that $\overset{(3)}{U}(t,s,\varepsilon) = \overset{(3)}{U}(t,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1}$, we have

$$\varepsilon^2 \frac{\partial \overset{(3)}{U}(t,s,\varepsilon)}{\partial s} = -\varepsilon^2 \overset{(3)}{U}(t,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1}\frac{d\overset{(3)}{U}(s,\varepsilon)}{ds}\overset{(3)}{U}(s,\varepsilon)^{-1} =$$
$$= -\overset{(3)}{U}(t,s,\varepsilon)A_{33}(s)\overset{(3)}{U}(s,\varepsilon)\overset{(3)}{U}(s,\varepsilon)^{-1} = -\overset{(3)}{U}(t,s,\varepsilon)A_{33}(s).$$

It follows from here that

$$\overset{(3)}{U}(t,s,\varepsilon) = -\varepsilon^2 \frac{\partial \overset{(3)}{U}(t,s,\varepsilon)}{\partial s}A_{33}(s)^{-1}. \tag{30}$$

Substituting this expression into (28), then integrating by parts the terms, containing δx and δy, due to the initial values $\delta y(0,\varepsilon)$, $\delta x(0,\varepsilon)$ and the estimates (23), (29) we obtain

$$\delta z(t,\varepsilon) = -A_{33}(t)^{-1}A_{31}(t)\delta x(t,\varepsilon) - A_{33}(t)^{-1}A_{32}(t)\delta y(t,\varepsilon) + \overset{(3)}{h}(\delta w,t,\varepsilon). \tag{31}$$

In view of the last expression, we get from (25) the problem

$$\varepsilon\frac{d\delta y}{dt} = (A_{21}(t) - A_{23}(t)A_{33}(t)^{-1}A_{31}(t))\delta x + (A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))\delta y +$$
$$+ \overset{(2)}{h}(\delta w,t,\varepsilon),\ \delta y(0,\varepsilon) = 0. \tag{32}$$

Using the fundamental matrix $\overset{(2)}{U}(t,s,\varepsilon)$ of the system

$$\varepsilon\frac{dY}{dt} = (A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))Y,$$

we write out the problem (32) in the integral form

$$\delta y(t,\varepsilon) = \frac{1}{\varepsilon}\int\limits_0^t \overset{(2)}{U}(t,s,\varepsilon)((A_{21}(s) - A_{23}(s)A_{33}(s)^{-1}A_{31}(s))\delta x(s,\varepsilon) + \overset{(2)}{h}(\delta w,s,\varepsilon))\,ds. \tag{33}$$

Due to stability of the matrix $A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t)$ the matrix $\overset{(2)}{U}(t,s,\varepsilon)$ has the estimate [10; 69].

$$\|\overset{(2)}{U}(t,s,\varepsilon)\| \leqslant c\exp(-\frac{æ(t-s)}{\varepsilon}),\ 0 \leqslant s \leqslant t \leqslant T. \tag{34}$$

Analogously to (30) we get the relation

$$\overset{(2)}{U}(t,s,\varepsilon) = -\varepsilon\frac{\partial \overset{(2)}{U}(t,s,\varepsilon)}{\partial s}(A_{22}(s) - A_{23}(s)A_{33}(s)^{-1}A_{32}(s))^{-1}.$$

Substituting this expression into (33), then integrating by parts the term containing δx, due to the initial value $\delta x(0,\varepsilon)$ and the estimates (23), (34) we obtain

$$\delta y(t,\varepsilon) = -(A_{22}(t) - A_{23}(t)A_{33}(t)^{-1}A_{32}(t))^{-1}(A_{21}(t) - A_{23}(t)A_{33}(t)^{-1}A_{31}(t))\delta x(t,\varepsilon) +$$
$$+ \overset{(2)}{h}(\delta w,t,\varepsilon). \tag{35}$$

Substituting the expressions (31), (35) into equation (24), we obtain the problem of the form

$$\frac{d\delta x}{dt} = C(t)\delta x + \overset{(1)}{h}(\delta w, t, \varepsilon),\ \delta x(0,\varepsilon) = 0. \tag{36}$$

The explicit expression for $C(t)$ can be easy written taking into account (24), (31), (35).

