DOI 10.31489/2022M4/4-19

UDC 517.95:519.6

M.M. Aripov[1], D. Utebaev[2,*], Zh.A. Nurullaev[1]

[1] *M. Ulugbek National University of Uzbekistan, Tashkent, Uzbekistan;*
[2] *Berdakh Karakalpak State University, Nukus, Uzbekistan*
(E-mail: mirsaidaripov@mail.ru, dutebaev_56@mail.ru, njusipbay@mail.ru)

On the convergence of difference schemes of high accuracy for the equation of ion-acoustic waves in a magnetized plasma

Multiparametric difference schemes of the finite element method of a high order of accuracy for the Sobolev-type equation of the fourth-order in time are studied. In particular, the first boundary value problem for the equation of ion-acoustic waves in a magnetized plasma is considered. A high-order accuracy of the scheme is achieved due to the special discretization of time and space variables. The presence of parameters in the scheme makes it possible to regularize the accuracy of the schemes and optimize the implementation algorithm. An a priori estimate in a weak norm is obtained by the method of energy inequality. Based on this estimate and the Bramble-Hilbert lemma, the convergence of the constructed algorithms in classes of generalized solutions is proved. An algorithm for implementing the difference scheme is proposed.

Keywords: Sobolev type equation, difference schemes, finite difference method, finite element method, stability, convergence, accuracy.

Introduction

As is known, the solution of complex applied problems requires the creation of more accurate numerical algorithms or the improvement of existing ones. This is especially seen in the study of complex non-stationary processes, for example, in boundary value problems for high-order partial differential equations. The study of such equations began with the research works of S.L. Sobolev. They are applied in solving problems of geophysics, oceanology, atmospheric physics, physics of magnetically ordered structures related to the propagation of waves in media with a strong dispersion, and many other problems [1–3]. For example, the equation of ion-acoustic waves in a magnetized plasma [3]

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}+\omega_{B_i}^2\right)\left(\Delta_3 u - r_D^{-2}u\right)+\omega_{p_i}^2\frac{\partial^2}{\partial t^2}\Delta_3 u+\omega_{p_i}^2\omega_{B_i}^2\frac{\partial^2 u}{\partial x_3^2}=f(x,t), \tag{1}$$

$$(x,t)\in Q_T=\Omega\cup\partial\Omega,\ \Omega=\left\{x \mid x=(x_1,x_2,x_3),\, 0<x_\alpha<l,\, \alpha=\overline{1,3}\right\},$$

refers to such equations. Here $u=(x,t)$ is the motion velocity, $\Delta_3=\partial^2u/\partial x_1^2+\partial^2u/\partial x_2^2+\partial^2u/\partial x_3^2$, $r_D^2=T_e^2/\left(4\pi e^2 n_0\right)$ is the Debye radius, $\omega_{B_i}=eB_0/(Mc)$ is the ion gyrofrequency, $\omega_{p_i}^2=4\pi e^2 n_0/M$ is the Langmuir frequency for ions, M is the mass, c is the speed of light in vacuum, B_0 is the external constant magnetic field, n_0 is the unperturbed particle density, e is the absolute value of the electron charge, T_e is the temperature of the electrons. In addition, similar equations appear in the mathematical modeling of internal waves in the ocean and atmosphere [4–6].

The study in [3] is devoted to analytical methods for solving problems of this type, where the problems of global and local solvability of initial-boundary value problems for linear and nonlinear equations are considered. Numerical methods for solving equations unresolved with respect to the time derivative are also considered. Non-stationary equations of the second order in time and pseudo-parabolic equations are considered. Here and in [7], these equations are reduced by some substitution

*Corresponding author.
E-mail: dutebaev_56@mail.ru

to two equations (one contains differentiation with respect to time, the other - with respect to space only); then, these equations are solved by the finite difference method on quasi-uniform grids. The second order of approximation in both variables is proved.

