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Inverse coefficient problem for differential equation in partial derivatives of a fourth order in time with integral over-determination

Derivatives in time of higher order (more than two) arise in various fields such as acoustics, medical ultrasound, viscoelasticity and thermoelasticity. The inverse problems for higher order derivatives in time equations connected with recovery of the coefficient are scarce and need additional consideration. In this article the inverse problem of determination is considered, which depends on time, lowest term coefficient in differential equation in partial derivatives of fourth order in time with initial and boundary conditions from an additional integral observation. Under some conditions of regularity, consistency and orthogonality of data by using of the contraction principle the unique solvability of the solution of the coefficient identification problem on a sufficiently small time interval has been proved.

Keywords: Inverse problems for PDEs, fourth order in time PDE, existence and uniqueness.

Introduction

Fourth order derivative in time arises in various fields. For instance, in the Taylor series expansion of the Hubble law [1], in the study of chaotic hyper jerk systems [2] and in the kinematic performance of long-dwell mechanisms of linkage type [3]. The fourth order in time equation, that is our motivation point, was introduced and first studied by Dell'Oro and Pata [4]

$$\partial_{\tau\tau\tau\tau}u(x,\tau)+\alpha\partial_{\tau\tau\tau}u(x,\tau)+\beta\partial_{\tau\tau}u(x,\tau)-\gamma\triangle\partial_{\tau\tau}u(x,\tau)-\rho\triangle u(x,\tau)=0,$$

where α,β,γ,ρ are real numbers. More recently, this model has attracted the attention of many authors [5–9].

We consider an inverse problem of recovering the time-dependent lowest term in the fourth order in time partial differential equation in the following type

$$\partial_{\tau\tau\tau\tau}u(x,\tau)+\partial_{\tau\tau}u(x,\tau)-\triangle\partial_{\tau\tau}u(x,\tau)-\triangle u(x,\tau)=a(\tau)u(x,\tau)+f(x,\tau) \tag{1}$$

subject to the initial conditions

$$u(x,0)=\xi_0(x),\ u_\tau(x,0)=\xi_1(x),\ u_{\tau\tau}(x,0)=\xi_2(x),u_{\tau\tau\tau}(x,0)=\xi_3(x) \tag{2}$$

and the boundary conditions

$$u(0,\tau)=u_x(1,\tau)=0, \tag{3}$$

and the additional condition

$$\int_0^1 u(x,\tau)dx=E(\tau), \tag{4}$$

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where $D_T = \{(x,\tau): 0 \le x \le 1,\ 0 \le \tau \le T\}$ for some fixed $T > 0$, $f(x,\tau)$ is the force function, $\xi_i(x)$, $i = 0,1,2,3$ are initial displacements, and $E(\tau)$ is the extra integral measurement to obtain the solution of the inverse problem.

The inverse coefficient problems for the first or second order in time (i.e. parabolic and hyperbolic equations, respectively) PDEs are studied satisfactorily. The inverse problems of the parabolic and hyperbolic PDEs investigated numerically and/or theoretically in [10–12] and [13, 14], respectively. The inverse problems of determining time or space dependent coefficients for the higher order in time (more than 2) PDEs attract many scientists. The inverse problem of recovering the solely space dependent and solely time dependent coefficients for the third order in time PDEs are studied by [15, 16], respectively. More recently, in [17] authors studied the inverse problem of determining time dependent potential and time dependent force terms from the third order in time partial differential equation theoretically and numerically by considering the critical parameter equal to zero.

Main purpose of this paper is the simultaneous identification of the time-dependent lowest coefficient $a(\tau)$, and $u(x,\tau)$, for the first time, from the equation (1), initial conditions (2), homogeneous boundary conditions (3) and additional condition (4) under some regularity and consistency conditions.

