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On the non-uniqueness of the solution to a boundary value problem of heat conduction with a load in the form of a fractional derivative

The paper deals with the second boundary value problem for the loaded heat equation in the first quadrant. The loaded term contains a fractional derivative in the Caputo sense of an order α, $2 < \alpha < 3$. The boundary value problem is reduced to an integro-differential equation with a difference kernel by inverting the differential part. It is proved that a homogeneous integro-differential equation has at least one non-zero solution. It is shown that the solution of the homogeneous boundary value problem corresponding to the original boundary value problem is not unique, and the load acts as a strong perturbation of the boundary value problem.

Keywords: second boundary value problem, loaded equation, Caputo fractional derivative, non-unique solvability, strong perturbation.

Introduction

Loaded differential equations today have a wide practical application in many areas of natural science. Moreover, loaded equations are a special class of equations that require separate consideration. In addition, loaded equations can act as one of the ways to introduce generalized solutions of wide classes of partial differential equations and as an effective method for finding approximate solutions to boundary value problems for differential equations. A significant contribution to the development of the theory of loaded equations was made by the work of A.M. Nakhushev [1] (and his other works), where definitions of loaded differential, loaded integro-differential, loaded functional equations and their numerous applications are given. In papers [2–5], the theory of loaded equations was further developed. [3] considers boundary value problems for a loaded differential operator, which are interpreted as perturbations of the corresponding differential operators. It is shown that the loaded part is a weak or strong perturbation, depending on the derivative order in the loaded term, as well as on the manifold on which the trace of the BVP solution is given.

There are many books devoted to fractional analysis today [6–21]. In recent years, an intensive study of loaded differential equations has been carried out, associated with various applied problems of mechanics, biology, ecology and chemistry, modeled using loaded equations. To date, many books have been devoted to fractional analysis (various applications in physics, mechanics, and simulation) [7], [14–20]. Among the variety of works, the monograph [6], covering a huge range of ideas. Monograph presents classical and modern results in the theory of fractional analysis, and gives their applications to integral and differential equations and function theory.

From a mathematical point of view, it is interesting to study the boundary value problems for the heat equation with a fractional load, when the loaded term is considered in the form of a fractional derivative or a fractional integral. In [21, 22] the load moves with a constant velocity, namely, it moves along the line $x = t$. The loaded term contains a fractional derivative in the Riemann-Liouville sense. The boundary value problem was reduced to the Volterra integral equation with a kernel containing a

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generalized hypergeometric series. The integral equation has a nonempty spectrum for certain values of the fractional derivative order and for the spectral parameter.

We also note that the boundary value problems of heat conduction and the Volterra integral equations arising in their study with singularities in the kernel, similar to the singularities in this paper, were considered in [23, 24].

In [25–28] fractionally loaded boundary value problems of heat conduction are investigated, the loaded term is represented in the form of the fractional derivative. The derivative order in the loaded term is less than the order of the differential part. In [25, 26] the loaded term is represented in the form of the Caputo fractional derivative with respect to the spatial variable. In [25], it is proved that there is continuity on the right in the order of the fractional derivative. There is no continuity on the left. In [26] there is continuity in the order of the derivative in the loaded term of the problem. In [27, 28], the loaded term has the form of a fractional Riemann-Liouville derivative with respect to the time variable. The kernel of the resulting integral equation contains a special function, for example, a generalized hypergeometric function in [25] or the Wright function in [27]. Conditions for the unique solvability of the integral equation are established by estimating the integral kernel. It is shown that the existence and uniqueness of solutions to the integral equation depends on the order of the fractional derivative in the loaded term.

In [29] the first boundary value problem for essentially loaded equation of heat conduction is considered. It is shown that if the point of load is fixed, then the stated boundary problem is uniquely solvable.

In this paper, the second boundary value problem for the loaded heat equation is considered in the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$. The load is presented as a Caputo fractional derivative. The fractional derivative is greater than the order of the differential part of the BVP. The boundary value problem is reduced to an integro-differential equation by representing the problem solution in terms of the Green's function. Solvability of the integro-differential equation depends on the fractional derivative order in the loaded term of the BVP. The integro-differential equation has an eigenfunction. The solution of the stated boundary problem is determined by the solution of the obtained integro-differential equation in explicit form. Since the uniqueness of the BVP solution is violated, in this case the load can be interpreted as a strong perturbation the BVP.