The studies in [8, 9] are devoted to numerical methods for solving initial-boundary value problems for equation (1). In [8], a mathematical model of ion-acoustic waves in plasma is considered in an external magnetic field. Issues of unique solvability of the Cauchy-Dirichlet problem are considered. Based on the theoretical results, an algorithm was developed for the numerical solution of the problem based on the modified Galerkin method. An implementation algorithm is given. A problem similar to an optimal control problem for the mathematical model (1), was considered in [9], where an algorithm for a numerical solution based on the modified Galerkin method and the Ritz method was developed.

In this article, the authors consider the issues of constructing and investigating difference schemes of high accuracy of initial-boundary value problems for the non-stationary equation of ion-acoustic waves in a magnetized plasma (1). First, we approximate the space variables, and the time variable is stored in differential form. As a result, we obtain a system of ordinary differential equations of large dimensions, solved by the difference scheme of the finite element method of the fourth-order accuracy. To obtain an accuracy estimate, a special technique for obtaining a priori estimates was used since the classical approach to studying the convergence of difference schemes based on the Taylor formula places high demands on the smoothness of the sought-for solution. Therefore, a number of results have recently been obtained on estimating the rate of convergence of difference schemes for equations of mathematical physics based on the Bramble-Hilbert lemma [10]. Such studies for various stationary and nonstationary problems were conducted in [11–15]. The notation from [16] is used in this article.

1 Statement of the problem

Let us rewrite equation (1) in the following form:

$$\frac{\partial^4}{\partial t^4}\left(\Delta_3 u - r_D^{-2}u\right) + \frac{\partial^2}{\partial t^2}[(\omega_{B_i}^2+\omega_{p_i}^2)\Delta_3 u - \omega_{B_i}^2 r_D^{-2}u] + \omega_{p_i}^2\omega_{B_i}^2\frac{\partial^2 u}{\partial x_3^2} = f(x,t), \tag{2}$$

$$(x,t)\in\Omega = \{x=(x_1,x_2,x_3):\ 0<x_k<l_k,\ k=1,2,3\}.$$

The initial and boundary conditions have the following form:

$$\left.\frac{\partial^\nu}{\partial t^\nu}u(x,t)\right|_{t=0} = u_{0,\nu},\quad \nu=\overline{0,3},\quad x\in\overline{\Omega}=\Omega\cup\partial\Omega, \tag{3}$$

$$u(x,t)|_{\partial\Omega}=0,\ \ t\in(0,T]. \tag{4}$$

The existence and uniqueness of solutions to such problems are considered in [1–3].

In our case, we will assume that $r_D^2\notin\sigma(\Delta)=\lambda_k$ is the set of eigenvalues of the homogeneous Dirichlet problem for the Laplace operator in domain Ω.

Let us formulate a generalized statement of problem (2)–(4). Function $u(x,t)$, which for each $t\in(0,T]$ belongs to $H=\left\{u\in W_2^1(\Omega),\ u=0,\ x\in\partial\Omega\right\}$ is called the generalized solution to the problem; it has derivative $\frac{\partial^4 u}{\partial t^4}\in W_2^1(\Omega)$, and satisfies the following relations almost everywhere for all $t\in(0,T]$:

$$a_3\left(\frac{d^4u(t)}{dt^4},\vartheta\right) + a_2\left(\frac{d^2u(t)}{dt^2},\vartheta\right) + a_1(u(t),\vartheta) = (f(t),\vartheta), \tag{5}$$

$$\left(\frac{d^ku}{dt^k}(0)-u_{0,k},\vartheta\right)=0,\ \ k=\overline{0,3},\ \ \forall\vartheta(x)\in H. \tag{6}$$

Here

$$a_1(u,\vartheta)=\omega_{p_i}^2\omega_{B_i}^2\int\limits_{\Omega}(u_{x_3}\vartheta_{x_3})dx,\ a_2(u,\vartheta)=\int\limits_{\Omega}\left[\sum_{k=1}^{3}(\omega_{B_i}^2+\omega_{p_i}^2)u_{x_k}\vartheta_{x_k}-\omega_{B_i}^2r_D^{-2}u\vartheta\right]dx,$$

$$a_3(u,\vartheta)=\int\limits_{\Omega}\left[\sum_{k=1}^{3}u_{x_k}\vartheta_{x_k}-r_D^{-2}u\vartheta\right]dx.$$