The fundamental matrix $\overset{(1)}{U}(t,s)$ of the system $\frac{dX}{dt} = C(t)X$ is bounded, i.e. $\|\overset{(1)}{U}(t,s)\| \leqslant c$. Therefore, writing the problem (36) in an integral form, we get

$$\delta x(t,\varepsilon) = \overset{(1)}{h}(\delta w, t, \varepsilon).$$

So, from the last relation and (31), (35) it follows that system (24)–(26) can be written in the form

$$\delta w = h(\delta w, t, \varepsilon), \tag{37}$$

where h satisfies to the conditions 1) and 2).

If we take in condition 2) $q < 1$ then h willbe a contraction mapping in $C_{[0,T]}$. According to the contractive mapping principle, equation (37) has a unique solution, and this solution can be found by the method of successive approximations.

In view of condition 2) and Lemma 3 in [8] we obtain the provable estimate (22).

The proof of Theorem 3 in [8] is based on the following theorem on the structure of coefficients J_j in (6) proven in [8] .

Theorem 5. The sum $\overline{J}_j + \Pi_{1(j-1)}J + Q_{1(j-1)}J$ of the performance indices in problems $\overline{P}_j$, $\Pi_{1(j-1)}P$, $Q_{1(j-1)}P$ is obtained by transforming the coefficient J_{2j} in expansion (6) and dropping terms, which are known after solving problems $\overline{P}_k$, $\Pi_{0k}P$, $Q_{0k}P$, $k = \overline{0, j-1}$, $\Pi_{1k}P$, $Q_{1k}P$, $k = \overline{0, j-2}$. The sum $\Pi_{0j}J + Q_{0j}J$ of the performance indices in problems $\Pi_{0j}P$, $Q_{0j}P$ is obtained by transforming the coefficient $J_{2(j+1)}$ in expansion (6) and dropping terms, which are known after solving problems $\overline{P}_k$, $k = \overline{0, j}$, $\Pi_{ik}P$, $Q_{ik}P$, $i = 0, 1$, $k = \overline{0, j-1}$.

Similarly, using Theorem 5, we can establish some generalization of Theorem 3.

Theorem 6. For sufficiently small $\varepsilon > 0$, $\{u_{*(n-1)}\}$ and the sequence with terms, obtained by supplementing to $\widetilde{u}_{*(n-1)}$ one or several terms from the expansions (4) for the optimal control u_* with ε^n are minimizing.

Detailing, $\{\widetilde{u}_{*(n-1)}\}$ and the sequence $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \overline{u}_{*n}\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \Pi_{0n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n Q_{0n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n \Pi_{1n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n Q_{1n}u_*\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + Q_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\Pi_{0n}u_* + Q_{1n}u_*)\}$, ..., $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{1n}u_*)\}$,..., $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_* + \Pi_{1n}u_*)\}$, $\{\widetilde{u}_{*(n-1)} + \varepsilon^n(\overline{u}_{*n} + \Pi_{0n}u_* + Q_{0n}u_* + Q_{1n}u_*)\}$ are minimizing.

3 Illustrative example

Let us consider the problem P_ε with $n_i = 1$, $u = (\overset{(1)}{u}, \overset{(2)}{u}, \overset{(3)}{u})'$, $\overset{(i)}{u} \in \mathbb{R}$, $i = \overline{1,3}$, of the form

$$\begin{gathered} J_\varepsilon(u) = \int_0^1 \Big(\frac{1}{2}(x(t,\varepsilon)^2 + y(t,\varepsilon)^2 + z(t,\varepsilon)^2 + \overset{(1)}{u}(t,\varepsilon)^2 + \overset{(2)}{u}(t,\varepsilon)^2 + \overset{(3)}{u}(t,\varepsilon)^2) + \\ + \varepsilon x(t,\varepsilon)\overset{(1)}{u}(t,\varepsilon) + \varepsilon y(t,\varepsilon)\overset{(3)}{u}(t,\varepsilon)\Big)\, dt \to \min_u, \end{gathered} \tag{38}$$

$$\frac{dx(t,\varepsilon)}{dt} = x(t,\varepsilon) + \overset{(1)}{u}(t,\varepsilon),$$
$$\varepsilon\frac{dy(t,\varepsilon)}{dt} = -y(t,\varepsilon) + \overset{(2)}{u}(t,\varepsilon) + \varepsilon\overset{(3)}{u}(t,\varepsilon), \tag{39}$$
$$\varepsilon^2\frac{dz(t,\varepsilon)}{dt} = -z(t,\varepsilon) + \overset{(3)}{u}(t,\varepsilon) + \varepsilon y(t,\varepsilon),$$
$$x(0,\varepsilon) = y(0,\varepsilon) = z(0,\varepsilon) = 10. \tag{40}$$