The article is organized as following: in Section 2, we first present the eigenvalues and eigenfunctions of the corresponding Sturm-Liouville spectral problem for equation (1), and two Banach spaces, which are related to the eigenvalues and eigenfunctions of the auxiliary Sturm-Liouville spectral problem, are introduced. Then, we transform the inverse problem into a system of Volterra integral equations by using the eigenfunction expansion method. Under some consistency and regularity conditions on data, the existence and uniqueness theorem of the solution of the inverse problem is proved via Banach fixed point theorem for sufficiently small times.

1 *Existence and Uniqueness*

In this section, we will set and prove the existence and uniqueness theorem of the solution of the inverse initial-boundary value problem for the fourth order in time equation by using Banach fixed point theorem.

The auxiliary spectral problem of the inverse problem (1)–(4) is

$$\begin{cases} Y''(x) + \lambda Y(x) = 0, & 0 \le x \le 1, \\ Y(0) = Y'(1) = 0. \end{cases} \tag{5}$$

The eigenvalues and corresponding eigenfunctions of these eigenvalues of the spectral problem (5) are $\mu_n = \left(\frac{2n+1}{2}\pi\right)^2$ and $Y_n(x) = \sqrt{2}\sin(\sqrt{\mu_n}x)$, $n = 0,1,2,...$, respectively. The system of eigenfunctions $Y_n(x)$ are biorthonormal on $[0,1]$, i.e.

$$\int_0^1 Y_n(x)Y_m(x)dx = \begin{cases} 1, & m = n \\ 0, & m \ne n \end{cases}.$$

Also the system of eigenfunctions $Y_n(x) = \sqrt{2}\sin(\sqrt{\mu_n}x)$, $n = 0,1,2,...$ forms a Riesz basis in $L_2[0,1]$.

Definition 1. Let the pair of functions $\{u(x,\tau), a(\tau)\}$ be from the class $C^{2,4}(D_T) \times C[0,T]$ and satisfies the equation (1) and conditions (2)–(4). Then the pair $\{u(x,\tau), a(\tau)\}$ is called the classical solution of the inverse problem (1)–(4).

Now, let us introduce two Banach spaces that are connected with the eigenvalues and eigenfunctions of the auxiliary spectral problem (5):

I.

$$B_T = \left\{ u(x,\tau) = \sum_{n=0}^{\infty} u_n(\tau) Y_n(x) : \ u_n(\tau) \in C[0,T], \right.$$

$$\left. J_T(u) = \left(\sum_{n=0}^{\infty} (\mu_n \left\| u_n(\tau) \right\|_{C[0,T]})^2 \right)^{1/2} < +\infty \right\},$$

where $u_n(\tau) = \sqrt{2} \int_0^1 u(x,\tau) \sin(\sqrt{\mu_n} x) dx$, and $J_T(u) := \|u(x,\tau)\|_{B_T}$ is the norm of the function $u(x,\tau)$.

II. $E_T = B_T \times C[0,T]$ is a Banach space with the norm

$$\|w(x,\tau)\|_{E_T} = \|u(x,\tau)\|_{B_T} + \|a(\tau)\|_{C[0,T]},$$

where $w(x,\tau) = \{u(x,\tau), a(\tau)\}$ is a vector function.

These spaces are suitable to investigate the solution of the inverse problem (1)–(4).

After giving these preliminary results, we can set and prove the existence and uniqueness theorem for the inverse problem (1)–(4):

Theorem 1. Let the assumptions

A$_1$ $\xi_0(x) \in C^1[0,1]$, $\xi_0''(x) \in L_2[0,1]$, $\xi_0(0) = \xi_0'(1) = 0$,

A$_2$ $\xi_1(x) \in C^1[0,1]$, $\xi_1''(x) \in L_2[0,1]$, $\xi_1(0) = \xi_1'(1) = 0$,

A$_3$ $\xi_2(x) \in C^1[0,1]$, $\xi_2''(x) \in L_2[0,1]$, $\xi_2(0) = \xi_2'(1) = 0$,