We denote $|u|_m=\sqrt{a_m(u,u)}$, $m=\overline{1,3}$, the energy seminorms in H, corresponding to bilinear forms $a_m(u,\vartheta)$. The energy space H_{A_m}, generated by seminorm $|u|_m$, is equivalent to space $H=\overset{\circ}{W}{}_2^1(\Omega)$ [17], therefore, the following estimates $0\le a_m(u,u)\le C_m\left\|u\right\|_1^2$, $m=\overline{1,3}$, are true, where C_m are the positive constants depending on ω, r_D.

2 Discretization in space

We discretize the problem in terms of space variables using the finite element method. Let $H_h\subset H$ be the set of elements of the form $\vartheta_h=\sum\limits_{m=1}^{M}a_m\Phi_m(x)$. Here $\{\Phi_m=\Phi_m(x)\}_{m=1}^{M}$ is the basis of piecewise polynomial functions that are a degree p polynomial on each finite element [18, 19].

Let us give an example of a basis based on third degree polynomials. Let us introduce a partition of domain Ω into $M=N_1*N_2*N_3$ parallelepipeds:

$$\Omega_{ijk}=\left\{(i-1)h_1\le x_1\le ih_1,\ (j-1)h_2\le x_2\le jh_2,\ \ (k-1)h_3\le x_3\le kh_3\right\},$$

$$i=\overline{1,N_1},\ j=\overline{1,N_2},\ k=\overline{1,N_3},\ h_s=l_s/N_s,\ s=1,2,3.$$

We choose a system of basis functions:

$$\Phi_{ijk}(x_1,x_2,x_3)=\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3),\ i=\overline{1,N_1},\ j=\overline{1,N_2},\ k=\overline{1,N_3},$$

where $\varphi_l(x)$ is the basis function built on the basis of the B_3-spline [18]. In this case $p=3$.

Let us put the semidiscrete problem for $t\in[0,T]$ in correspondence with (5), (6):

$$a_3\left(\frac{d^4u_h(t)}{dt^4},\vartheta_h\right)+a_2\left(\frac{d^2u_h(t)}{dt^2},\vartheta_h\right)+\ a_1\left(u_h,\vartheta_h\right)=\left(f(t),\vartheta_h\right),\tag{7}$$

$$\left(\frac{d^\nu u_h}{dt^\nu}(0)-u_{0,\nu},\vartheta_h\right)=0,\nu=\overline{0,3},\forall\vartheta_h(x)\in H_h.\tag{8}$$

Problem (7), (8) corresponds to the following Cauchy problem:

$$D\frac{d^4u_h(t)}{dt^4}+B\frac{d^2u_h(t)}{dt^2}+Au_h(t)=f_h(t),\ \frac{d^\nu u_h}{dt^\nu}(0)=u_{1,\nu,h},\ \ \nu=\overline{0,3}.\tag{9}$$

Operators $D,\ B,\ A$ operate from H_h to H_h. They correspond to stiffness matrices $D=a_3(\varphi_l,\varphi_m)_{l,m=1}^{M}$, $B=a_2(\varphi_l,\varphi_m)_{l,m=1}^{M}$, $A=a_1(\varphi_l,\varphi_m)_{l,m=1}^{M}$. Besides, $u_{k,h}=P_hu_k(x)$, $k=\overline{0,3}$, where P_h is the projection operator $P_hH=H_h$.

The boundary conditions are approximated exactly.

