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Systems of integral equations with a degenerate kernel and an algorithm for their solution using the Maple program

In the mathematical literature, a scalar integral equation with a degenerate kernel is well described (see below (1)), where all the written functions are scalar quantities). The authors are not aware of publications where systems of integral equations of (1) type with kernels in the form of a product of matrices would be considered in detail. It is usually said that the technique for solving such systems is easily transferred from the scalar case to the vector one (for example, in the monograph A.L. Kalashnikov "Methods for the approximate solution of integral equations of the second kind" (Nizhny Novgorod: Nizhny Novgorod State University, 2017), a brief description of systems of equations with degenerate kernels is given, where the role of degenerate kernels is played by products of scalar rather than matrix functions). However, as the simplest examples show, the generalization of the ideas of the scalar case to the case of integral systems with kernels in the form of a sum of products of matrix functions is rather unclear, although in this case the idea of reducing an integral equation to an algebraic system is also used. At the same time, the process of obtaining the conditions for the solvability of an integral system in the form of orthogonality conditions, based on the conditions for the solvability of the corresponding algebraic system, as it seems to us, has not been previously described. Bearing in mind the wide applications of the theory of integral equations in applied problems, the authors considered it necessary to give a detailed scheme for solving integral systems with degenerate kernels in the multidimensional case and to implement this scheme in the Maple program. Note that only scalar integral equations are solved in Maple using the *intsolve* procedure. The authors did not find a similar procedure for solving systems of integral equations, so they developed their own procedure.

Keywords: integral operator, degenerate kernel, Maple program procedure, scalar integral equation.

1 Fredholm integral equations with a degenerate kernel (general theory)

Consider the integral system

$$y(t) = \lambda \sum_{j=1}^{m} A_j(t) \int_0^T B_j(s) y(s) ds + h(t). \tag{1}$$

Let the expressions $A_j(t)$ and $B_j(s)$, forming the kernel of the integral operator in it be matrix functions (their smoothness and dimensions are specified below). Just as in the one-dimensional case [1–4], such systems can be reduced to algebraic systems using the following operations. Denote

$$w_j = \int_0^T B_j(s) y(s) ds, \ \ j = \overline{1,m}. \tag{2}$$

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Then instead of (1) we get the equality

$$y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t). \tag{3}$$

Multiplying in turn equality (3) on the left by matrices $B_1(t), ..., B_m(t)$ and integrating the resulting equalities with respect to $t \in [0, T]$, we get

$$\int_0^T B_i(t) y(t) dt = \lambda \sum_{j=1}^{m} \left(\int_0^T B_i(t) A_j(t) dt \right) w_j + \int_0^T B_i(t) h(t) dt, \ \ i = \overline{1, m}.$$

Using (2), we obtain the algebraic system of equations

$$w_i = \lambda \sum_{j=1}^{m} c_{ij} w_j + H_i \ \ (i = \overline{1, m}), \tag{4}$$

where indicated: $c_{ij} = \int_0^T B_i(t) A_j(t) dt$, $H_i = \int_0^T B_i(t) h(t) dt, \ i, j = \overline{1, m}$. Now let us refine the conditions on the matrices $A_j(t)$, $B_j(t), j = \overline{1, m}$. It is clear that these matrices must be integrable on the $[0, T]$. We assume that all their elements are continuous on the segment $[0, T]$. In addition, there must be matrices $A_j B_j, B_i A_j, \sum_{i=1}^{m} B_i A_j, B_j y, B_j h$, so their sizes must be consistent for all $i, j = \overline{1, m}$. This can be achieved if we take all matrices $A_i(t)$ of the same size $n \times p$ and all matrices $B_j(t)$ of the same size $p \times n$, where p is any natural number. Then the vector w_i will be a column of the size $p \times 1$, c_{ij} *is* $(p \times p)$-matrix, H_j *is* $(p \times 1)$-vector, $i, j = \overline{1, n}$. Introduce the vectors $w = \{w_1, \ldots, w_m\}$, $H = \{H_1, \ldots, H_m\}$ of the size $(pm) \times 1$ and the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{pmatrix}.$$

This matrix is square in size $(mp) \times (mp)$. Now system (4) can be written as follows:

$$w = \lambda C w + H \Leftrightarrow (I - \lambda C) w = H. \tag{5}$$

For $\lambda = 0$ system (5) has an obvious solution $w = H$, so we will assume that $\lambda \neq 0$. In this case, system (5) can be rewritten as

$$(\mu I - C) w = \mu H \ \ \left(\mu = \frac{1}{\lambda} \right). \tag{6}$$

Now let's establish a connection between system (6) and system (1). These systems are equivalent in the following sense: *if there exists the solution* $y = y(t) \in C([0, T], \mathbb{C}^n)$ *of the system (1), then there exists the corresponding solution*

$$w = \{ \int_0^T B_1(s) y(s) ds, \ \ldots, \int_0^T B_m(s) y(s) ds \}$$

of the system (6). Conversely, if there exists the solution $w = \{w_1, \ldots, w_m\} \in \mathbb{C}^{mp}$ *of the system (6), then there is the solution* $y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t)$*of the original system (1).*

The last statement needs proof, but we will not carry it out. Let us find out in which case different solutions of the system (6) generate different solutions of the integral system (1). So, let $w = \{w_1, \ldots, w_m\}$ and $\tilde{w} = \{\tilde{w}_1, \ldots, \tilde{w}_m\}$ there be different solutions of the system (6). Then the solutions $y(t)$ and $\tilde{y}(t)$ of the integral system (1), corresponding to them, will coincide, if

$$\sum_{j=1}^{m} A_j(t)w_j \equiv \sum_{j=1}^{m} A_j(t)\tilde{w}_j \Leftrightarrow \sum_{j=1}^{m} A_j(t)(w_j - \tilde{w}_j) \equiv 0 \ \ (\forall t \in [0,T]). \tag{7}$$

If we denote by $A_j^{(k)}$ the k-th column of the matrix A_j, and by $w_j^{(k)}$, $\tilde{w}_j^{(k)}$ the k-th components of the vectors w_j and $\tilde{w}_j$ respectively, then identity (7) can be written in the form

$$\sum_{j=1}^{m}\sum_{k=1}^{p} A_j^{(k)}(t) \cdot (w_j^{(k)} - \tilde{w}_j^{(k)}) \equiv 0 \ (\forall t \in [0,T]). \tag{8}$$