By setting $\varepsilon = 0$ in (38)–(40), taking into account the equalities $\Pi_{i0}x(\tau_i) = 0$, $i = 0, 1$, we obtain degenerate problem $\overline{P}_0$:

$$\overline{J}_0(\overline{u}_0) = \frac{1}{2}\int_0^1 (\overline{x}_0(t)^2 + \overline{y}_0(t)^2 + \overline{z}_0(t)^2 + \overset{(1)}{\overline{u}}_0(t)^2 + \overset{(2)}{\overline{u}}_0(t)^2 + \overset{(3)}{\overline{u}}_0(t)^2)\,dt \to \min_{\overline{u}_0},$$

$$\frac{d\overline{x}_0(t)}{dt} = \overline{x}_0(t) + \overset{(1)}{\overline{u}}_0(t),\ \overline{x}_0(0) = 10,$$
$$0 = -\overline{y}_0(t) + \overset{(2)}{\overline{u}}_0(t),$$
$$0 = -\overline{z}_0(t) + \overset{(3)}{\overline{u}}_0(t).$$

The form of this problem follows also from (7)–(9) with $j = 0$. It is not difficult to find the solution

$$\overline{x}_0(t) = 2a((\sqrt{2}+1)e^{\sqrt{2}t} + (\sqrt{2}-1)e^{-\sqrt{2}(t-2)}),\ \overset{(1)}{\overline{u}}_0(t) = 2a(e^{\sqrt{2}t} - e^{-\sqrt{2}(t-2)}),$$
$$\text{where } a = 5/((\sqrt{2}-1)e^{2\sqrt{2}} + \sqrt{2}+1),$$
$$\overline{y}_0(t) = \overline{z}_0(t) = \overset{(2)}{\overline{u}}_0(t) = \overset{(3)}{\overline{u}}_0(t) = 0.$$

Using (10)–(12) at $j = 0$ and taking into account that $\Pi_{00}x(\tau_0) = 0$, we write out problem $\Pi_{00}P$ in the following form:

$$\Pi_{00}J(\Pi_{00}u) = \frac{1}{2}\int_0^{+\infty} (\Pi_{00}y(\tau_0)^2 + \Pi_{00}z(\tau_0)^2 + \Pi_{00}\overset{(1)}{u}(\tau_0)^2 + \Pi_{00}\overset{(2)}{u}(\tau_0)^2 + \Pi_{00}\overset{(3)}{u}(\tau_0)^2)\,d\tau_0 \to \min_{\Pi_{00}u},$$

$$\frac{d\Pi_{00}y(\tau_0)}{d\tau_0} = -\Pi_{00}y(\tau_0) + \Pi_{00}\overset{(2)}{u}(\tau_0),\ \overline{y}_0(0) + \Pi_{00}y(0) = 10,$$
$$0 = -\Pi_{00}z(\tau_0) + \Pi_{00}\overset{(3)}{u}(\tau_0).$$

Using (13)–(15) at $j = 0$ and taking into account that $Q_{00}x(\sigma_0) = 0$, we write out problem $Q_{00}P$ in the following form:

$$Q_{00}J(Q_{00}u) = Q_{00}y(0)\overline{\eta}_0(1) + \frac{1}{2}\int_{-\infty}^0 (Q_{00}y(\sigma_0)^2 + Q_{00}z(\sigma_0)^2 + Q_{00}\overset{(1)}{u}(\sigma_0)^2 +$$
$$+ Q_{00}\overset{(2)}{u}(\sigma_0)^2 + Q_{00}\overset{(3)}{u}(\sigma_0)^2)\,d\sigma_0 \to \min_{Q_{00}u},$$

$$\frac{dQ_{00}y(\sigma_0)}{d\sigma_0} = -Q_{00}y(\sigma_0) + Q_{00}\overset{(2)}{u}(\sigma_0),$$
$$0 = -Q_{00}z(\sigma_0) + Q_{00}\overset{(3)}{u}(\sigma_0),$$

$$Q_{00}y(-\infty) = 0.$$

In this example, $\overline{\psi}_j(t) = (\overline{\xi}_j(t), \overline{\eta}_j(t), \overline{\zeta}_j(t))'$ means a costate variable for the problem $\overline{P}_j$, $\Pi_{ij}\psi(\tau_i) = (\Pi_{ij}\xi(\tau_i), \Pi_{ij}\eta(\tau_i), \Pi_{ij}\zeta(\tau_i))'$ is a costate variable for the problem $\Pi_{ij}P$, and $Q_{ij}\psi(\sigma_i) = (Q_{ij}\xi(\sigma_i), Q_{ij}\eta(\sigma_i), Q_{ij}\zeta(\sigma_i))'$ is a costate variable for the problem $Q_{ij}P$, $i = 0, 1$.