3 Discretization in time

Following [20], problem (9) is approximated by the finite element method. Its generalized solution is defined as a continuous function $u(t) \in C^2[0,T]$ satisfying the following integral identity for arbitrary function $\vartheta(t) \in C^2(t_b, t_f)$

$$\int_{t_b}^{t_f} (D\ddot{u}\ddot{\vartheta} - B\dot{u}\dot{\vartheta} + Au\vartheta)dt + \left[D\dddot{u}\vartheta - D\ddot{u}\dot{\vartheta} + B\dot{u}\vartheta\right]\Big|_{t_b}^{t_f} = \int_{t_b}^{t_f} (f, \vartheta)\, dt, \tag{10}$$

where $0 \le t_b \le t_f \le T,\ \dot{u} = du/dt,\ \ddot{u} = d^2u/dt^2,\ \dddot{u} = d^3u/dt^3$.

On the segment $[0, T]$, we introduce uniform grid $\overline{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, ...;\ \tau > 0\}$. On each of intervals (t_n, t_{n+1}), we seek an approximate solution to problem (9) in the form of fifth degree polynomials

$$y(t) = \varphi_{00}^n(t)y^n + \varphi_{01}^n(t)y^{n+1} + \varphi_{10}^n(t)\dot{y}^n + \varphi_{11}^n(t)\dot{y}^{n+1} + \varphi_{20}^n(t)\ddot{y}^n + \varphi_{21}^n(t)\ddot{y}^{n+1}, \tag{11}$$

where $y^n = y(t_n),\ y^{n+1} = y(t_{n+1}),\ \dot{y}^n = dy(t_n)/dt,\ \dot{y}^{n+1} = dy(t_{n+1})/dt,\ \ddot{y}^n = d^2y(t_n)/dt^2$, $\ddot{y}^{n+1} = d^2y(t_{n+1})/dt^2,\ \varphi_{00}^n(t) = -6\xi^5 + 15\xi^4 + 6\xi^5 - 10\xi^3 + 1,\ \varphi_{01}^n(t) = 6\xi^5 - 15\xi^4 + 10\xi^3$, $\varphi_{10}^n(t) = \tau(-3\xi^5 + 8\xi^4 - 6\xi^3 + \xi),\ \varphi_{11}^n(t) = \tau(-3\xi^5 + 7\xi^4 - 4\xi^3),\ \varphi_{20}^n(t) = \tau^2(-\xi^5/2 + 3\xi^4/2$ $-3\xi^3/2 + \xi^2/2),\ \varphi_{21}^n(t) = \tau(\xi^5/2 - \xi^4 + \xi^3/2),\ \xi = (t - t_n)/\tau$.

Choosing weight functions $\vartheta(t)$, in the form of linear combinations of interpolation functions and substituting them into (10), we obtain the following parametric difference scheme

$$\begin{aligned} &D_\eta \dot{y}_t - \eta\tau^2 A y^{(0.5)} - D\ddot{y}^{(0.5)} = \varphi_1, \\ &D_\gamma y_t - D_\gamma \dot{y}^{(0.5)} + \eta\tau^2 D\ddot{y}_t = \varphi_2, \\ &D_\alpha \dot{y}_t - D_\beta \ddot{y}^{(0.5)} - \eta\tau^2 A y^{(0.5)} = \varphi_3, \end{aligned} \tag{12}$$

where

$$D_m = D - m\tau^2 B,\ m = \alpha, \beta, \gamma, \eta,\ \varphi_1 = -\frac{\tau}{6}\int_{t_n}^{t_{n+1}} f(t)dt = -\frac{\tau^2}{6}\int_0^1 f(t_n + \tau\xi)d\xi,$$

$$\varphi_2 = -\frac{7\tau}{60}\int_{t_n}^{t_{n+1}} f(t)\vartheta_2^{(\gamma,\eta)}(t)dt = -\frac{7\tau^2}{60}\int_0^1 f(t_n + \tau\xi)[s_1\vartheta_2^{(1)}(\xi) + s_2\vartheta_2^{(5)}(\xi)]d\xi,$$

$$\varphi_3 = -\frac{10}{\tau}\int_{t_n}^{t_{n+1}} f(t)\vartheta_3^{(\alpha,\beta,\eta)}(t)dt = -10\int_0^1 f(t_n + \tau\xi)[s_3\vartheta_3^{(2)} + s_4\vartheta_3^{(4)}]d\xi,\ \vartheta_2^{(\gamma,\eta)} = s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(5)},$$