The article is structured as follows. Section 1 includes some necessary concepts, definitions, auxiliary assertions, and preliminary assumptions about the classes of the BVP solution and the data included in the problem under study. In Section 2, we set the BVP that we are going to solve. In Section 3, the problem is reduced to an integro-differential equation with a difference kernel. In Section 4 we solve the resulting integro-differential equation by Laplace integral transform method. We write out the solution of the resulting equation in explicit form and formulate the corresponding results on the non-uniqueness of the solution to the BVP and the solution to the associated integro-differential equation.

1 Preliminaries

We first give some definitions and useful information.

Definition 1 ([6]). Let $f(t) \in L_1[a,b]$. Then, the Riemann-Liouville derivative of the order β is defined as follows

$$ {}_{r}D^{\beta}_{a,t}f(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\beta-n+1}}d\tau, \ \beta, a \in R, \ n-1 < \beta < n. \tag{1} $$

Definition 2. Let $f(t) \in \mathrm{AC}^n[a,b]$ (i.e. $f^{(n-1)}(t)$ is an absolutely continuous function). Then, the Caputo derivative of the order β is defined as follows

$$ {}_{c}D^{\beta}_{a,\,t}\,f(t) = \frac{1}{\Gamma(n-\beta)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}d\tau; \ \beta, \ a \in \Re, \ n-1 < \beta < n, \tag{2} $$

From formula (1) it follows that

$$ {}_rD^0_{a,t}f(t) = f(t), \quad {}_rD^n_{a,t}f(t) = f^{(n)}(t), \quad n \in N. $$

We study a BVP for the loaded heat equation, when the loaded term is represented in the form of a fractional derivative. To study the formulated boundary problem, we need a formula for inverting the differential part of the equation.

It's known [30; 57] that in the domain $Q = \{(x,t) \mid x > 0, \quad t > 0\}$ the following boundary value problem of heat conduction

$$ u_t = a^2 u_{xx} + F\left(x,t\right), $$

$$ u\left.\right|_{t=0} = f(x), \quad u_x\left.\right|_{x=0} = g(x), $$

has the solution $u(x,t)$ described by the formula

$$ u\left(x,t\right) = \int_0^\infty G\left(x,\xi,t\right)\, f(\xi)\, d\xi - a\int_0^t G(x,\,0,\,t-\tau)\,g(\tau)\,d\tau + $$

$$ + \int_0^t\int_0^\infty G\left(x,\xi,t-\tau\right)F\left(\xi,\tau\right)d\xi d\tau, \tag{3} $$

where

$$ G(x,\,\xi,\,t) = \frac{1}{2\sqrt{\pi\,a\,t}}\left\{\exp\left(-\frac{\left(x-\xi\right)^2}{4\,a\,t}\right) + \exp\left(-\frac{\left(x+\xi\right)^2}{4\,a\,t}\right)\right\}. $$

The following equality holds true for the Green function $G\left(x,\xi,t\right)$

$$ \int_0^\infty G\left(x,\xi,t\right)d\xi = 1. \tag{4} $$

It follows from the definitions that for the existence of a derivative of $f(t)$ in the sense of Riemann-Liouville (1) it is sufficient that $f(t)$ belongs to the class of summable functions, for the existence of a derivative in the sense of Caputo (2) it is sufficient that the $n-1$st derivative of the function $f(t)$ be an absolutely continuous function, where n-1 is the integer part of the derivative order, i.e. $f(t) \in \mathrm{AC}^n[a,b]$ and there is the next relation formula for these derivatives

$$ {}_cD^\beta_{a,t}\,f(t) = {}_rD^\beta_{a,t}\left[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]. $$