Since $w \neq \tilde{w}$, then at least one of the differences $w_j^{(k)} - \tilde{w}_j^{(k)}$ is not equal to zero, therefore the identity (8) means that the columns of the matrix

$$S(t) = (A_1^{(1)}(t), \ldots, A_1^{(p)}(t); A_2^{(1)}(t), \ldots, A_2^{(p)}(t); \ldots; A_m^{(1)}(t), \ldots, A_m^{(p)}(t))$$

are linearly dependent on the segment $[0,T]$. Hence, if the columns of the matrix $S(t)$ are linearly independent on the segment $[0,T]$, then it follows from the identity (8) that everything $w_j^{(k)} \equiv \tilde{w}_j^{(k)}$, and therefore $y(t) \equiv \tilde{y}(t)$. So, in the case of linear independence on the segment $[0,T]$ of the columns of the matrix $S(t)$, the correspondence $w \to y(t)$ will be one-to-one ($w \leftrightarrow y(t)$), therefore, in this case, we can replace the study of the solvability of the system (1) with the study of the solvability of the algebraic system of equations (6) (or what is the same system (5)). Henceforth, we will assume that *the columns of the matrix $S(t)$ are linearly independent on a segment* $[0,T]$. Systems of type (6) are well studied in linear algebra. It is known that if $\mu = \frac{1}{\lambda}$ is not an eigenvalue of the matrix C, then the homogeneous system $(\mu I - C)w = 0$ has only a trivial solution $w = 0$. This means that the corresponding integral system (1) has a solution for any right side $h(t) \in \mathrm{C}([0,T], \mathbb{C}^n)$, which can be written as

$$y(t) = \frac{1}{\mu}\sum_{j=1}^{m} A_j(t)w_j + h(t) \ \ (w \equiv \{w_1, \ldots, w_m\}).$$

If $\mu = \frac{1}{\lambda}$ $(\lambda \neq 0)$ is an eigenvalue of the geometric multiplicity r of the matrix C, then the homogeneous system $(\mu I - C)w = 0$ has the basic system $w^{(1)}, \ldots, w^{(r)}$ of solutions, and its general solution can be written as

$$w = \alpha_1 w^{(1)} + \ldots + \alpha_r w^{(r)},$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary constants. In this case, the conjugate homogeneous system $(\bar{\mu} I - C^*)z = 0$ also has a basic system $z^{(1)}, \ldots, z^{(r)}$ of solutions, consisting of r vectors. In order for the inhomogeneous system (6) to have a solution, it is necessary and sufficient that its right side be orthogonal to all vectors of the basis system of solutions of the adjoint homogeneous system:

$$\left(\mu H, z^{(j)}\right) = 0 \Leftrightarrow \left(H, z^{(j)}\right) = 0, \, j = \overline{1,r}. \tag{9}$$

In this case, the inhomogeneous system (6) has the following solution:

$$w = \alpha_1 w^{(1)} + \ldots + \alpha_r w^{(r)} + \tilde{w}, \tag{10}$$

where $\alpha_1, \ldots, \alpha_r$ are arbitrary constants, $w = \tilde{w}$ is a particular solution of the system (6) (or, what is the same, of the system (5)). Let's see which condition for the original integral system (1) is equivalent to condition (9). For this, we write a homogeneous adjoint equation for (1):

$$\hat{y}(t) = \bar{\lambda} \sum_{j=1}^{m} \int_0^T B_j^*(t) A_j^*(s) \hat{y}(s) ds \Leftrightarrow$$

$$\Leftrightarrow \hat{y}(t) = \bar{\lambda} \sum_{j=1}^{m} B_j^*(t) \int_0^T A_j^*(s) \hat{y}(s) ds. \quad (11)$$

Denoting $z_j = \int_0^T A_j^*(s)\hat{y}(s)ds$, $j = \overline{1, m}$, we rewrite system (11) as

$$\hat{y}(t) = \bar{\lambda} \sum_{j=1}^{m} B_j^*(t) z_j. \quad (12)$$

Multiplying both parts of (12) on the left by $A_i^*(t)$ and integrating over $t \in [0, T]$, we obtain

$$\begin{aligned} &\int_0^T A_i^*(t)\hat{y}(t)dt = \bar{\lambda} \sum_{j=1}^{m} \left(\int_0^T A_i^*(t) B_j^*(t) dt\right) z_j \Leftrightarrow \\ &\Leftrightarrow z_i = \bar{\lambda} \sum_{j=1}^{m} d_{ij} z_j, \; i = \overline{1, m}, \end{aligned} \quad (12_1)$$

where indicated: $d_{ij} = \int_0^T A_i^*(t) B_j^*(t) dt$, $i, j = \overline{1, m}$. It is easy to see that $d_{ij} = c_{ji}^*$, where c_{ij} are the matrices involved in system (4). The matrix of the system (12_1) has the form

$$C^* = \begin{pmatrix} c_{11}^* & c_{21}^* & \cdots & c_{m1}^* \\ c_{12}^* & c_{22}^* & \cdots & c_{m2}^* \\ \cdots & \cdots & \cdots & \cdots \\ c_{1m}^* & c_{2m}^* & \cdots & c_{mm}^* \end{pmatrix},$$

therefore, the algebraic system corresponding to the homogeneous conjugate integral equation (11) will be as follows:

$$\begin{aligned} &z = \bar{\lambda} C^* z \Leftrightarrow (I - \bar{\lambda} C^*) z = 0 \Leftrightarrow \\ &\Leftrightarrow (\bar{\mu} I - C^*) z = 0 \; (\mu = \tfrac{1}{\lambda}, \lambda \neq 0). \end{aligned} \quad (13)$$

All solutions of the adjoint equation (11) are found from (12), where $z = \{z_1, \ldots, z_m\}$ is the solution of the system (13). Orthogonality (9) means that (take into account that

$$H = \{\int_0^T B_1(t)h(t)dt, \ldots, \int_0^T B_m(t)h(t)dt, \; z^{(j)} = \{z_1^{(j)}, \ldots, z_m^{(j)}\})$$

$$\sum_{i=1}^{m} \int_0^T (B_i(t)h(t)dt, \, z_i^{(j)}) dt = 0 \Leftrightarrow \sum_{i=1}^{m} \int_0^T (h(t), B_i^*(t) z_i^{(j)}) dt = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_0^T (h(t), \bar{\lambda} \sum_{i=1}^{m} B_i^*(t) z_i^{(j)}) dt = 0, j = \overline{1, r}.$$