A$_4$ $\xi_3(x) \in C^1[0,1]$, $\xi_3''(x) \in L_2[0,1]$, $\xi_3(0) = \xi_3'(1) = 0$,

A$_5$ $E(\tau) \in C^4[0,T]$, $E(\tau) \neq 0 \ \forall \tau \in [0,T]$, $E^{(i)}(0) = \int_0^1 \xi_i(x) dx$, $i = 0,1,2,3$ and $E^{(i)}(\tau) = \frac{d^i}{d\tau^i} E(\tau)$,

A$_6$ $f(x,\tau) \in C(\overline{D}_T)$, $f_x, f_{xx} \in C[0,1]$, $\forall \tau \in [0,T]$, $f(0,\tau) = f_x(1,\tau) = 0$,

be satisfied, and $\Delta = (1+\mu_n)^2 - 4\mu_n > 0$. Then, the inverse problem (1)–(4) has a unique solution for small T.

Proof. For arbitrary $a(\tau) \in C[0,T]$, to construct the formal solution of the inverse problem (1)–(4), we will use the Fourier (Eigenfunction expansion) method. In accordance with this, let us consider

$$u(x,\tau) = \sum_{n=0}^{\infty} u_n(\tau) Y_n(x), \tag{6}$$

is a solution of the inverse problem (1)–(4), where $Y_n(x) = \sqrt{2} \sin(\sqrt{\mu_n} x)$, $n = 0,1,2,...$ are the eigenfunctions and $\mu_n = \left(\frac{2n+1}{2}\pi\right)^2$, $n = 0,1,2,...$ are the eigenvalues of the corresponding spectral problem.

Since $u(x,\tau)$ is the formal solution of the inverse problem (1)-(4), we get the following Cauchy problems with respect to $u_n(\tau)$ from the equation (1) and initial conditions (2);

$$\begin{cases} u_n^{(4)}(\tau) + (1+\mu_n) u_n''(\tau) + \mu_n u_n(\tau) = F_n(\tau; a, u), \\ u_n(0) = \xi_{0n}, \ u_n'(0) = \xi_{1n}, \ u_n''(0) = \xi_{2n}, u_n'''(0) = \xi_{3n}, \ n = 0,1,2,.... \end{cases} \tag{7}$$

Here $F_n(\tau; a, u) = a(\tau)u_n(\tau) + f_n(\tau)$, $u_n(\tau) = \sqrt{2}\int_0^1 u(x,\tau)\sin(\sqrt{\mu_n}x)dx$, $f_n(\tau) = \sqrt{2}\int_0^1 f(x,\tau)\sin(\sqrt{\mu_n}x)dx$, and $\xi_{in} = \sqrt{2}\int_0^1 \xi_i(x)\sin(\sqrt{\mu_n}x)dx$, $i = 0,1,2,3$, $n = 0,1,2,....$

These Cauchy problems have the quartic characteristic polynomial

$$P_4(k) = k^4 + (1+\mu_n)k^2 + \mu_n.$$

If we convert this quartic equation to a quadratic equation by changing the variable $s = k^2$, we obtain

$$P_2(s) = s^2 + (1+\mu_n)s + \mu_n.$$

It is easy to see that $\Delta = (1+\mu_n)^2 - 4\mu_n = (\mu_n - 1)^2$ and that is always positive. Then $P_2(s) = 0$ has two real roots

$$s_1 = -1, \ s_2 = -\mu_n.$$

Thus $P_4(k) = 0$ has four complex conjugate roots

$$k_{1,2} = \pm i, \ k_{3,4} = \pm i\sqrt{\mu_n}.$$

Solving (7) by using the these roots of the characteristic polynomial, we obtain

$$u_n(\tau) = \frac{\mu_n\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{0n} + \frac{\mu_n^{3/2}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{1n} +$$
$$+ \frac{\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{2n} + \frac{\sqrt{\mu_n}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{3n} +$$
$$+ \int_0^\tau \frac{\sqrt{\mu_n}\sin((\tau - s)) - \sin(\sqrt{\mu_n}(\tau - s))}{\mu_n^{3/2} - \sqrt{\mu_n}}F_n(s; a, u)ds. \tag{8}$$