So we assume that the solution $u(x,t)$ belongs to the class

$$ u(x,t) \in \mathrm{AC}^3\left(t \in [0,T]\right), \tag{5} $$

The right side of the BVP equation vanishes at $t < 0$ and belongs to the class

$$ f(x,t) \in L_\infty\left(A\right) \cap C\left(B\right), \tag{6} $$

where $A = \{(x,t)\,|\,x > 0,\ t \in [0,T]\}$, $B = \{(x,t)\,|\,x > 0,\ t \geq 0\}$, $T - \mathit{const} > 0$, also we assume

$$ f_1\left(x,t\right) = \int_0^t\int_0^\infty G\left(x,\xi,t-\tau\right)f\left(\xi,\tau\right)d\xi d\tau \in \mathrm{AC}^3\left(t \in [0,T]\right). \tag{7} $$

The classes in which the problem is studied are determined from the natural requirement for the existence and convergence of improper integrals that arise in the study.

2 Statement of the fractionally loaded BVP of heat conduction

In a domain $Q = \{(x, t) : x > 0, t > 0\}$ we consider a BVP

$$u_t - u_{xx} + \lambda\Big\{{}_cD^{\alpha}_{0t}u(x,t)\Big\}\Big|_{x=\gamma(t)} = f(x,t), \tag{8}$$

$$u\,(x,0) = 0, \quad u_x\,(0,t) = 0, \tag{9}$$

where λ is a complex parameter,

$$_cD^{\alpha}_{0t}u(x,t) = \frac{1}{\Gamma(3-\beta)}\int_0^t \frac{u_{\tau^3(x,\tau)}}{(t-\tau)^{\alpha-2}}d\tau$$

is Caputo derivative (2) of an order α, $2 < \alpha < 3$, $\gamma(t)$ is a continuous increasing function, $\gamma(0) = 0$ or $\gamma(t)$ is a positive *const*.

The solution of the problem and the right side of the equation belong to the classes (5) and (6), respectively.

3 Reducing the problem to a Volterra integro-differential equation of the second kind

Lemma 1. Boundary value problem (8)–(9) is reduced to a Volterra integro-differential equation of the second kind.

Proof. We invert the differential part of problem (8)–(9) by formula (3):

$$u(x,t) = -\lambda\int_0^t\int_0^\infty \Big\{{}_cD^{\alpha}_{0t}u(x,t)\Big\}\Big|_{x=\gamma(t)} G(x,\xi,t-\tau)d\xi d\tau +$$

$$+\int_0^t\int_0^\infty G\,(x,\xi,t-\tau)\,f\,(\xi,\tau)\,d\xi\,d\tau.$$

Taking into account relation (4) and introducing the notation

$$f_1\,(x,t) = \int_0^t\int_0^\infty G\,(x,\xi,t-\tau)\,f\,(\xi,\tau)\,d\xi\,d\tau,$$

we get the following representation of the solution to problem (8)–(9):

$$u(x,t) = -\lambda\int_0^t \mu(\tau)d\tau + f_1(x,t), \tag{10}$$

where

$$\mu(t) = \Big\{{}_cD^{\alpha}_{0t}u(x,t)\Big\}\Big|_{x=\gamma(t)}. \tag{11}$$

From representation (10) we take the derivative of the order $2 < \alpha < 3$ with respect to the variables t on both sides and put $x = \gamma(t)$. On the left side, we get the function $\mu\,(t)$. We also introduce the notation according to formula (11).

Then BVP (8)–(9) is reduced to the integro-differential equation:

$$\mu(t) + \lambda\int_0^t K_\alpha\,(t,\tau)\,\mu''(\tau)d\tau = f_2(t), \tag{12}$$

with conditions $\mu(0) = \mu'(0) = 0$, where

$$K_\alpha\,(t,\tau) = \frac{1}{(3-\alpha)(t-\tau)^{\alpha-2}} \tag{13}$$

and

$$f_2(t) = \Big\{{}_cD^{\alpha}_{0t}f_1(x,t)\Big\}\Big|_{x=\gamma(t)}. \tag{14}$$

Lemma 1 on reducing the BVP to an integro-differential equation is proved.