According to (12), we obtain from this that the orthogonality conditions (9) are equivalent to the conditions

$$\int_0^T (h(t), \hat{y}^{(j)}(t)) dt = 0, j = \overline{1, r}, \tag{14}$$

where $\hat{y}^{(1)}(t), \dots, \hat{y}^{(m)}(t)$ is the basic system of solutions of the conjugate homogeneous integral system (11). Thus, if $\mu = \frac{1}{\lambda}(\lambda \neq 0)$ is an eigenvalue of the geometric multiplicity r of the kernel of equation (1), then for the solvability of the integral system (1) in the space $\mathrm{C}([0, T], \mathbb{C}^n)$, it is necessary and sufficient that orthogonality conditions (14) hold. In this case, the general solution of the equation (1) can be written as

$$y(t) = \alpha_1 y^{(1)}(t) + \dots + \alpha_r y^{(r)}(t) + \tilde{y}(t),$$

where $\alpha_1, \dots, \alpha_r$ are arbitrary constants (the same as in (10)), $y^{(1)}(t), \dots, y^{(r)}(t)$ is the basic system of solutions of the corresponding homogeneous equation, and $y(t) = \lambda \int\limits_0^T K(t, s) y(s) ds$ $(\lambda = \frac{1}{\mu})$ is a particular solution of the inhomogeneous system (1).

2 Computational implementation of finding solutions to the integral system (1) with a degenerate kernel

It was shown above that in order to obtain a solution to the integral system (1), it is necessary to find vectors $w = \{w_1, ..., w_m\}$ from system (5) and substitute its components into formula (3). However, despite the simplicity of this scheme, its implementation is associated with considerable computational difficulties. Let's list them:

1) calculation of integrals $c_{ij} = \int\limits_0^T B_i(t) A_j(t) dt \left(i, j = \overline{1, m}\right)$ and compilation of matrices $C = (c_{ij})$, $C^* = (d_{ij})$ of the size $(mp) \times (mp)$;

2) calculation of integrals $H_i = \int\limits_0^T B_i(t) h(t) dt \left(i = \overline{1, m}\right)$ and composing the vector $H = \{H_1, ..., H_m\}$ of the size $(pm) \times 1$;

3) find the solution of the adjoint system $\left(I - \bar{\lambda} C^*\right) z = 0$;

4) verification of the orthogonality conditions $\left(H, z^{(j)}\right) = 0 \ \left(j = \overline{1, m}\right)$, where $z^{(1)}, ..., z^{(m)}$ are the basic solutions of the adjoint system;

5) when the orthogonality conditions are met, the calculation of the solution $w = w_1, ..., w_m$ of the algebraic system $(I - \lambda C) w = H$;

6) constructing the solution to the original integral system (1): $y(t) = \lambda \sum_{j=1}^{m} A_j(t) w_j + h(t)$.

Overcoming these difficulties manually will take a long time, so there is a need to overcome them with the help of some program on the computer. The *intsolve* program in Maple allows you to quickly and efficiently solve scalar integral equations with a degenerate kernel [5–9]. We do not know an analogue of such a program for systems of integral equations, so we considered it necessary to develop it ourselves. For the sake of simplicity of presentation of such a program, consider the case of a second-order system

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \int_0^1 \begin{bmatrix} a_1(t) b_1(s) & a_2(t) b_2(s) \\ a_3(t) b_3(s) & a_4(t) b_4(s) \end{bmatrix} \cdot \begin{bmatrix} y(s) \\ z(s) \end{bmatrix} ds + \begin{bmatrix} m(t) \\ n(t) \end{bmatrix} \tag{15}$$

(the unit in the upper limit of the integral is not essential here; it can be replaced by an arbitrary number T). There is no doubt that this system is a system with a degenerate kernel, but it is not so easy to represent it in the form (1), i.e., to write the kernel as a sum of products of matrices $A_i(t)$ and $B_i(s)$. Therefore, below we choose a way to represent the kernel as a sum of products of matrices with separated variables, based on the expansion of any matrix in a standard basis:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The easiest way to do this is with Maple. First, note that in Maple, indexes can be written both in square brackets and directly in the usual form. For example, a with an index j can be written both in the form a_j and in the form $a[j]$. If we denote by $e1$ and $e2$ unit vectors $e1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then this decomposition can be written as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot e[1] \,.\, (e[1])^{\%T} + b \cdot e[1] \,.\, (e[2])^{\%T} + c \cdot e[2] \,.\, (e[1])^{\%T} + d \cdot e[2] \,.\, (e[2])^{\%T},$$

where $\%T$ is the sign of the transposition, the dot in the middle means the multiplication of a scalar by a vector, and the dot below is the matrix multiplication of vectors. For example,

$$\alpha \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \alpha y \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} . \begin{bmatrix} z & u \end{bmatrix} = \begin{bmatrix} xz & xu \\ yz & yu \end{bmatrix}.$$

Now the kernel of the integral operator in (15) can be written as

$$(a[1](t) \cdot e[1]) \,.\, \left(b[1](s) \cdot (e[1])^{\%T}\right) + (a[2](t) \cdot e[1]) \,.\, \left(b[2](s) \cdot (e[2])^{\%T}\right) +$$
$$+ (a[3](t) \cdot e[2]) \,.\, \left(b[3](s) \cdot (e[1])^{\%T}\right) + (a[4](t) \cdot e[2]) \,.\, \left(b[4](s) \cdot (e[2])^{\%T}\right),$$

and system (15) itself in the form

$$\begin{aligned} u(t) = {} & (a[1](t) \cdot e[1]) \,.\, \int_0^1 \left(b[1](s) \cdot (e[1])^{\%T} .u(s)\right) \mathrm{d}s + \\ & + (a[2](t) \cdot e[1]) \,.\, \int_0^1 \left(b[2](s) \cdot (e[2])^{\%T}\right) .u(s)\, \mathrm{d}s + \\ & + (a[3](t) \cdot e[2]) \,.\, \int_0^1 \left(b[3](s) \cdot (e[1])^{\%T}\right) .u(s)\, \mathrm{d}s + \\ & + (a[4](t) \cdot e[2]) \,.\, \int_0^1 \left(b[4](s) \cdot (e[2])^{\%T}\right) .u(s)\, \mathrm{d}s + h(t), \end{aligned} \tag{16}$$

where $u(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$, $h(t) = \begin{bmatrix} m(t) \\ n(t) \end{bmatrix}$. In this notation, the kernel of the integral operator is represented as the sum of products of matrices with separated variables:

$$A1(t) = a[1](t) \cdot e[1]\,;\, B1(s) = b[1](s) \cdot (e[1])^{\%T}\,;\, A2(t) = a[2](t) \cdot e[1]\,;$$
$$B2(s) = b[2](s) \cdot (e[2])^{\%T}\,;\, A3(t) = a[3](t) \cdot e[2]\,;\, B3(s) = b[3](s) \cdot (e[1])^{\%T}\,;$$
$$A4(t) = a[4](t) \cdot e[2]\,;\, B4(s) = b[4](s) \cdot (e[2])^{\%T}.$$