Substitute the expression (8) into (6) to determine $u(x,\tau)$. Then we get

$$u(x,\tau) = \sum_{n=0}^\infty \left[\frac{\mu_n\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{0n} + \frac{\mu_n^{3/2}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{1n} + \right.$$
$$+ \frac{\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\xi_{2n} + \frac{\sqrt{\mu_n}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\xi_{3n} + \tag{9}$$
$$\left. + \int_0^\tau \frac{\sqrt{\mu_n}\sin((\tau - s)) - \sin(\sqrt{\mu_n}(\tau - s))}{\mu_n^{3/2} - \sqrt{\mu_n}}F_n(s; a, u)ds\right]Y_n(x).$$

Let us derive the equation of $a(\tau)$. If we integrate the equation (1) from $x = 0$ to $x = 1$ with respect to x, and consider the additional condition (4), then we have:

$$a(\tau) = \frac{1}{E(\tau)}\left[E^{(4)}(\tau) + E''(\tau) - f_{int}(\tau) + \sum_{n=0}^\infty \sqrt{\mu_n}\left(u_n''(\tau) + u_n(\tau)\right)\right], \tag{10}$$

where $f_{int}(\tau) = \int_0^1 f(x,\tau)dx$. If we consider $u_n(\tau)$ which is defined in (8) and its second derivative into the last equation, we get

$$a(\tau) = \frac{1}{E(\tau)}\left[E^{(4)}(\tau) + E''(\tau) - f_{int}(\tau) + \sum_{n=0}^\infty \sqrt{\mu_n}\left(\cos(\sqrt{\mu_n}\tau)\xi_{0n} + \frac{\sin(\sqrt{\mu_n}\tau)}{\sqrt{\mu_n}}\xi_{1n} + \right.\right.$$
$$\left.\left. + \cos(\sqrt{\mu_n}\tau)\xi_{2n} + \frac{\sin(\sqrt{\mu_n}\tau)}{\sqrt{\mu_n}}\xi_{3n} + \int_0^\tau \frac{1}{\sqrt{\mu_n}}\sin(\sqrt{\mu_n}(\tau - s))F_n(s; a, u)ds\right)\right]. \tag{11}$$

We convert the inverse problem (1)–(4) into the system of Volterra integral equations (9)–(10) with respect to $u(x,\tau)$ and $a(\tau)$ by considering

$$u_n(\tau) = \int_0^1 u(x,\tau) Y_n(x) dx,\ n = 0, 1, 2, ...$$

is the solution of the system of differential equations (7). Analogously, we can prove that if $\{u(x,\tau), a(\tau)\}$ is a solution of the inverse problem (1)–(4), then $u_n(\tau),\ n = 0, 1, 2, ...$satisfy the system of differential equations (7). For proof of this assertion please see [18]. From this assertion we can conclude that proving the uniqueness of the solution of the inverse problem (1)–(4), it suffices to prove the uniqueness of the solution of the system (9) and (11).

To prove the existence of a unique solution of the system (9) and (11) we need to rewrite this system into operator form and to show that this operator a contraction operator. To this end let us denote $w(x,\tau) = [u(x,\tau), a(\tau)]^T$ is a 2×1 vector function and rewrite the system of equations (9) and (11) in the following operator equation

$$w = \mathbf{\Pi}(w), \tag{12}$$

where $\mathbf{\Pi}(w) \equiv [\mathit{\Pi}_1, \mathit{\Pi}_2]^T$ and $\mathit{\Pi}_1$ and $\mathit{\Pi}_1$ are equal to the right hand sides of (9) and (11), respectively.