Using (16)–(18) at $j = 0$ and taking into account that $\Pi_{10}x(\tau_1) = \Pi_{10}y(\tau_1) = 0$, we write out problem $\Pi_{10}P$ in the following form:

$$\Pi_{10}J(\Pi_{10}u) = \frac{1}{2}\int\limits_0^{+\infty} (\Pi_{10}z(\tau_1)^2 + \Pi_{10}\overset{(1)}{u}(\tau_1)^2 + \Pi_{10}\overset{(2)}{u}(\tau_1)^2 + \Pi_{10}\overset{(3)}{u}(\tau_1)^2)\, d\tau_1 \to \min_{\Pi_{10}u},$$

$$\frac{d\Pi_{10}z(\tau_1)}{d\tau_1} = -\Pi_{10}z(\tau_1) + \Pi_{10}\overset{(3)}{u}(\tau_1),$$
$$\overline{z}_0(0) + \Pi_{00}z(0) + \Pi_{10}z(0) = 10.$$

Using (19)–(21) at $j = 0$ and taking into account that $Q_{10}x(\sigma_1) = Q_{10}y(\sigma_1) = 0$, we write out problem $Q_{10}P$ in the following form:

$$Q_{10}J(Q_{10}u) = Q_{10}z(0)(\overline{\zeta}_0(1) + Q_{00}\zeta(0)) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{10}z(\sigma_1)^2 + Q_{10}\overset{(1)}{u}(\sigma_1)^2 +$$
$$+ Q_{10}\overset{(2)}{u}(\sigma_1)^2 + Q_{10}\overset{(3)}{u}(\sigma_1)^2)\, d\sigma_1 \to \min_{Q_{10}u},$$

$$\frac{dQ_{10}z(\sigma_1)}{d\sigma_1} = -Q_{10}z(\sigma_1) + Q_{10}\overset{(3)}{u}(\sigma_1),$$
$$Q_{10}z(-\infty) = 0.$$

Taking into account the solution of problem $\overline{P}_0$ and solving problems $\Pi_{i0}P$, $Q_{i0}P$, $i = 0, 1$, we get the zero order approximation of an asymptotic solution of problem (38)–(40) of the form (4).

$$\widetilde{x}_0(t,\varepsilon) = \overline{x}_0(t),\ \overset{(1)}{\widetilde{u}}{}_0(t,\varepsilon) = \overset{(1)}{\overline{u}}{}_0(t),\ \widetilde{y}_0(t,\varepsilon) = 10e^{-\sqrt{2}t/\varepsilon},$$
$$\overset{(2)}{\widetilde{u}}{}_0(t,\varepsilon) = 10(1-\sqrt{2})e^{-\sqrt{2}t/\varepsilon},\ \widetilde{z}_0(t,\varepsilon) = 10e^{-\sqrt{2}t/\varepsilon^2},\ \overset{(3)}{\widetilde{u}}{}_0(t,\varepsilon) = 10(1-\sqrt{2})e^{-\sqrt{2}t/\varepsilon^2}.$$

Further, in the expressions of problems for finding asymptotics terms of the first order approximation we take into account the found asymptotics terms of the zero order approximation. We omit zero terms, in particular, $\overset{(2)}{\overline{u}}{}_0(t)$, $\overset{(3)}{\overline{u}}{}_0(t)$, $\overline{y}_0(t)$, $\overline{z}_0(t)$, $\Pi_{00}\overset{(1)}{u}(\tau_0)$, $\Pi_{00}\overset{(3)}{u}(\tau_0)$, $\Pi_{00}z(\tau_0)$, $\Pi_{10}\overset{(1)}{u}(\tau_1)$, $\Pi_{10}\overset{(2)}{u}(\tau_1)$. Note, that problems $Q_{00}P$ and $Q_{10}P$ have the zero solution.