Let us rewrite system (16) in the form

$$u(t) = A1(t) \,.\, \int_0^1 B1(s) .u(s)\, \mathrm{d}s + A2(t) \,.\, \int_0^1 B2(s) .u(s)\, \mathrm{d}s +$$
$$+ A3(t) \,.\, \int_0^1 B3(s) .u(s)\, \mathrm{d}s + A4(t) \,.\, \int_0^1 B4(s) .u(s)\, \mathrm{d}s + h(t).$$

Enter numbers

$$\begin{gathered}\int\limits_0^1 B1(s).u(s)\,\mathrm{d}s = w1, \int\limits_0^1 B2(s).u(s)\,\mathrm{d}s = w2,\\ \int\limits_0^1 B3(s).u(s)\,\mathrm{d}s = w3, \int\limits_0^1 B4(s).u(s)\,\mathrm{d}s = w4.\end{gathered} \tag{17}$$

Then system (17) takes the form

$$u(t) = A1(t)\cdot w1 + A2(t)\cdot w2 + A3(t)\cdot w3 + A4(t)\cdot w4 + h(t). \tag{18}$$

We multiply this equality successively by matrices $B1(t), B2(t), B3(t), B4(t)$ on the left and integrate the results over $t \in [0,1]$; we get

$$\begin{gathered}
w1 = \left(\int\limits_0^1 B1(t)\;.A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int\limits_0^1 B1(t)\;.A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
+\left(\int\limits_0^1 B1(t)\;.A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int\limits_0^1 B1(t)\;.A4(t)\,\mathrm{d}t\right)\cdot w4 + \int\limits_0^1 B1(t).h(t)\,\mathrm{d}t;\\
w2 = \left(\int\limits_0^1 B2(t)\;.A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int\limits_0^1 B2(t)\;.A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
\left(\int\limits_0^1 B2(t)\;.A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int\limits_0^1 B2(t)\;.A4(t)\,\mathrm{d}t\right)\cdot w4 + \int\limits_0^1 B2(t).h(t)\,\mathrm{d}t;\\
w3 = \left(\int\limits_0^1 B3(t)\;.A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int\limits_0^1 B3(t)\;.A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
+\left(\int\limits_0^1 B3(t)\;.A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int\limits_0^1 B3(t)\;.A4(t)\,\mathrm{d}t\right)\cdot w4 + \int\limits_0^1 B3(t).h(t)\,\mathrm{d}t;\\
w4 = \left(\int\limits_0^1 B4(t)\;.A1(t)\,\mathrm{d}t\right)\cdot w1 + \left(\int\limits_0^1 B4(t)\;.A2(t)\,\mathrm{d}t\right)\cdot w2 +\\
+\left(\int\limits_0^1 B4(t)\;.A3(t)\,\mathrm{d}t\right)\cdot w3 + \left(\int\limits_0^1 B4(t)\;.A4(t)\,\mathrm{d}t\right)\cdot w4 + \int\limits_0^1 B4(t).h(t)\,\mathrm{d}t.
\end{gathered} \tag{19}$$

Since the matrices $A[i]$, $B[j]$ are known and their product $B[j].A[i]$ is a scalar quantity, then (19) is a system of linear algebraic equations with respect to the unknowns $w1, w2, w3, w4$. Solving this system in Maple and substituting the found unknowns in (18), we find the solution of the original integral system (15).

Example 1. Solve a system of integral equations

$$\begin{gathered}
y(t) = \int\limits_0^1 6tsy(s)\;\mathrm{d}s + \int\limits_0^1 3t^2 sz(s)\;\mathrm{d}s + t^2 + 1,\\
z(t) = \int\limits_0^1 (3+t)(5s+3)\,y(s)\;\mathrm{d}s + \int\limits_0^1 (8t+5)\,s^3 z(s)\;\mathrm{d}s + 4t.
\end{gathered} \tag{20}$$

Solution. Enter the coefficients

$$\begin{gathered}
a_1(t) := 6\,t; b_1(t) := t; a_2(t) := \;3\,t^2; b_2(t) := t; a_3(t) := 3+\;t;\\
b_3(t) := 5\,t+3; a_4(t) := 8\,t+5; b_4(t) := \;t^3; m(t) := t^2+1; n(t) := 4\,t;
\end{gathered}$$

Enter vectors

$$e[1] := \begin{bmatrix}1\\0\end{bmatrix}; \quad e[2] := \begin{bmatrix}0\\1\end{bmatrix}; u(t) := \begin{bmatrix}y(t)\\z(t)\end{bmatrix}; h(t) := \begin{bmatrix}m(t)\\n(t)\end{bmatrix};$$

Enter matrices:

$$A1(t) := a[1](t) \cdot e[1]; B1(s) := b[1](s) \cdot (e[1])^{\%T}; A2(t) := a[2](t) \cdot e[1];$$
$$B2(s) := b[2](s) \cdot (e[2])^{\%T}; A3(t) := a[3](t) \cdot e[2]; B3(s) := b[3](s) \cdot (e[1])^{\%T};$$
$$A4(t) := a[4](t) \cdot e[2]; B4(s) := b[4](s) \cdot (e[2])^{\%T};$$