Using integration by parts under the assumptions $(A_1) - (A_6)$, we obtain following equalities

$$\xi_{0n} = \frac{1}{\mu_n}\alpha_{0n},\ \xi_{1n} = \frac{1}{\mu_n}\alpha_{1n},\ \xi_{2n} = \frac{1}{\mu_n}\alpha_{2n},\ \xi_{3n} = \frac{1}{\mu_n}\alpha_{3n},\ f_n(\tau) = \frac{1}{\mu_n}\omega_n(\tau),$$

where $\omega_n(\tau) = -\sqrt{2}\int_0^1 f_{xx}(x,\tau)\sin(\sqrt{\mu_n}x)dx,\ \alpha_{in} = -\sqrt{2}\int_0^1 \xi_i''(x)\sin(\sqrt{\mu_n}x)dx,\ i = 0, 1, 2, 3.$ Since $\sqrt{2}\sin(\sqrt{\mu_n}x)$ forms a biorthonormal system of functions on $[0,1]$, by using Bessel's inequality we get

$$\sum_{n=0}^{\infty} |\alpha_{in}|^2 \le \left\|\xi_i''\right\|^2_{L_2[0,1]},\ i = 0, 1, 2, 3,\ \sum_{n=0}^{\infty} |\omega_n(\tau)|^2 \le \|f_{xx}(\cdot,\tau)\|^2_{L_2[0,1]}. \tag{13}$$

Before showing that Φ is a contraction operator, let us find the estimates for the coefficients arising in the operator equations (9) and (11):

$$\begin{aligned} \left|\frac{\mu_n\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\right| &\le \frac{\mu_n + 1}{\mu_n - 1} = d_n^1, \left|\frac{\mu_n^{3/2}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\right| \le \frac{\mu_n^{3/2} + 1}{\mu_n^{3/2} - \sqrt{\mu_n}} = d_n^2, \\ \left|\frac{\cos(\tau) - \cos(\sqrt{\mu_n}\tau)}{\mu_n - 1}\right| &\le \frac{2}{\mu_n - 1} = d_n^3, \left|\frac{\sqrt{\mu_n}\sin(\tau) - \sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2} - \sqrt{\mu_n}}\right| \le \frac{\sqrt{\mu_n} + 1}{\mu_n^{3/2} - \sqrt{\mu_n}} = d_n^4. \end{aligned}$$

Since the sequences $d_n^i,\ i = 1, 2, 3, 4$ are convergent, they are bounded. Consider that

$$d_n^i \le m_i, \text{ for each } i = 1, 2, 3, 4, \tag{14}$$

where m_i are real constants.

Now we can show in two steps that $\mathbf{\Pi}$ is a contraction operator by considering the assumptions and estimates are given above.

I) First let us demonstrate that $\mathbf{\Pi}$ is a continuous map which maps the space E_T onto itself continuously. That is to say, our aim is to show $\mathit{\Pi}_1(z) \in B_T$ and $\mathit{\Pi}_2(z) \in C[0,T]$ for arbitrary $w = [u(x,\tau), a(\tau)]^T$ such that $u(x,\tau) \in B_T$, $a(\tau) \in C[0,T]$.

Let us start with showing that $\mathit{\Pi}_1(z) \in B_T$, i.e. we need to verify

$$J_T(\mathit{\Pi}_1) = \left(\sum_{n=0}^{\infty}(\mu_n \|\mathit{\Pi}_{1,n}(\tau)\|_{C[0,T]})^2\right)^{1/2} < +\infty,$$

where

$$\mathit{\Pi}_{1,n}(\tau)=\frac{\mu_n\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{0n}+\frac{\mu_n^{3/2}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{1n}+$$

$$+\frac{\cos(\tau)-\cos(\sqrt{\mu_n}\tau)}{\mu_n-1}\xi_{2n}+\frac{\sqrt{\mu_n}\sin(\tau)-\sin(\sqrt{\mu_n}\tau)}{\mu_n^{3/2}-\sqrt{\mu_n}}\xi_{3n}+$$

$$+\int_0^\tau\frac{\sqrt{\mu_n}\sin((\tau-s))-\sin(\sqrt{\mu_n}(\tau-s))}{\mu_n^{3/2}-\sqrt{\mu_n}}F_n(s;a,u)ds.$$