4 Study of the integro-differential equation. Main result

Lemma 2. The homogeneous integro-differential equation (12) has a non-trivial solution.

Proof. We denote the Laplace transforms of $\mu(t)$ and $f_2(t)$ as

$$\overline{\mu(p)} = L[\mu(t)] = \int_0^{\infty} e^{-pt}\mu(t)dt, \quad \overline{f_2(p)} = L[f_2(t)].$$

Since

$$L\left[\frac{1}{t^{\alpha-2}}\right] = \frac{\Gamma(3-\alpha)}{p^{3-\alpha}},$$

then applying Laplace transform to equation (12) with the condition $\mu(0) = \mu'(0) = 0$, we get

$$\overline{\mu(p)} = \frac{\overline{f_2(p)}}{1+\lambda p^{\alpha-1}}. \tag{15}$$

Consider equation (15) for $\overline{f_2(p)} \equiv 0$.

$$\overline{\mu(p)}(1+\lambda p^{\alpha-1}) = 0. \tag{16}$$

Let's solve the equation:

$$1+\lambda p^{\alpha-1} = 0. \tag{17}$$

For $\lambda \in \mathbb{C}$ and $2 < \alpha < 3$. Then $\alpha - 1$ is a real number. Let's consider cases.
I. $\alpha \in Q$. In case for $\alpha \in Q$, there can be finite number $p_1, p_2, ..., p_n$ are solutions to equation (17). Then nonzero solutions to (16) are

$$\overline{\mu_k(p)} = C_k\delta(p-p_k),$$

here $\delta(x)$ is the delta function, $C_k = const$; p_k are solutions of equation (17), $k = 1, ..., n$, n is a denominator of the rational number $\alpha - 1$. Here and below, the numbers are in the left half-plane of the complex plane, i.e. $Re,\ p_k < 0$.

I. $\alpha \in Q$. Applying the inverse Laplace transform to the last equation, we get

$$\mu_k(t) = C_k\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \delta(p-p_k)e^{pt}dp = C_k e^{p_k t}.$$

Integral is taken along the line $Re\, p = \sigma$ and is considered in the form of the main value. That's why, if $p = p_k$ are the solutions of equation (17), then eigenfunctions of equation (12) have the following form

$$\mu_k(t) = C_k e^{p_k t}. \tag{18}$$

Remark 1. The power of the complex number z with rational power $z^{\frac{m}{n}}$ is defined as:

$$z^{\frac{m}{n}} = (\sqrt[n]{z})^m.$$

II. $\alpha \in R$ *but* $\alpha \notin Q$. Then $\alpha - 1$ is an irrational number

Remark 2. Power of the complex number z with real irrational index of power $0 < s < \alpha - 1$ is defined as the limit

$$z^s = \lim_{n\to\infty} z^{\frac{\alpha_n}{\beta_n}};$$
$$\frac{\alpha_n}{\beta_n} \to s,$$

here α_n and β_n are sequences of natural numbers.

Based on Remark 2 we can claim that equation (17) has at least one solution p_0 for $\lambda \in C$ and $2 < \alpha < 3$.

Then equation (12) has at least one eigenfunction (18). The number of eigenfunctions depends on the values of parameters α and λ.

Now let's find a solution of nonhomogeneous equation (12) ($f_2(t) \neq 0$).

Equation (15) can be rewritten:

$$\overline{\mu(p)} = \overline{f_2(p)} - \lambda \frac{p^{\alpha-1}}{1+\lambda p^{\alpha-1}} \overline{f_2(p)}. \tag{19}$$

Now we apply the inverse Laplace transform to equation (19)

$$L^{-1}\left[\frac{p^{\alpha-1}}{1+\lambda p^{\alpha-1}}\right] = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{A(p)}{1+\lambda p^{\alpha-1}} dp = R_\lambda(t,\alpha), \tag{20}$$

here $A(p) = p^{\alpha-1}e^{pt}$.