Using (7)–(9) at $j = 1$, we write out problem $\overline{P}_1$ in the following form:

$$\overline{J}_1(\overline{u}_1) = \int\limits_0^1 (\frac{1}{2}(\overline{x}_1(t)^2 + \overline{y}_1(t)^2 + \overline{z}_1(t)^2 + \overset{(1)}{\overline{u}_1}(t)^2 + \overset{(2)}{\overline{u}_1}(t)^2 + \overset{(3)}{\overline{u}_1}(t)^2) + \overline{x}_0(t)\overset{(1)}{\overline{u}_1}(t) + \overline{x}_1(t)\overset{(1)}{\overline{u}_0}(t))\, dt \to \min_{\overline{u}_1},$$

$$\frac{d\overline{x}_1(t)}{dt} = \overline{x}_1(t) + \overset{(1)}{\overline{u}_1}(t),\ \overline{x}_1(0) + \Pi_{01}x(0) = 0,$$
$$0 = -\overline{y}_1(t) + \overset{(2)}{\overline{u}_1}(t),$$
$$0 = -\overline{z}_1(t) + \overset{(3)}{\overline{u}_1}(t).$$

Using (10)–(12) at $j = 1$ and taking into account that $\Pi_{00}x(\tau_0) = 0$, we write out problem $\Pi_{01}P$ in the following form:

$$\Pi_{01}J(\Pi_{01}u) = \int\limits_0^{+\infty} (\frac{1}{2}(\Pi_{01}x(\tau_0)^2 + \Pi_{01}y(\tau_0)^2 + \Pi_{01}z(\tau_0)^2 + \Pi_{01}\overset{(1)}{u}(\tau_0)^2 + \Pi_{01}\overset{(2)}{u}(\tau_0)^2 + \Pi_{01}\overset{(3)}{u}(\tau_0)^2) +$$

$$+\Pi_{01}x(\tau_0)\Pi_{00}\overset{(1)}{u}(\tau_0) + \Pi_{01}\overset{(3)}{u}(\tau_0)(\Pi_{00}y(\tau_0) - \Pi_{00}\eta(\tau_0)))\, d\tau_0 \to \min_{\Pi_{01}u},$$

$$\frac{d\Pi_{01}x(\tau_0)}{d\tau_0} = 0,\ \Pi_{01}x(+\infty) = 0,$$

$$\frac{d\Pi_{01}y(\tau_0)}{d\tau_0} = -\Pi_{01}y(\tau_0) + \Pi_{01}\overset{(2)}{u}(\tau_0),\ \overline{y}_1(0) + \Pi_{01}y(0) + \Pi_{11}y(0) = 0,$$

$$0 = -\Pi_{01}z(\tau_0) + \Pi_{01}\overset{(3)}{u}(\tau_0) + \Pi_{00}y(\tau_0).$$

In view of (13)–(15) at $j = 1$, problem $Q_{01}P$ is defined by the relations

$$Q_{01}J(Q_{01}u) = Q_{01}y(0)\overline{\eta}_1(1) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{01}x(\sigma_0)^2 + Q_{01}y(\sigma_0)^2 + Q_{01}z(\sigma_0)^2 + Q_{01}\overset{(1)}{u}(\sigma_0)^2 +$$

$$+Q_{01}\overset{(2)}{u}(\sigma_0)^2 + Q_{01}\overset{(3)}{u}(\sigma_0)^2)\, d\tau_0 \to \min_{Q_{01}u},$$

$$\frac{dQ_{01}x(\sigma_0)}{d\sigma_0} = 0,\ \frac{dQ_{01}y(\sigma_0)}{d\sigma_0} = -Q_{01}y(\sigma_0) + Q_{01}\overset{(2)}{u}(\sigma_0),$$

$$0 = -Q_{01}z(\sigma_0) + Q_{01}\overset{(3)}{u}(\sigma_0),$$

$$Q_{01}x(-\infty) = Q_{01}y(-\infty) = 0.$$

Taking into account that $\Pi_{11}x(\tau_1) = 0$, in view of (16)-(18) at $j = 1$, problem $\Pi_{11}P$ is defined by the following way:

$$\Pi_{11}J(\Pi_{11}u) = \frac{1}{2}\int\limits_0^{+\infty} (\Pi_{11}y(\tau_1)^2 + \Pi_{11}z(\tau_1)^2 + \Pi_{11}\overset{(1)}{u}(\tau_1)^2 + \Pi_{11}\overset{(2)}{u}(\tau_1)^2 + \Pi_{11}\overset{(3)}{u}(\tau_1)^2)\, d\tau_1 \to \min_{\Pi_{11}u},$$

$$\frac{d\Pi_{11}y(\tau_1)}{d\tau_1} = 0,\ \Pi_{11}y(+\infty) = 0,$$

$$\frac{d\Pi_{11}z(\tau_1)}{d\tau_1} = -\Pi_{11}z(\tau_1) + \Pi_{11}\overset{(3)}{u}(\tau_1),\ \overline{z}_1(0) + \Pi_{01}z(0) + \Pi_{11}z(0) = 0.$$