$$\vartheta_2^{(1)} = \tau(\xi - 1/2), \vartheta_2^{(5)} = \tau(3\xi^5 + 15\xi^4/2 - 5\xi^3 + \xi/2), s_1 = 3 - 120\gamma, s_2 = 14 - 840\gamma,$$

$$\vartheta_3^{(\alpha,\beta,\eta)} = s_3\vartheta_3^{(2)} + s_4\vartheta_3^{(4)}, \vartheta_3^{(2)} = \tau^2\xi(\xi - 1)/2, \vartheta_3^{(4)} = \tau^2\xi^2(\xi - 1)^2/4, s_3 = 140\alpha + 15,\ s_4 = 1400\alpha + 140,$$

here $\alpha,\ \beta,\ \gamma,\ \eta$ - are some constants.

The first initial condition is approximated exactly. The remaining initial conditions are approximated as in [16], by the fourth-order approximation, using the Taylor series and initial equations:

$$\dot{y}(0) = u_{0,1} + \frac{\tau}{2}\left(E - \frac{\tau^2}{12}D^{-1}B\right)u_{0,2} + \frac{\tau^2}{6}u_{0,3} + \frac{\tau^3}{24}D^{-1}[f(0) - Au_{0,0}],$$

$$\ddot{y}(0) = u_{0,2} + \tau u_{0,3} + \frac{\tau^2}{2} D^{-1}[f(0) - Bu_{0,2} - Au_{0,0}] + \frac{\tau^3}{4} D^{-1}[\dot{f}(0) - B\dot{u}_{0,2} - A\dot{u}_{0,0}].$$

It is easy to check that the scheme has the fourth order of approximation error on smooth solutions, i.e. $\psi_1 = O(\tau^4)$, $\psi_2 = O(\tau^4)$, $\psi_3 = O(\tau^4)$ if the following conditions are met

$$\alpha - \beta = 1/12, \ \eta = 1/12, \tag{13}$$

γ is an arbitrary constant.

4 Estimation of accuracy in space

Theorem 1. Let $u(x,t)$, $\frac{\partial u}{\partial t}(x,t) \in L_2\{[0,T]; W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1\}$. If the narrowing of space H_h to a separate finite element is a k degree polynomial, then for solving problem $u_h(t) \in H_h$ (9) approximating problem (2)–(4), the following accuracy estimate holds

$$\left\| \frac{\partial u}{\partial t}(x,t) - \frac{\partial u_h}{\partial t}(x,t) \right\|_1 + \|u(x,t) - u_h(x,t)\|_1 + \int_0^t \left\| \frac{\partial^2 u}{\partial t^2}(x,t') - \frac{\partial^2 u_h}{\partial t^2}(x,t') \right\|_1 dt' +$$

$$+ \int_0^t \left\| \frac{\partial u}{\partial t}(x,t') - \frac{\partial u_h}{\partial t}(x,t') \right\|_1 dt' \le Mh^k \left(\sqrt{\int_0^t \|u(x,t')\|_{k+1}^2 dt'} + \sqrt{\int_0^t \left\| \frac{\partial u}{\partial t}(x,t') \right\|_{k+1}^2 dt'} \right),$$

$$\forall t \in [0,T], \ M = M(r_D, \omega) > 0.$$

Proof. We integrate identity (5) over t from t_n to $t_{n+1} = t_n + \tau$, and applying the integration-by-parts formula, we obtain:

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{u}(t), \ddot{\vartheta}) - a_2(\dot{u}(t), \dot{\vartheta}) + a_1(u(t), \vartheta) \Big](t)dt + a_3(\dddot{u}(t), \vartheta)|_{t_n}^{t_{n+1}} - a_3\ (\ddot{u}(t), \dot{\vartheta})\Big|_{t_n}^{t_{n+1}} +$$
$$+ a_2(\dot{u}(t), \vartheta)|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (f(t), \vartheta)dt, \ \forall \vartheta(x) \in H_h. \tag{14}$$