After some manipulations under the assumptions $(A_1)-(A_6)$, using the estimates (14) we obtain

$$(J_T(\mathit{\Pi}_1))^2=\sum_{n=0}^{\infty}(\mu_n\left\|\mathit{\Pi}_{1,n}(\tau)\right\|_{C[0,T]})^2\le$$

$$\le 6\sum_{i=0}^{3}m_{i+1}^2\sum_{n=0}^{\infty}|\alpha_{in}|^2+6m_4^2T^2\sum_{n=0}^{\infty}\left(\max_{0\le\tau\le T}|\omega_n(\tau)|\right)^2+6T^2\left\|a(\tau)\right\|_{C[0,T]}^2\sum_{n=0}^{\infty}\left(\mu_n\left\|u_n(\tau)\right\|_{C[0,T]}\right)^2.$$

Since $u(x,\tau),a(\tau)$ belong to the spaces B_T, and $C[0,T]$, respectively, the series at the right hand side of $(J_T(\mathit{\Pi}_1))^2$ are convergent from the Bessel's inequality (considering the estimates (13)). $J_T(\mathit{\Pi}_1)$ is convergent (i.e. $J_T(\mathit{\Pi}_1)<+\infty$) because $(J_T(\mathit{\Pi}_1))^2$ is bounded above. Thus we can conclude that $\mathit{\Pi}_1(z)$ belongs to the space B_T.

Now let us verify $\mathit{\Pi}_2(w)\in C[0,T]$. From the equation (10) we have

$$|\mathit{\Pi}_2(w)|\le\frac{1}{\min\limits_{0\le\tau\le T}|E(\tau)|}\left[\left|E^{(4)}(\tau)\right|+\left|E''(\tau)\right|+|f_{int}(\tau)|+\sum_{n=0}^{\infty}\sqrt{\lambda_n}\left(\left|u_n''(\tau)\right|+|u_n(\tau)|\right)\right].$$

Using the Cauchy-Schwartz inequality and the estimates are given in (13) and (14) we obtain

$$\max_{0\le\tau\le T}|\mathit{\Pi}_2(z)|\le\frac{1}{\min\limits_{0\le\tau\le T}|E(\tau)|}\left[\max_{0\le\tau\le T}\left|E^{(4)}(\tau)\right|+\max_{0\le\tau\le T}\left|E''(\tau)\right|+\max_{0\le\tau\le T}|f_{int}(\tau)|+\right.$$

$$+\left(\sum_{n=0}^{\infty}\frac{1}{\mu_n}\right)^{1/2}\left\{\left(\sum_{n=0}^{\infty}|\alpha_{0n}|^2\right)^{1/2}+\left(\sum_{n=0}^{\infty}|\alpha_{2n}|^2\right)^{1/2}\right\}+\left(\sum_{n=0}^{\infty}\frac{1}{\mu_n^2}\right)^{1/2}\left\{\left(\sum_{n=0}^{\infty}|\alpha_{1n}|^2\right)^{1/2}+\right.$$

$$\left.\left.+\left(\sum_{n=0}^{\infty}|\alpha_{3n}|^2\right)^{1/2}+T\left(\sum_{n=0}^{\infty}\left(\max_{0\le\tau\le T}|\omega_n(\tau)|\right)^2\right)^{1/2}+T\left\|a(\tau)\right\|_{C[0,T]}^2\left(\sum_{n=0}^{\infty}\left(\mu_n\left\|u_n(\tau)\right\|_{C[0,T]}\right)^2\right)^{1/2}\right]. \tag{15}$$

Considering the estimates (13) and $\sum_{n=0}^{\infty}\frac{1}{\mu_n}$, $\sum_{n=0}^{\infty}\frac{1}{\mu_n^2}$ are convergent, the majorizing series (15) are also convergent. According to Weierstrass M-test, $\mathit{\Pi}_2(z)$ is continuous and belongs to the space $C[0,T]$.