We compose and solve a system of equations for unknowns $w1, w2, w3, w4$:

$$w1 = \left(\int_0^1 B1(t).A1(t)\,\mathrm{d}t\right) \cdot w1 + \left(\int_0^1 B1(t).A2(t)\,\mathrm{d}t\right) \cdot w2 +$$
$$\left(\int_0^1 B1(t).A3(t)\,\mathrm{d}t\right) \cdot w3 + \left(\int_0^1 B1(t).A4(t)\,\mathrm{d}t\right) \cdot w4 + \int_0^1 B1(t).h(t)\,\mathrm{d}t$$

$$w2 = \left(\int_0^1 B2(t).A1(t)\,\mathrm{d}t\right) \cdot w1 + \left(\int_0^1 B2(t).A2(t)\,\mathrm{d}t\right) \cdot w2 +$$
$$\left(\int_0^1 B2(t).A3(t)\,\mathrm{d}t\right) \cdot w3 + \left(\int_0^1 B2(t).A4(t)\,\mathrm{d}t\right) \cdot w4 + \int_0^1 B2(t).h(t)\,\mathrm{d}t$$

$$w3 = \left(\int_0^1 B3(t).A1(t)\,\mathrm{d}t\right) \cdot w1 + \left(\int_0^1 B3(t).A2(t)\,\mathrm{d}t\right) \cdot w2$$
$$+\left(\int_0^1 B3(t).A3(t)\,\mathrm{d}t\right) \cdot w3 + \left(\int_0^1 B3(t).A4(t)\,\mathrm{d}t\right) \cdot w4 + \int_0^1 B3(t).h(t)\,\mathrm{d}t$$

$$w4 = \left(\int_0^1 B4(t).A1(t)\,\mathrm{d}t\right) \cdot w1 + \left(\int_0^1 B4(t).A2(t)\,\mathrm{d}t\right) \cdot w2 +$$
$$\left(\int_0^1 B4(t).A3(t)\,\mathrm{d}t\right) \cdot w3 + \left(\int_0^1 B4(t).A4(t)\,\mathrm{d}t\right) \cdot w4 + \int_0^1 B4(t).h(t)\,\mathrm{d}t$$

Calculate the solution of the original integral system

$$u(t) = A1(t) \cdot w1 + A2(t) \cdot w2 + A3(t) \cdot w3 + A4(t) \cdot w4 + h(t);$$

Answer.

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{602}{381}t^2 - \frac{80}{127}t + 1 \\ -\frac{832}{381} + \frac{265}{381}t \end{bmatrix}.$$

Verification. Let us introduce the obtained solutions:

$$y(t) := -\frac{602}{381}t^2 - \frac{80}{127}t + 1;\ z(t) := -\frac{832}{381} + \frac{265}{381}t.$$

Let us calculate the difference between the left and right parts of the original system:

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} \int_0^1 6tsy(s)\ \mathrm{d}s + \int_0^1 3t^2sz(s)\ \mathrm{d}s + t^2 + 1 \\ \int_0^1 (3+t)(5s+3)\,y(s)\ \mathrm{d}s + \int_0^1 (8t+5)\,s^3z(s)\ \mathrm{d}s + 4t \end{bmatrix}.$$

Got a vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus, the solution to system (20) is the vector function

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -\frac{602}{381}t^2 - \frac{80}{127}t + 1 \\ -\frac{832}{381} + \frac{265}{381}t \end{bmatrix}.$$

Remark 1. We have considered the two-dimensional case of the integral system (20). It is clear that the described algorithm obviously extends to any integral systems of type (20) of order higher than the second.

3 Systems of integro-differential equations with a degenerate kernel and their reduction to integral systems

Systems of equations of the form

$$\frac{dy}{dt} = A(t)y + \lambda \int_0^T K(t,s)y(s)ds + h(t), y(0,\varepsilon) = y^0, t \in [0,T], \tag{21}$$

where $y = \{y_1(t), \dots, y_n(t)\}$is an unknown function, $h(t) = \{h_1(t), \dots, h_n(t)\}$ is the known function (inhomogeneity), $A(t)$, $K(t,s)$ are known matrices of size $n\times\ n$, are called *systems of integro-differential equations of the Fredholm type (or simply integro-differential systems).* They can be reduced to an integrated system. It is done like this.

Let us assume that $Y(t)$ is the fundamental matrix of solutions of the differential system $\frac{dz}{dt} = A(t)z$. Taking $H(t) \equiv \lambda \int_0^T K(t,s)y(s)ds + h(t)$ for the inhomogeneity of the differential system $dy/dt = A(t)y + H(t)$, we find its "solution"

$$\begin{aligned} y(t) = Y(t)y^0 + \lambda \int_0^t Y(t)Y^{-1}(\zeta)(\int_0^T K(\zeta,s)y(s)ds)d\zeta + \\ + \int_0^t Y(t)Y^{-1}(\zeta)h(\zeta)d\zeta. \end{aligned} \tag{22}$$

Denoting

$$h_0(t) \equiv Y(t)y^0 + \int_0^t Y(t)Y^{-1}(\zeta)h(\zeta)d\zeta \tag{23}$$

and changing the order of integration in the iterated integral (22), we have

$$y(t) = \lambda \int_0^T (\int_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta)y(s)ds + h_0(t). \tag{24}$$

We have obtained an integral system (24) with a kernel

$$G(t,s) \equiv \int_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta. \tag{25}$$

It is easy to show that the system (24) is equivalent to the system (21). The following result is obtained.

Lemma 1. If $Y(t)$ is a fundamental matrix of solutions of a homogeneous system $\dot{z} = A(t)z$ (it is assumed that it exists on a segment $[0,T]$), then the integro-differential system (21) is equivalent to the integral Fredholm type:

$$y(t) = \lambda \int_0^1 G(t,s)y(s)ds + h_0(t), \tag{26}$$

where $h_0(t) \equiv Y(t)y^0 + \int_0^t Y^{-1}(\zeta)h(\zeta)d\zeta$, and the kernel $G(t,s)$ has the form (25).

For equations (26) of the Fredholm type, statements about solvability look rather complicated.

Theorem 1. Let in the system (21) the matrices $A(t) \in \mathrm{C}([0,T],\mathbb{C}^{n\times n})$,$K(t,s) \in \mathrm{C}(0 \le s,t \le$ $\le T,\mathbb{C}^{n\times n})$,$h(t) \in \mathrm{C}([0,T],\mathbb{C}^{n})$. Then the following statements are true:

a) if λ is not a characteristic value of the kernel (25), then the integro-differential system (21) is solvable for any right-hand side $h(t)$ and, moreover, uniquely; in this case, its solution is given by the formula

$$y(t) = h_0(t) + \lambda \int\limits_0^T R_\lambda(t,s)h_0(s)ds,$$

where$R_\lambda(t,s)$ *is the resolvent of the kernel (25),* $h_0(t)$ *is the function (23);*

b) if λ is the characteristic value of the kernel (25) of rank r, then system (21) is solvable in the space $\mathrm{C}^1([0,T],\mathbb{C}^n)$ if and only if the inhomogeneity (23) is orthogonal to all solutions of the homogeneous adjoint system $z(t) = \bar{\lambda}\int\limits_0^T \overline{G^T(s,t)}z(s)ds$, i.e.

$$\int\limits_0^T (h_0(t), z^{(j)}(t))dt = 0,\, j = \overline{1,r},$$

where $z^{(1)}(t),\ldots,z^{(r)}$ is the basic system of solutions of the homogeneous adjoint system. In this case, the solution of the integro-differential system (21) is given by the formula

$$y(t) = \sum_{j=1}^{r} \alpha_j y^{(j)}(t) + \tilde{y}(t),$$

where $y^{(1)}(t),\ldots,y^{(r)}(t)$ is the basic system of solutions of the homogeneous system, $y(t) = \lambda\int\limits_0^T G(t,s)y(s)ds$, $\tilde{y}(t)$ is a particular solution of the integral system (26), and $\alpha_1,\ldots,\alpha_r$ are arbitrary constants.