The integral in (20) is considered as the main value and the integration is taken along the contour which doesn't include p_k on the left side. Then the solution of equation (12) can be written as: the solution of equation (12)

$$\mu(t) = f_2(t) + \sum_k C_k e^{p_k t} - \lambda \int_0^t R_\lambda(t-\tau;\alpha) f_2(\tau) d\tau, \tag{21}$$

here p_k are the roots of equation (17), C_k are arbitrary constants and $R_\lambda(t;\alpha)$ is defined as in (20). The zeros of the denominator of the integral function in (20) are the numbers p_k so that $A(p_k)) \neq 0$. Therefore

$$R_\lambda(t,\alpha) = \sum_k res_{p=p_k} \frac{A(p)}{1+\lambda p^{\alpha-1}} = \sum_k \frac{A(p_k)}{\lambda(\alpha-1)p_k^{\alpha-2}} = \sum_k \frac{p_k e^{p_k t}}{\lambda(\alpha-1)}.$$

Then (21) can be rewritten as

$$\mu(t) = f_2(t) + \sum_k C_k e^{p_k t} - \sum_k \frac{p_k}{\alpha-1}\int_0^t e^{p_k(t-\tau)} f_2(\tau) d\tau. \tag{22}$$

Thus, the following theorem has been proved.

Theorem. Integro-differential equation (12) with kernel and right side defined by the formulas (13) ($2 < \alpha < 3$) and (14), respectively, has a solution defined by the formula (22), moreover, the corresponding homogeneous equation (12) (when $f_2(t) = 0$) has a nonzero solution

$$\mu(t) = \sum_k e^{p_k t},$$

where here p_k are the roots of equation (17) and $Re\, p_k < 0$.

Conclusion

So function (22) is the solution of equation (12). Then the solution of BVP (8)–(9) has the form of (10)

$$u(x,t) = -\lambda \int_0^t \mu(\tau) d\tau + f_1(x,t),$$

where the function $f_1(x,t)$ is defined by the formula (7).

In such a way it can be claimed that term with a load in equation for BVP (8)–(9) is considered a strong perturbation, since according to (22) and (10) the homogeneous BVP (8)–(9) (when $f(x,t) = 0$) has non-zero solutions in the form of:

$$u(x,t) = \sum_k \frac{\lambda}{p_k}(e^{p_k t} - 1),$$

here p_k are solutions of equation (17) and $Re\, p_k < 0$.

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Бөлшек туындысы түріндегі жүктемемен жылу өткізгіштіктің шекаралық есебін шешудің бірегей еместігі туралы

Бірінші квадрантта бөлшектік-жүктелген жылуөткізгіштік теңдеуі үшін екінші шеттік есеп қарастырылған. Жүктеме қосылғышы $2 < \alpha < 3$ ретті Капуто бөлшек туындысы ретінде берілген. Шеттік есеп дифференциалдық бөлігін ауыстыру арқылы айырма өзекті интегро-дифференциалдық теңдеуге келтіріледі. Біртекті интегро-дифференциалдық теңдеудің кем дегенде бір нөлдік емес шешімі бар екені дәлелденді. Біртекті шекаралық есептің шешімі бірегей емес, ал жүктеме шекаралық есептің қатты ауытқуы болып табылатыны көрсетілген.

Кілт сөздер: екінші шеттік есеп, жүктелген теңдеу, Капуто бөлшектік туындысы, көп мағыналы шешілім, қатты ауытқу.

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О неединственности решения краевой задачи теплопроводности с нагрузкой в виде дробной производной

В статье рассмотрена вторая краевая задача для нагруженного уравнения теплопроводности в первом квадранте. Нагруженное слагаемое содержит дробную производную в смысле Капуто порядка $2 < \alpha < 3$. Краевая задача сводится к интегро-дифференциальному уравнению с разностным ядром изменения дифференциальной части. Доказано, что однородное интегро-дифференциальное уравнение имеет хотя бы одно ненулевое решение. Показано, что решение однородной краевой задачи, соответствующей исходной краевой задаче, неединственно, а нагрузка выступает как сильное возмущение краевой задачи.

Ключевые слова: вторая краевая задача, нагруженное уравнение, дробная производная Капуто, неоднозначная разрешимость, сильное возмущение.