Therefore, we show that $\mathbf{\Pi}$ maps E_T onto itself continuously.

II) Since $\mathbf{\Pi}$ maps E_T onto itself continuously, let us show that $\mathbf{\Pi}$ is contraction mapping operator. Assume that let w_1 and w_2 be any two elements of E_T such that $w_i(x,\tau)=\left[u^{(i)}(x,\tau),a^{(i)}(\tau)\right]^T$, $i=1,2$. From the definition of the space E_T, we have $\left\|\mathbf{\Pi}(w_1)-\mathbf{\Pi}(w_2)\right\|_{E_T}=\left\|\mathit{\Pi}_1(w_1)-\mathit{\Pi}_1(w_2)\right\|_{B_T}+\left\|\mathit{\Pi}_2(w_1)-\mathit{\Pi}_2(w_2)\right\|_{C[0,T]}$. For the convenience of this norm, let us consider the following differences:

$$\mathit{\Pi}_1(w_1)-\mathit{\Pi}_1(w_2)=$$
$$=\sum_{n=0}^{\infty}\int_0^\tau\frac{\sqrt{\mu_n}\sin((\tau-s))-\sin(\sqrt{\mu_n}(\tau-s))}{\mu_n^{3/2}-\sqrt{\mu_n}}\left(F_n(s;a^1,u^1)-F_n(s;a^2,u^2)\right)dsY_n(x)$$

$$\Pi_2(w_1) - \Pi_2(w_2) = \frac{1}{E(t)} \sum_{n=0}^{\infty} \int_0^{\tau} \frac{1}{\sqrt{\mu_n}} \sin(\sqrt{\mu_n}(\tau - s)) \left(F_n(s; a^1, u^1) - F_n(s; a^2, u^2)\right) ds.$$

After some manipulations in last equations under the assumptions (A_1)–(A_6) and using the estimates (13)–(14), we obtain

$$\|\Pi_1(z_1) - \Pi_1(z_2)\|_{B_T} \leq \sqrt{2} m_4 T \left[\left\| a^{(1)} \right\|_{C[0,T]} \left\| u^{(1)} - u^{(2)} \right\|_{B_T} + \left\| u^{(2)} \right\|_{B_T} \left\| a^{(1)} - a^{(2)} \right\|_{C[0,T]} \right],$$

$$\|\Pi_2(z_1) - \Pi_2(z_2)\|_{C[0,T]} \leq$$

$$\leq \frac{T}{\min\limits_{0 \leq t \leq T} |E(t)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \left[\left\| a^{(1)} \right\|_{C[0,T]} \left\| u^{(1)} - u^{(2)} \right\|_{B_T} + \left\| u^{(2)} \right\|_{B_T} \left\| a^{(1)} - a^{(2)} \right\|_{C[0,T]} \right].$$

From the last inequalities it follows that

$$\|\mathbf{\Pi}(z_1) - \mathbf{\Pi}(z_2)\|_{E_T} \leq C(T, a^{(1)}, u^{(2)}) \|z_1 - z_2\|_{E_T},$$

where $C(T, a^{(1)}, u^{(2)}) = T \left(\|a^{(1)}\|_{C[0,T]} + \|u^{(2)}\|_{B_T} \right) \left(\sqrt{2} m_4 + \frac{1}{\min\limits_{0 \leq \tau \leq T} |E(\tau)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \right)$.

