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Well-posedness of the initial-boundary value problems for the time-fractional degenerate diffusion equations

This paper deals with the solving of initial-boundary value problems for the one-dimensional linear time-fractional diffusion equations with time-degenerate diffusive coefficients t^β with $\beta > 1-\alpha$. The solutions to initial-boundary value problems for the one-dimensional time-fractional degenerate diffusion equations with Riemann-Liouville fractional integral $I_{0+,t}^{1-\alpha}$ of order $\alpha \in (0,1)$ and with Riemann-Liouville fractional derivative $D_{0+,t}^{\alpha}$ of order $\alpha \in (0,1)$ in the variable, are shown. The solutions to these fractional diffusive equations are presented using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$. The solution to the problems is discovered by the method of separation of variables, through finding two problems with one variable. Rather, through finding a solution to the fractional problem depending on the parameter t, with the Dirichlet or Neumann boundary conditions. The solution to the Sturm-Liouville problem depends on the variable x with the initial fractional-integral Riemann-Liouville condition. The existence and uniqueness of the solution to the problem are confirmed. The convergence of the solution was evidenced using the estimate for the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ from and by Parseval's identity.

Keywords: time-fractional diffusion equation, method of separation variables, Kilbas-Saigo function.

Introduction

Many mathematicians have attracted most interest to the fractional diffusion equations. Inverse source problems for degenerate time-fractional PDE were studied in [1]. In [2, 3], Al-Refai and Luchko analyzed the initial-boundary value problems for the linear and non-linear fractional diffusion equations with the Riemann-Liouville time-fractional derivative. Various types of fractional derivatives and their properties were investigated in the monograph [4–8]. Fractional calculus can be applied in mechanics, physics, mathematics, etc. [8–12]. Note that degenerate fractional evolutionary equations were investigated in [13, 14]. In [15], maximum and minimum principles for time-fractional diffusion equations involving fractional derivatives were proposed. Luchko studied initial-boundary value problems for a generalized diffusion equation with a distributed order [16].

In our previous work [17], we studied the Cauchy-Dirichlet and Cauchy-Neumann problems for the Caputo time-fractional diffusion equation. This paper considers the Cauchy-Dirichlet and Cauchy-Neumann problems for the diffusion equation with Riemann–Liouville time-fractional derivative. A solution is discovered by using the Kilbas-Saigo function and by the method of separation of variables. The existence, convergence, and uniqueness of the solution are proved.

1 Dirichlet problem

Let us consider the one-dimensional case of the time-fractional diffusion equation

$$D_{0+,t}^{\alpha} u(t,x) - t^\beta u_{xx}(t,x) = 0, \ \ (t,x) \in (0,\infty)\times(0,1), \tag{1}$$

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with the Dirichlet boundary condition

$$u(t,0) = u(t,1) = 0, \;\; t > 0, \;\; x \in [0,1], \tag{2}$$

and the Cauchy initial condition

$$I^{1-\alpha}_{0+,t} u(0,x) = \phi(x), \;\; x \in [0,1], \tag{3}$$

where $\beta > 1-\alpha$, $D^{\alpha}_{0+,t}$ is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ defined by [5; 79]

$$D^{\alpha}_{0+,t} f(t) = \frac{d}{dt} I^{1-\alpha}_{0+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)ds}{(t-s)^{\alpha}}.$$

Here $I^{1-\alpha}_{0+,t}$ is the Riemann-Liouville fractional integral given by [5; 80]

$$I^{1-\alpha}_{0+,t} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{\alpha}}.$$

Let $H^2(0,1)$ is a Hilbert space, defined by

$$H^2(0,1) = \{u : \;\; u \in L^2(0,1); \;\; u_{xx} \in L^2(0,1)\},$$

where the norm is

$$||u||^2_{H^2(0,1)} = \sum_{k=1}^{\infty} \lambda_k^2 |(u, e_k)|^2 < \infty.$$

Definition 1. The solution to problem (1)–(3) is $t^{1-\alpha} u \in C((0,\infty); L^2(0,1))$, which satisfies $t^{1-\alpha-\beta} D^{\alpha}_{0+,t} u$, $t^{1-\alpha} u_{xx} \in C((0,\infty); L^2(0,1))$.

Theorem 1. Let $\phi(x) \in H^2(0,1)$, then there exists a unique solution u to problem (1)–(3), which has the form

$$u(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, 1+\frac{\beta-1}{\alpha}} (-\pi^2 k^2 t^{\beta+\alpha}) \sin \pi k x, \;\; (t,x) \in (0,\infty) \times (0,1),$$

where $\phi_k = 2\int_0^1 \phi(x) \sin \pi k x dx$, $k \in N$ and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function defined as [8, Remark 5.1]

$$E_{\alpha,m,l}(z) = \sum_{i=0}^{\infty} c_i z^i, \;\; c_0 = 1, \;\; c_i = \Pi_{j=0}^{i-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \;\; i \geq 1. \tag{4}$$

For the function $E_{\alpha,m,m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha})$ the following estimate holds [4, Proposition 3.6]

$$E_{\alpha,m,m-\frac{1}{\alpha}}(-\lambda_k t^{\beta+\alpha}) \leq \frac{1}{\left(1 + \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))} \lambda_k t^{\beta+\alpha}\right)^{1+\frac{1}{m}}}, \;\; m = \frac{\beta+\alpha}{\alpha}, \;\; t > 0. \tag{5}$$

Proof Theorem 1.