Now let the kernel in the original equation (21) be degenerate, i.e.

$$K(t,s) = \sum_{j=1}^{m} A_j(t)B_j(s), \tag{27}$$

where all $A_j(t)$ are matrices of the size $n\times p$, and all $B_j(s)$ are matrices of the size $p\times n$, $j = \overline{1,m}$ (we assume that the columns of the matrix $S(t) = (A_1(t),\ldots,A_m(t))$ are linearly independent on the segment $[0,T]$). Then the kernel of equation (26) will have the form

$$\begin{aligned} G(t,s) &= \int\limits_0^t Y(t)Y^{-1}(\zeta)K(\zeta,s)d\zeta = \\ &= \textstyle\sum_{j=1}^m Y(t)\left(\int\limits_0^t Y^{-1}(\zeta)A_j(\zeta)d\zeta\right)B_j(s) \equiv \sum_{j=1}^m \Phi_j(t)B_j(s), \end{aligned} \tag{28}$$

where denoted: $\Phi_j(t) \equiv Y(t)\int\limits_0^t Y^{-1}(\zeta)A_j(\zeta)d\zeta, j = \overline{1,m}$. Hence, the degenerate kernel (27) of the original integro-differential system (21) generates the degenerate kernel (28) of the integral system (24), equivalent to it, therefore, to construct a solution to system (24), we can apply the procedure developed above. We will show how this is done using the Maple program in the following example.

Example 2. Let's try to get the solution of the system

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}\,t}\, y(t) = -y(t) + \int_0^1 a_1(t)\, b_1(s)\, y(s)\ \mathrm{d}s + \int_0^1 a_2(t)\, b_2(s)\, z(s)\ \mathrm{d}s + m(t)\,,\\
&\tfrac{\mathrm{d}}{\mathrm{d}\,t}\, z(t) = -2\, z(t) + \int_0^1 a_3(t)\, b_3(s)\, y(s)\ \mathrm{d}s + \int_0^1 a_4(t)\, b_4(s)\, z(s)\ \mathrm{d}s + n(t)\,,\\
&y(0) = a, z(0) = b,
\end{aligned} \tag{29}$$

where, for the sake of simplicity, the following data are taken:

$$\begin{aligned}
&a1(t) = t; a2(t) = t^2; a3(t) = 2\,t; a4(t) = t+1; b1(t) = 3\,t;\\
&b2(t) = 2\,t^2; b3(t) = t; b4(t) = t-1; m(t) = 2\,t; n(t) = t^2; a = 1;\ b = 3.
\end{aligned}$$

Solution.[*]
restart:
with(linalg):
Enter the coefficients:

$$\begin{aligned}
&a1(t) := t; a2(t) := t^2; a3(t) := 2\,t; a4(t) := t+1; b1(t) := 3\,t;\\
&b2(t) := 2\,t^2; b3(t) := t; b4(t) := t-1; m(t) := 2\,t; n(t) := t^2; a := 1;\ b := 3;
\end{aligned}$$

Enter kernel:

$$\begin{aligned}
&(a1(t)\cdot e[1])\,.\left(b1(s)\cdot (e[1])^{\%T}\right) + (a2(t)\cdot e[1])\,.\left(b2(s)\cdot (e[2])^{\%T}\right)\\
&+ (a3(t)\cdot e[2])\,.\left(b3(s)\cdot (e[1])^{\%T}\right) + (a4(t)\cdot e[2])\,.\left(b4(s)\cdot (e[2])^{\%T}\right);
\end{aligned}$$

Enter vectors:

$$h(t) := \begin{bmatrix} m(t)\\ n(t)\end{bmatrix};$$
$$u(t) := \begin{bmatrix} y(t)\\ z(t)\end{bmatrix};$$

Then the integro-differential system (29) takes the form:

$$\begin{aligned}
&map(diff,\ u(t), t) = \begin{bmatrix} -1 & 0\\ 0 & -2\end{bmatrix} .u(t) + (a1(t)\cdot e[1])\,.\int_0^1 \left(b1(s)\cdot (e[1])^{\%T} .u(s)\right) \mathrm{d}s\\
&+ (a2(t)\cdot e[1])\,.\int_0^1 \left(b2(s)\cdot (e[2])^{\%T}\right) .u(s)\,\mathrm{d}s\\
&+ (a3(t)\cdot e[2])\,.\int_0^1 \left(b3(s)\cdot (e[1])^{\%T}\right) .u(s)\,\mathrm{d}s\\
&+ (a4(t)\cdot e[2])\,.\int_0^1 \left(b4(s)\cdot (e[2])^{\%T}\right) .u(s)\,\mathrm{d}s + h(t);
\end{aligned}$$

Enter matrices:

$$\begin{aligned}
&A1(t) := a1(t)\cdot e[1]; B1(s) := b1(s)\cdot (e[1])^{\%T}; A2(t) := a2(t)\cdot e[1];\\
&B2(s) := b2(s)\cdot (e[2])^{\%T}; A3(t) := a3(t)\cdot e[2]; B3(s) := b3(s)\cdot (e[1])^{\%T};\\
&A4(t) := a4(t)\cdot e[2]; B4(s) := b4(s)\cdot (e[2])^{\%T};
\end{aligned}$$

[*] Maple does not put punctuation marks.