Using (19)–(21) at $j = 1$ and taking into account that $Q_{11}x(\sigma_1) = 0$, we obtain problem $Q_{11}P$ in the form:

$$Q_{11}J(Q_{11}u) = Q_{11}z(0)(\overline{\zeta}_1(1) + Q_{01}\zeta(0)) + \frac{1}{2}\int\limits_{-\infty}^{0} (Q_{11}y(\sigma_1)^2 + Q_{11}z(\sigma_1)^2 + Q_{11}\overset{(1)}{u}(\sigma_1)^2 +$$

$$+Q_{11}\overset{(2)}{u}(\sigma_1)^2 + Q_{11}\overset{(3)}{u}(\sigma_1)^2)\, d\sigma_1 \to \min_{Q_{11}u},$$

$$\frac{dQ_{11}y(\sigma_1)}{d\sigma_1} = 0,\ Q_{11}y(-\infty) = 0,$$
$$\frac{dQ_{11}z(\sigma_1)}{d\sigma_1} = -Q_{11}z(\sigma_1) + Q_{11}\overset{(3)}{u}(\sigma_1),\ Q_{11}z(-\infty) = 0.$$

Solving problems $\overline{P}_1$, $\Pi_{i1}P$, $Q_{i1}P$, $i = 0, 1$, we get the first order approximation of asymptotic solution of problem (38)–(40):

$$\widetilde{x}_1(t,\varepsilon) = \widetilde{x}_0(t,\varepsilon) - \varepsilon a((2+\sqrt{2})te^{\sqrt{2}t} - (2-\sqrt{2})te^{-\sqrt{2}(t-2)}),$$
$$\overset{(1)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(1)}{\widetilde{u}}_0(t,\varepsilon) - \varepsilon a((2+\sqrt{2}+\sqrt{2}t)e^{\sqrt{2}t} - (2-\sqrt{2}-\sqrt{2}t)e^{-\sqrt{2}(t-2)}),$$
$$\widetilde{y}_1(t,\varepsilon) = \widetilde{y}_0(t,\varepsilon),\ \overset{(2)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(2)}{\widetilde{u}}_0(t,\varepsilon),$$
$$\widetilde{z}_1(t,\varepsilon) = \widetilde{z}_0(t,\varepsilon) + \varepsilon 5((-\sqrt{2}+1)e^{-\sqrt{2}t/\varepsilon} + (\sqrt{2}-1)e^{-\sqrt{2}t/\varepsilon^2}),$$
$$\overset{(3)}{\widetilde{u}}_1(t,\varepsilon) = \overset{(3)}{\widetilde{u}}_0(t,\varepsilon) + \varepsilon 5(-(\sqrt{2}+1)e^{-\sqrt{2}t/\varepsilon} + (2\sqrt{2}-3)e^{-\sqrt{2}t/\varepsilon^2}).$$

The exact solution of problem (38)–(40) was calculated by means of Maple 2022.

The exact solution and asymptotic approximations to the solution of problem (38)–(40) at $\varepsilon = 0.25$ are presented in Figures 1–6, where the black line denotes the exact solution, the yellow line means the solution of the degenerate problem, the red line – the zero order approximation and the green one – the first order approximation. Please note that the degenerate and the zero order asymptotics solutions for the trajectory $x(t,\varepsilon)$ and the control $\overset{(1)}{u}(t,\varepsilon)$ are equal, the zero and the first order approximations for the trajectory $y(t,\varepsilon)$ and the control $\overset{(2)}{u}(t,\varepsilon)$ are the same.

Values of the performance index $J_\varepsilon(u)$ corresponding to the optimal control u_* and it's approximations $\overline{u}_0$, $\widetilde{u}_0$, $\widetilde{u}_1$ are presented in Table. We give here three decimal points using ordinary rules of approximating. From this table it is seen that for a less values of ε there is a better proximity between values of the performance index for asymptotic approximations to the optimal control and it's minimal value and $J_\varepsilon(\overline{u}_0) > J_\varepsilon(\widetilde{u}_0) > J_\varepsilon(\widetilde{u}_1) > J_\varepsilon(u_*)$, this corresponds to Theorems 2, 3.