Likewise, from (7) we obtain

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{u}_h, \ddot{\vartheta}_h) - a_2(\dot{u}_h, \dot{\vartheta}_h) + a_1(u_h, \vartheta_h) \Big](t)dt + a_3(\dddot{u}_h(t), \vartheta_h)|_{t_n}^{t_{n+1}} - a_3\ (\ddot{u}_h, \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} +$$

$$+ a_2(\dot{u}_h, \vartheta_h)|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (f(t), \vartheta_h)dt, \ \forall \vartheta_h(x) \in H_h.$$

Choosing $\vartheta = \vartheta_h \in H_h \subset H$ from (14) and subtracting both obtained identities, we have:

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{z}_h, \ddot{\vartheta}_h) - a_2(\dot{z}_h, \dot{\vartheta}_h) + a_1(z_h, \vartheta_h) \Big](t)dt + a_3(\dddot{z}_h, \vartheta_h)|_{t_n}^{t_{n+1}} - a_3\ (\ddot{z}_h, \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} +$$
$$+ a_2(\dot{z}_h, \vartheta_h)|_{t_n}^{t_{n+1}} = 0, \ \forall \vartheta_h(x) \in H_h, \tag{15}$$

where $z_h = u - u_h,\ e_h = u - u_I,\ \xi_h = u_I - u_h,\ u_I = u_I(x,t)$ is the solution interpolant $u(x,t)$ in x [19]. Let us choose a test function

$$\vartheta_h(t) = -\int\limits_t^s \xi_h(t')dt' \in H_h,\ t < s;\ \vartheta_h(t) = 0,\ t \ge s,\ \dot{\vartheta}_h(t) = \xi_h(t),\ \vartheta_h(s) = \dot{\vartheta}_h(s) = 0.$$

Then, with $z_h = \xi_h + e_h$, identity (15) can be written in the following form:

$$\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\ddot{\xi}_h, \dot{\xi}_h) + a_2(\dot{\xi}_h, \xi_h) + a_1(\dot{\vartheta}_h, \vartheta_h)\Big](t)dt + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\Big|_{t_n}^{t_{n+1}} =$$

$$= -\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\ddot{e}_h, \dot{\xi}_h) + a_2(\dot{e}_h, \xi_h) + a_1(e_h, \vartheta_h)\Big](t)dt.$$

Hence, given the following relations:

$$a_3(\ddot{\xi}_h,\, \dot{\xi}_h) = \frac{1}{2}\frac{d}{dt}a_3(\dot{\xi}_h,\, \dot{\xi}_h),\ a_2(\dot{\xi}_h,\, \xi_h) = \frac{1}{2}\frac{d}{dt}a_2(\xi_h,\, \xi_h),\ a_1(\dot{\vartheta}_h,\, \vartheta_h) = \frac{1}{2}\frac{d}{dt}a_1(\vartheta_h,\, \vartheta_h),$$

$$a_3(\ddot{e}_h,\, \dot{\xi}_h) = \frac{d}{dt}a_3(\dot{e}_h,\, \dot{\xi}_h) - a_3(\dot{e}_h,\, \ddot{\xi}_h),\ a_2(\dot{e}_h,\, \xi_h) = \frac{d}{dt}a_2(e_h,\, \xi_h) - a_2(e_h,\, \dot{\xi}_h),$$

we obtain

$$E_h(t_{n+1}) + 0.5a_1(\vartheta_h, \vartheta_h)(t_{n+1}) + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\Big|_{t_n}^{t_{n+1}} =$$

$$= E_h(t_n) + 0.5a_1(\vartheta_h, \vartheta_h)(t_n) - [a_3(\dot{e}_h, \dot{\xi}_h)(t_{n+1}) - a_3(\dot{e}_h, \dot{\xi}_h)(t_n) + a_2(e_h, \xi_h)(t_{n+1}) -$$