Since $E(\tau) \in C^4[0,T]$, $E(\tau) \neq 0$ $\forall \tau \in [0,T]$, $a^{(1)}(\tau) \in C[0,T]$, $u^{(2)}(x,\tau) \in B_T$ and m_4 is a finite constant, $\left(\|a^{(1)}\|_{C[0,T]} + \|u^{(2)}\|_{B_T} \right) \left(\sqrt{2} m_4 + \frac{1}{\min\limits_{0 \leq \tau \leq T} |E(\tau)|} \left(\sum_{n=0}^{\infty} \frac{1}{\mu_n} \right)^{1/2} \right)$ is bounded above. Thus $C(T, a^{(1)}, u^{(2)})$ tends to zero as $T \to 0$. In other words, for sufficiently small T we have $0 < C(T, a^{(1)}, u^{(2)}) < 1$. This means that the operator $\mathbf{\Pi}$ is a contraction mapping operator.

From the first and second steps, the operator $\mathbf{\Pi}$ is contraction mapping operator that is a continuous and onto map on E_T. Then according to Banach fixed point theorem the solution of the operator equation (12) exists and it is unique.

2 Conclusion

The paper considers the inverse problem of determining the time dependent lowest term coefficient in fourth order in time partial differential equation with initial and boundary conditions from an additional observation. The unique solvability of the solution of the inverse problem on a sufficiently small time interval has been proved by using of the contraction principle. The proposed work is a novel and has never been solved theoretically nor numerically before. Our results shed light on the methodology for the existence and uniqueness of the inverse problem for the fourth order in time PDEs in two dimensions.

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Интегралдық түрлендіруі бар уақыт бойынша дербес туындылы төртінші ретті дифференциалдық теңдеу үшін коэффициентті кері есебі

Уақыт бойынша жоғары ретті (екіден көп) туындылар акустика, медициналық ультрадыбыста, тұтқырлық және жылу серпімділігі сияқты әртүрлі салаларда пайда болады. Коэффициентті қалпына келтіруге байланысты уақыт бойынша теңдеулердегі жоғары туындылар үшін кері есептер аз және қосымша қарауды қажет етеді. Мақалада дифференциалдық теңдеудегі уақытқа тәуелді кіші коэффициентке қосымша интегралды бақылау жүргізіп, уақыт бойынша бастапқы және шекаралық шарттары бар төртінші ретті дербес туындылы анықтаудың кері есебі қарастырылған. Сығымдау принципін қолдана отырып, шарттардың регулярлығы, қарама-қайшы болмауы және ортогоналдылығының кейбір жағдайларында коэффициенттерді жеткілікті аз уақыт аралығында анықтау есебін шешудің бір мәнді шешімділігі дәлелденді.

Кілт сөздер: дербес туындылы дифференциалдық теңдеулер үшін кері есептер, уақыт бойынша төртінші ретті дербес туындылы дифференциалдық теңдеулер, бар болуы және жалғыздығы.

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Обратная коэффициентная задача для дифференциального уравнения в частных производных четвертого порядка по времени с интегральным переопределением

Производные по времени более высокого порядка (больше двух) возникают в различных областях, таких как акустика, медицинский ультразвук, вязкоупругость и термоупругость. Обратные задачи для высших производных в уравнениях по времени, связанные с восстановлением коэффициента, немногочисленны и требуют дополнительного рассмотрения. В статье рассмотрена обратная задача определения, зависящая от времени, младшего коэффициента в дифференциальном уравнении в частных производных четвертого порядка по времени с начальными и граничными условиями по дополнительному интегральному наблюдению. При некоторых условиях регулярности, непротиворечивости и ортогональности данных с использованием принципа сжатия доказана однозначная разрешимость решения задачи определения коэффициентов на достаточно малом интервале времени.

Ключевые слова: обратные задачи для УрЧП, УрЧП четвертого порядка по времени, существование и единственность.