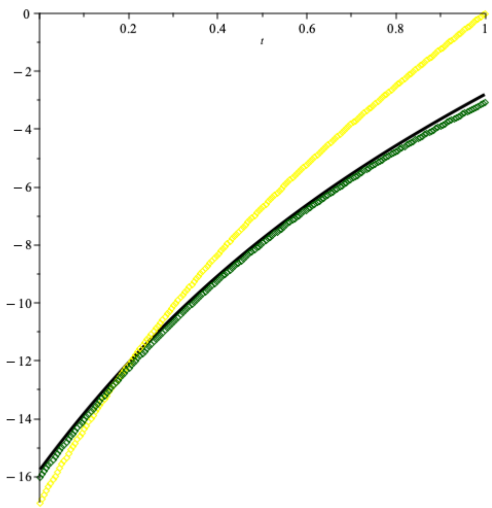

Figure 1. Control $\overset{(1)}{u}(t,\varepsilon)$.

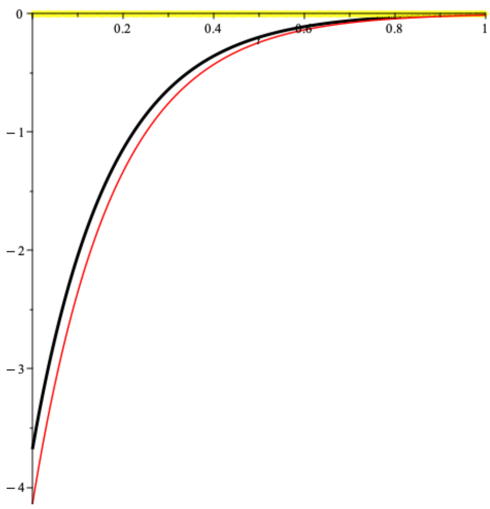

Figure 2. Control $\overset{(2)}{u}(t,\varepsilon)$.

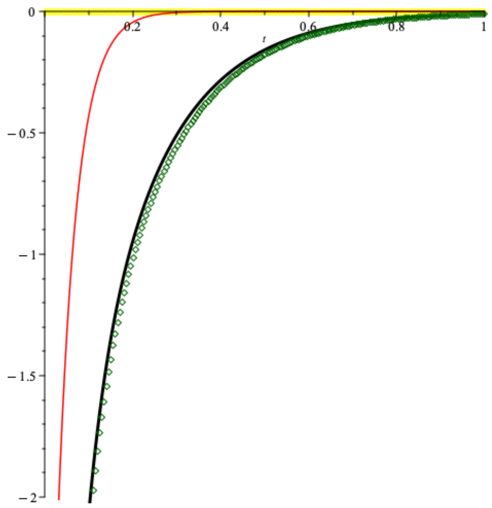

Figure 3. Control $\overset{(3)}{u}(t,\varepsilon)$.

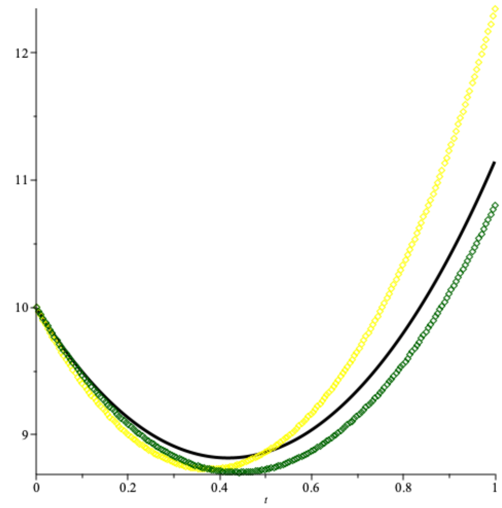

Figure 4. Trajectory $x(t,\varepsilon)$.

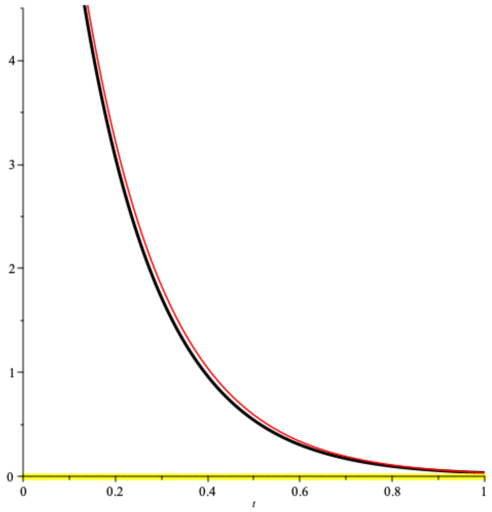

Figure 5. Trajectory $y(t,\varepsilon)$.