Existence of solution. Since the Sturm–Liouville operator has eigenvalues $\{\lambda_k > 0, \;\; k \in N\}$ on $L^2(0,1)$ and the corresponding orthonormal eigenfunctions $\{X_k(x), \;\; k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we can write the solution to problem (1)–(3) as follows

$$u(t,x) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \;\; (t,x) \in (0,\infty) \times (0,1), \tag{6}$$

$$\phi(x)=\sum_{k=1}^{\infty}\phi_k X_k(x),\;\; x\in(0,1),$$

where

$$\phi_k=2\int_0^1\phi(x)X_k(x)dx.$$

Substituting (6) to diffusion equation (1)–(3), we gain the next problem

$$D^{\alpha}_{0+,t}T_k(t)+\lambda_k t^{\beta}T_k(t)=0,\;\; t>0, \tag{7}$$

$$I^{1-\alpha}_{0+,t}T_k(0)=\phi_k, \tag{8}$$

$$X_k''(x)+\lambda_k X_k(x)=0, \tag{9}$$

$$X_k(0)=X_k(1)=0. \tag{10}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Dirichlet problem (9)–(10) are $X_k(x)=\sin\pi kx$ and $\lambda_k=(\pi k)^2$, respectively. It is known that a unique solution to problem (7)–(8) is [5; 227]

$$T_k(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha}). \tag{11}$$

Substituting (11) and the orthonormal eigenfunctions $X_k(x)=\sin\pi kx$ to (6), we can get the solution to problem (1)–(3) in the next form

$$u(t,x)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx, (t,x)\in(0,\infty)\times(0,1). \tag{12}$$

Convergence of solution. Applying (5) to (11), we get

$$T_k(t)\le\frac{|\phi_k||t^{\alpha-1}|}{\Gamma(\alpha)\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2k^2t^{\beta+\alpha}\right)^{1+\frac{1}{m}}},\;\; m=\frac{\beta+\alpha}{\alpha}.$$

By Parseval's identity, it follows from (12) that

$$\sup_{t\ge0}||t^{1-\alpha}u(t,\cdot)||^2_{L^2(0,1)}=\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2}\sum_{k=1}^{\infty}|\phi_k|^2\left|E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\right|^2\le$$

$$\le\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2}\sum_{k=1}^{\infty}\frac{|\phi_k|^2}{\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2k^2t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}}\le$$

$$\le\sup_{t\ge0}\frac{1}{|\Gamma(\alpha)|^2\left(1+\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha(m+1))}\pi^2t^{\beta+\alpha}\right)^{2(1+\frac{1}{m})}}\sum_{k=1}^{\infty}|\phi_k|^2\le\sum_{k=1}^{\infty}|\phi_k|^2=||\phi(\cdot)||^2_{L^2(0,1)}. \tag{13}$$

Solving $D^{\alpha}_{0+,t}u$ and u_{xx} we get

$$D^{\alpha}_{0+,t}u(t,x)=\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\phi_k D^{\alpha}_{0+,t}t^{\alpha-1}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx=$$

$$= -\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\pi^2k^2\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx, \tag{14}$$

and

$$u_{xx}(t,x) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin''\pi kx =$$

$$= -\frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=1}^{\infty}\pi^2k^2\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\sin\pi kx. \tag{15}$$

Applying (13)–(15) we get

$$\sup_{t\geq 0}||t^{1-\alpha-\beta}\mathcal{D}_t^{\alpha}u(t,\cdot)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty}\pi^4k^4|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty,$$

and

$$\sup_{t\geq 0}||t^{1-\alpha}u_{xx}(t,\cdot)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty}\pi^4k^4|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty.$$

Uniqueness of the solution. Suppose that u_1 and u_2 are solutions to problem (1)–(3) and we choose $u = u_1 - u_2$ in such a way, that u satisfies the diffusion equation (1) and boundaries, initial conditions (2), (3). Define

$$T_k(t) = \int_0^1 u(t,x)\sin\pi kx dx, \quad k\in N, \ t>0. \tag{16}$$

Applying $D^{\alpha}_{0+,t}$ to left-side (16) equation by using (1) we obtain

$$\begin{aligned} D^{\alpha}_{0+,t}T_k(t) &= \int_0^1 D^{\alpha}_{0+,t}u(t,x)\sin\pi kx dx \\ &= t^{\beta}\int_0^1 u_{xx}(t,x)\sin\pi kx dx \\ &= t^{\beta}\int_0^1 u(t,x)\sin''\pi kx dx \\ &= -t^{\beta}\pi^2k^2\int_0^1 u(t,x)\sin\pi kx dx \\ &= -t^{\beta}\pi^2k^2T_k(t), \quad k\in N, \ t>0. \end{aligned}$$

From (2), (3) we have

$$I^{1-\alpha}_{0+,t}T_k(0) = 0,$$

which means that $u(t,x)\equiv 0$. Hence $u_1(t,x) = u_2(t,x)$, therefore the problem (1)–(3) has a unique solution.