Then the IDE system (29) can be rewritten as:

$$map\left(diff,\ u\left(t\right),t\right)=\left[\begin{array}{cc}-1 & 0\\ 0 & -2\end{array}\right].u\left(t\right)+A1\left(t\right).\int\limits_{0}^{1}\ B1\left(s\right).u\left(s\right)\mathrm{d}s$$
$$+A2\left(t\right).\int\limits_{0}^{1}\ B2\left(s\right).u\left(s\right)\mathrm{d}s+A3\left(t\right).\int\limits_{0}^{1}\ B3\left(s\right).u\left(s\right)\mathrm{d}s$$
$$+A4\left(t\right).\int\limits_{0}^{1}B4\left(s\right).u\left(s\right)\mathrm{d}s+h\left(t\right);$$

We find the fundamental decision matrix:

$$dsolve\left(\left\{\frac{\mathrm{d}}{\mathrm{d}\,t}\,y\left(t\right)=-y\left(t\right),\frac{\mathrm{d}}{\mathrm{d}\,t}\,z\left(t\right)=-2\,z\left(t\right)\right\}\right);$$

$$Y\left(t\right):=\left[\begin{array}{cc}\mathrm{e}^{-t} & 0\\ 0 & \mathrm{e}^{-2\,t}\end{array}\right];$$

Change the inhomogeneity:

$$h1\left(t\right):=Y\left(t\right).\left[\begin{array}{c}a\\ b\end{array}\right]+map\left(int,Y\left(t\right).Y^{-1}\left(x\right).h\left(x\right),x=0..t\right);$$

Enter matrices:

$$F1\left(t\right):=map\left(int,Y\left(t\right).Y^{-1}\left(x\right).A1\left(x\right),x=0..t\right);$$
$$F2\left(t\right):=map\left(int,Y\left(t\right).Y^{-1}\left(x\right).A2\left(x\right),x=0..t\right);$$
$$F3\left(t\right):=map\left(int,Y\left(t\right).Y^{-1}\left(x\right).A3\left(x\right),x=0..t\right);$$
$$F4\left(t\right):=map\left(int,Y\left(t\right).Y^{-1}\left(x\right).A4\left(x\right),x=0..t\right);$$

Denote $wj=\int\limits_{0}^{1}Bj\left(s\right).y\left(s\right)\ \mathrm{d}s,j=\overline{1,4}:$

Then the equivalent integral system can be written as:

$$u\left(t\right)=F1\left(t\right)\cdot w1+F2\left(t\right)\cdot w2+F3\left(t\right)\cdot w3+F4\left(t\right)\cdot w4+h1\left(t\right):$$

Multiply this equation on the left sequentially by the matrices $B1\left(t\right),B2\left(t\right),B3\left(t\right),B4\left(t\right)$ and integrate the results obtained over $t\in[0,1]$. We obtain the system of algebraic equations:

$$eq1:=w1=map\left(\begin{array}{l}int,w1\cdot B1\left(t\right).F1\left(t\right)\ +w2\cdot B1\left(t\right).F2\left(t\right)\\ +w3\cdot B1\left(t\right).F3\left(t\right)\ +w4\cdot B1\left(t\right).F4\left(t\right)\\ +B1\left(t\right).h1\left(t\right),t=0..1\end{array}\right);$$
$$eq2:=w2=map\left(\begin{array}{l}int,w1\cdot B2\left(t\right).F1\left(t\right)\ +w2\cdot B2\left(t\right).F2\left(t\right)\\ +w3\cdot B2\left(t\right).F3\left(t\right)\ +w4\cdot B2\left(t\right).F4\left(t\right)\\ +B2\left(t\right).h1\left(t\right),t=0..1\end{array}\right);$$
$$eq3:=w3=map\left(\begin{array}{l}int,w1\cdot B3\left(t\right).F1\left(t\right)\ +w2\cdot B3\left(t\right).F2\left(t\right)\\ +w3\cdot B3\left(t\right).F3\left(t\right)\ +w4\cdot B3\left(t\right).F4\left(t\right)\\ +B3\left(t\right).h1\left(t\right),t=0..1\end{array}\right);$$
$$eq4:w4=map\left(\begin{array}{l}int,w1\cdot B4\left(t\right).F1\left(t\right)\ +w2\cdot B4\left(t\right).F2\left(t\right)\\ +w3\cdot B4\left(t\right).F3\left(t\right)\ +w4\cdot B4\left(t\right).F4\left(t\right)\\ +B4\left(t\right).h1\left(t\right),t=0..1\end{array}\right);$$

Let's solve this system:

$$solve\left(\left[eq1,eq2,eq3,eq4\right],\left\{w1,w2,w3,w4\right\}\right):$$

and activate the found solutions with the assignment operator (:=).

We write down the solution of the original integro-differential system (29):

$$F1\,(t)\cdot w1 + F2\,(t)\cdot w2 + F3\,(t)\cdot w3 + F4\,(t)\cdot w4 + h1\,(t)\,;$$

$$\begin{aligned} y\,(t) := &-\tfrac{9}{20}\tfrac{\left(394416\mathrm{e}^{-1}\mathrm{e}^{-2}+10800\mathrm{e}^{-1}-139299\mathrm{e}^{-2}-10975\right)\left(\mathrm{e}^{t}t-\mathrm{e}^{t}+1\right)\mathrm{e}^{-t}}{9864\mathrm{e}^{-1}\mathrm{e}^{-2}+8664\mathrm{e}^{-1}-3561\mathrm{e}^{-2}-1831} \\ &-\tfrac{1}{10}\tfrac{\left(739476\mathrm{e}^{-1}\mathrm{e}^{-2}-105660\mathrm{e}^{-1}-154209\mathrm{e}^{-2}+17035\right)\left(\mathrm{e}^{t}t^{2}-2\mathrm{e}^{t}t+2\mathrm{e}^{t}-2\right)\mathrm{e}^{-t}}{9864\mathrm{e}^{-1}\mathrm{e}^{-2}+8664\mathrm{e}^{-1}-3561\mathrm{e}^{-2}-1831}+ \\ &\mathrm{e}^{-t}+\left(2\mathrm{e}^{t}t-2\mathrm{e}^{t}+2\right)\mathrm{e}^{-t}; \end{aligned} \tag{30}$$

$$\begin{aligned} z\,(t) := &-\tfrac{3}{40}\tfrac{\left(394416\mathrm{e}^{-1}\mathrm{e}^{-2}+10800\mathrm{e}^{-1}-139299\mathrm{e}^{-2}-10975\right)\left(2t\mathrm{e}^{2t}-\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{9864\mathrm{e}^{-1}\mathrm{e}^{-2}+8664\mathrm{e}^{-1}-3561\mathrm{e}^{-2}-1831} \\ &-\tfrac{1}{40}\tfrac{\left(98208\mathrm{e}^{-1}\mathrm{e}^{-2}+57696\mathrm{e}^{-1}-26187\mathrm{e}^{-2}-11719\right)\left(2t\mathrm{e}^{2t}+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{9864\mathrm{e}^{-1}\mathrm{e}^{-2}+8664\mathrm{e}^{-1}-3561\mathrm{e}^{-2}-1831} \\ &+3\mathrm{e}^{-2t}+\left(\tfrac{1}{2}t^{2}\mathrm{e}^{2t}-\tfrac{1}{2}t\mathrm{e}^{2t}+\tfrac{1}{4}\mathrm{e}^{2t}-\tfrac{1}{4}\right)\mathrm{e}^{-2\,t}; \end{aligned}$$