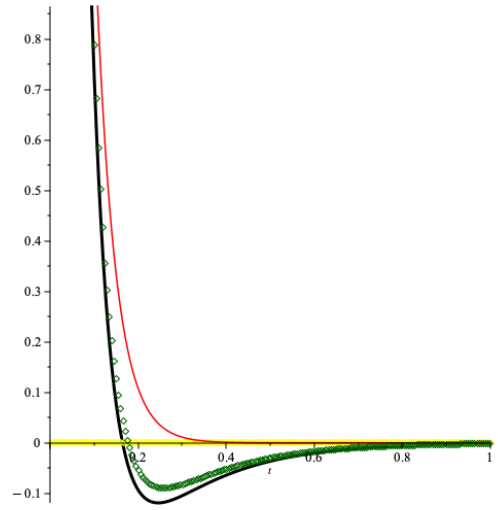

Figure 6. Trajectory $z(t,\varepsilon)$.

T a b l e

Values of the performance index

ε	$J_\varepsilon(\overline{u}_0)$	$J_\varepsilon(\widetilde{u}_0)$	$J_\varepsilon(\widetilde{u}_1)$	$J_\varepsilon(u_*)$
0.25	76.413	74.184	72.268	72.200
0.125	79.716	78.991	78.559	78.555

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Г.А. Курина[1,2], М.А. Калашникова[3]

[1] *Воронеж мемлекеттік университеті, Воронеж, Ресей;*
[2] *Ресей Ғылым академиясының «Информатика және басқару» федералды зерттеу орталығы, Мәскеу, Ресей;*
[3] *Auксо, Воронеж, Ресей*

Үшқарқынды айнымалылар мен әлсіз бейсызықты ауытқуы бар сызықтық-квадраттық басқару есептері үшін минимумдаушы тізбектер

Мақала үшқарқынды күй айнымалылары бар әлсіз бейсызықты жүйенің траекторияларында әлсіз бейсызықты ауытқу квадраттық сапа критерийін азайту есебі үшін минимизациялық тізбектерді құруға арналған. Бұл жағдайда шешімнің асимптотикалық ыдырауын есептің шарттарына тікелей ауыстырудан және шешімдері бастапқы бейсызықты басқару есебінің шешімінің асимптотикалық ыдырауының мүшелері болып табылатын тиімді басқару есептерінің үйірін (қарастырылып отырған жағдайда сызықты-квадраттық) құрудан тұратын шешімнің асимптотикалық құрылысының тікелей схемасы қолданылады. Тиімді басқаруға кейбір асимптотикалық жуықтауды басқару ретінде пайдаланған кезде тиімді траекторияның күй теңдеуінің траекториясына жақындығы бағаланады. Минимумдаушы тізбектерді құрудың схемасын егжей-тегжейлі көрсететін мысал келтірілген.

Кілт сөздер: үшқарқынды айнымалылар, тиімді басқарудың бейсызықты есептері, асимптотикалық бағалаулар, минимумдаушы тізбектер.

Г.А. Курина[1,2], М.А. Калашникова[3]

[1] *Воронежский государственный университет, Воронеж, Россия;*
[2] *Федеральный исследовательский центр «Информатика и управление» Российской академии наук, Москва, Россия;*
[3] *Ауксо, Воронеж, Россия*

Минимизирующие последовательности для линейно-квадратичной задачи управления с трехтемповыми переменными и слабыми нелинейными возмущениями

Статья посвящена построению минимизирующих последовательностей для задачи минимизации слабо нелинейно возмущенного квадратичного критерия качества на траекториях слабо нелинейной системы с трехтемповыми переменными состояния. При этом использована так называемая прямая схема построения асимптотики решения, заключающаяся в непосредственной подстановке постулируемого асимптотического разложения решения в условия задачи и построении серии задач оптимального управления (в рассматриваемом случае линейно-квадратичных), решения которых являются членами асимптотического разложения решения исходной нелинейной задачи управления. Получена оценка близости оптимальной траектории к траектории уравнения состояния при использовании в качестве управления некоторого асимптотического приближения к оптимальному управлению. Приведен пример, детально иллюстрирующий предложенную схему построения минимизирующих последовательностей.

Ключевые слова: трехтемповые переменные, нелинейные задачи оптимального управления, асимптотические оценки, минимизирующие последовательности.

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