2 Cauchy-Neumann problem

Let us consider time-fractional diffusion equation

$$D^{\alpha}_{0+,t}u(t,x) - t^{\beta}u_{xx}(t,x) = 0, \quad (t,x)\in(0,\infty)\times(0,1), \tag{17}$$

with the Neumann boundary condition

$$u_x(t,0) = u_x(t,1) = 0, \quad t>0, \quad x\in[0,1], \tag{18}$$

and the Cauchy initial condition

$$I_{0+,t}^{1-\alpha}u(0,x)=\phi(x). \tag{19}$$

Definition 2. The solution to problem (17)–(19) is $t^{1-\alpha}u \in C((0,\infty);\ L^2(0,1))$, which satisfies $t^{1-\alpha-\beta}D_{0+,t}^{\alpha}u,\ t^{1-\alpha}u_x,\ t^{1-\alpha}u_{xx}\in C((0,\infty);L^2(0,1))$.

Theorem 2. Let $\phi(x)\in H^2(0,1)$. The unique solution to problem (17)–(19) is the function u, which has form

$$u(t,x)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\phi_k E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\cos\pi kx,\ \ (t,x)\in(0,\infty)\times(0,1),$$

where $\phi_0=\int_0^1\phi(x)dx$ and $\phi_k=2\int_0^1\phi(x)\cos\pi kxdx,\ \ k\in N$ and $E_{\alpha,m,l}(z)$ is a Kilbas-Saigo function, which is defined by the formula (4) and (5).

It can be easily proven by the idea of Theorem 1.

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Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептің қисындылығы

Мақалада t^β, $\beta > 1 - \alpha$ диффузиялық коэффициенттері бар бір өлшемді сызықтық бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептерді шешу қарастырылған. $\in (0,1)$ үшін бөлшек ретті Риман-Лиувилль интегралы $I_{0+,t}^{1-\alpha}$ және $\alpha \in (0,1)$ үшін бөлшек ретті Риман-Лиувилль туындысы $D_{0+,t}^{\alpha}$ бар бір өлшемді уақыт бойынша бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есептердің шешімдері көрсетілген. Бөлшек ретті диффузиялық теңдеулердің шешімдері $E_{\alpha,m,l}(z)$ Килбас-Сайго функциясы арқылы берілген. Есептердің шешімі айнымалыларды ажырату әдісі арқылы, бір айнымалысы бар екі есепті табу арқылы анықталады. Демек, Дирихле немесе Нейман шекаралық шарттарымен t параметріне тәуелді бөлшек ретті есебінің шешімін және x параметріне тәуелді Штурм-Лиувилл есебіне қойылған бастапқы шарты бөлшек ретті Риман-Лиувилл интегралы арқылы берілген есептің шешімін табу арқылы. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген. Шешімнің жинақтылығы Kilbas-Saigo $E_{\alpha,m,l}(z)$ функциясының бағалауы көмегімен және Парсевал теңдігін қолдану арқылы дәлелденді.

Кілт сөздер: бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, айнымалыларын ажырату әдісі, Килбас-Сайго функциясы.

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Корректность начально-краевых задач для дробных вырожденных диффузионных уравнений

Статья посвящена решению начально-краевых задач для одномерных дробных вырожденных линейных диффузионных уравнений коэффициентами диффузии t^β при $\beta > 1-\alpha$, начально-краевых задач для одномерных уравнений вырождающейся диффузии с дробным временем с дробным интегралом Римана-Лиувилля $I_{0+,t}^{1-\alpha}$ порядка $\alpha \in (0,1)$ и с дробной производной Римана-Лиувилля $D_{0+,t}^{\alpha}$ порядка $\alpha \in (0,1)$ по переменной. Решения этих дробных диффузионных уравнений представлены с помощью функции Килбаса-Сайго $E_{\alpha,m,l}(z)$, их получили методом разделения переменных, путем нахождения двух задач с одной переменной. Вернее, путем нахождения решения дробной задачи, зависящей от параметра t, с граничными условиями Дирихле или Неймана, и решение задачи Штурма–Лиувилля, зависящей от переменной x с начальным дробно-интегральным условием Римана-Лиувилля. Доказаны существование и единственность решения задачи. Сходимость в решения подтверждена с помощью оценки функции Килбаса-Сайго $E_{\alpha,m,l}(z)$ и тождества Парсеваля.

Ключевые слова: дробно-вырожденное диффузионное уравнение, метод разделения переменных, функция Килбаса-Сайго.

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