The verification is carried out by substituting the solution into the difference between the left and right parts of the system (29):

$$\begin{aligned} &\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}y\,(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}z\,(t) \end{bmatrix} - \begin{bmatrix} -y\,(t)+2t \\ -2z\,(t)+t^{2} \end{bmatrix} \\ &-\left(\begin{array}{l} \begin{bmatrix} t \\ 0 \end{bmatrix}\cdot\left(\int\limits_0^1 3sy\,(s)\ \mathrm{d}s\right)+\begin{bmatrix} t^{2} \\ 0 \end{bmatrix}\cdot\left(\int\limits_0^1 2s^{2}z\,(s)\ \mathrm{d}s\right) \\ +\begin{bmatrix} 0 \\ 2t \end{bmatrix}\cdot\left(\int\limits_0^1 sy\,(s)\ \mathrm{d}s\right)+\begin{bmatrix} 0 \\ t+1 \end{bmatrix}\cdot\left(\int\limits_0^1 (s-1)\,z\,(s)\ \mathrm{d}s\right) \end{array}\right) \end{aligned}$$

$\xrightarrow{\text{simplify symbolic}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Consequently, functions (30) satisfy system (29).

Remark 2. When entering data in a Maple file, take into account that exponents and signs of differentials are entered as operators.

In conclusion, we note that the developed procedure, with some modifications, will be used to study linear and nonlinear singularly perturbed systems of integral and integro-differential equations with rapidly oscillating coefficients and inhomogeneities [10–14].

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Өзгешеленген ядросы бар интегралдық теңдеулер жүйесі және оларды Maple бағдарламасымен шешу алгоритмі

Математикалық әдебиеттерде өзгешеленген ядросы бар скалярлық интегралдық теңдеулер жақсы сипатталған (төменде (1) қараңыз, мұнда барлық жазылған функциялар скаляр шамалар). Авторларға матрицалардың көбейтіндісі түріндегі ядролары бар (1) типті интегралдық теңдеулер жүйесі егжей-тегжейлі қарастырылатын жарияланымдар белгісіз. Әдетте мұндай жүйелерді шешу әдістемесі скаляр жағдайдан векторлық жағдайға оңай ауыстырылады деп айтылады (мысалы, А.Л. Калашниковтың "Методы приближенного решения интегральных уравнений второго рода" (Нижний Новгород: ННГУ, 2017) монографиясында өзгешеленген ядролы теңдеулер жүйесінің қысқаша сипаттамасы берілген, мұнда өзгешеленген ядроның рөлін матрицалық функциялар емес, скалярлық функциялардың көбейтіндісі атқарады делінген). Алайда, қарапайым мысалдар көрсеткендей, матрицалық функциялардың көбейтіндісінің қосындысы түріндегі ядролы интегралдық жүйелер жағдайына скалярлық жағдайдың идеяларын жалпылау біршама түсініксіз, дегенмен бұл жағдайда интегралдық жүйені алгебралық теңдеулер жүйесіне келтіру идеясы қолданылады. Сонымен қатар, сәйкес алгебралық жүйенің шешімділік шарттарына сүйене отырып, ортогоналдылық шарттары түріндегі интегралдық жүйенің шешімділік шарттарын алу процесі бұрын сипатталмаған. Қолданбалы есептердегі интегралдық теңдеулер теориясының кең қолданылуын ескере отырып, авторлар көпөлшемді жағдайда ядролары өзгешеленген интегралдық жүйелерді шешудің егжей-тегжейлі схемасын беруді және бұл схеманы Maple бағдарламасында енгізуді қажет деп санады. Maple бағдарламасында тек скалярлық интегралдық теңдеулер *intsolve* процедурасы арқылы шешілетінін ескеріңіз. Авторлар интегралдық теңдеулер жүйесін шешудің ұқсас процедурасын таппады, сондықтан олар өздерінің процедурасын жасады.

Кілт сөздер: интегралдық оператор, өзгешеленген ядро, Maple бағдарламасының процедурасы, скалярлық интегралдық теңдеу.

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Системы интегральных уравнений с вырожденным ядром и алгоритм их решения с помощью программы Maple

В математической литературе хорошо описано скалярное интегральное уравнение с вырожденным ядром (см. ниже (1), где все записанные функции являются скалярными величинами). Авторам неизвестны публикации, в которых подробно рассматривались бы системы интегральных уравнений типа (1) с ядрами в виде произведения матриц. Обычно говорят, что техника решения таких систем легко переводится со скалярного случая на векторный (например, в монографии А.Л. Калашникова «Методы приближенного решения интегральных уравнений второго рода» (Нижний Новгород: ННГУ, 2017). Дано краткое описание систем уравнений с вырожденными ядрами, где роль вырожденных ядер играют произведения скалярных, а не матричных функций). Однако, как показывают простейшие примеры, обобщение идей скалярного случая на случай целочисленных систем с ядрами в виде суммы произведений матриц-функций весьма неясно, хотя в этом случае используется идея сведения интеграла уравнения к алгебраической системе. В то же время процесс получения условий разрешимости интегральной системы в виде условий ортогональности на основе условий разрешимости соответствующей алгебраической системы, как нам кажется, ранее не описывался. Учитывая широкое применение теории интегральных уравнений в прикладных задачах, авторы сочли необходимым привести подробную схему решения интегральных систем с вырожденными ядрами в многомерном случае и реализовать эту схему в программе Maple. Обратите внимание, что в Maple с помощью процедуры *intsolve* решаются только скалярные интегральные уравнения. Авторы не нашли аналогичной методики решения систем интегральных уравнений, поэтому разработали собственную методику.

Ключевые слова: интегральный оператор, вырожденное ядро, программная процедура Maple, скалярное интегральное уравнение.

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