$$-a_2(e_h, \xi_h)(t_n)] + \int\limits_{t_n}^{t_{n+1}} \Big[a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)\Big](t)dt,$$

where $E_h(t) = 0.5[a_3(\dot{\xi}_h, \dot{\xi}_h) + a_2(\xi_h, \xi_h)]$. Now let us sum this equation over $n = \overline{1, m-1}$, where m corresponds to the time point $s = m\tau$:

$$\begin{gathered} E_h(s) + 0.5a_1(\vartheta_h, \vartheta_h)(s) + [a_3(\dddot{z}_h, \vartheta_h) - a_3(\ddot{z}_h, \dot{\vartheta}_h) + a_2(\dot{z}_h, \vartheta_h)]\,|_0^s = \\ = E_h(0) + 0.5a_1(\vartheta_h, \vartheta_h)(0) - [a_3(\dot{e}_h, \dot{\xi}_h)(s) - a_3(\dot{e}_h, \dot{\xi}_h)(0) + a_2(e_h, \xi_h)(s) - \\ -a_2(e_h, \xi_h)(0)] + \int\limits_0^s [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)](t)dt. \end{gathered} \tag{16}$$

Taking into account the properties of functions $\vartheta_h(t)$ and initial conditions $z_h(0) = \dot{z}_h(0) = \ddot{z}_h(0) = \dddot{z}_h(0) = 0,\ \xi_h(0) = \dot{\xi}_h(0) = \ddot{\xi}_h(0) = \dddot{\xi}_h(0) = 0$, from (16) we obtain

$$E_h(s) + 0.5a_1(\vartheta_h, \vartheta_h)(0) = \int\limits_{t_n}^{t_{n+1}} [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, \vartheta_h)](t)dt. \tag{17}$$

Let us introduce one more function

$$w_h(t) = \int\limits_0^t \xi_h(t')dt' \in H_h,\ t < s;\ w_h(t) = 0,\ t \geq s.$$

Then, $\vartheta_h(t) = w_h(t) - w_h(s)$ and from (17) we have the energy identity:

$$E_h(s) + 0.5a_1(w_h, w_h)(s) = \int\limits_0^s [a_3(\dot{e}_h, \ddot{\xi}_h) + a_2(e_h, \dot{\xi}_h) + a_1(e_h, w_h(t) - w_h(s))]dt. \tag{18}$$

Let us estimate the terms on the right-hand side of (18):

$$\int\limits_0^s a_3\,(\dot{e}_h, \ddot{\xi}_h)dt \leq \varepsilon_1 \int\limits_0^s a_3(\ddot{\xi}_h, \ddot{\xi}_h)dt + \frac{1}{4\varepsilon_1}\int\limits_0^s a_3(\dot{e}_h, \dot{e}_h)dt,$$

$$\int\limits_0^s a_2(e_h, \dot{\xi}_h)dt \leq \varepsilon_2 \int\limits_0^s a_2(\dot{\xi}_h, \dot{\xi}_h)dt + \frac{1}{4\varepsilon_2}\int\limits_0^s a_2(e_h, e_h)dt,$$

$$\int\limits_0^s a_1(e_h, w_h(t) - w_h(s))dt \leq \varepsilon_3 \int\limits_0^s a_1(w_h(t), w_h(t))dt + s\varepsilon_3 a_1(w_h(s), w_h(s)) + \frac{1}{2\varepsilon_3}\int\limits_0^s a_1(e_h, e_h)dt.$$

Choosing $\varepsilon_1 = \ \varepsilon_2 = 1/2$, and ε_3 from condition $\frac{\varepsilon_1}{2} + \varepsilon_3 T \leq \frac{3}{4}$, from (18) we have the following estimate:

$$\begin{gathered} E_h(s) + \int\limits_0^s [a_3(\ddot{\xi}_h, \ddot{\xi}_h) + a_2(\dot{\xi}_h, \dot{\xi}_h)\,](t)dt + a_1(w_h,\, w_h)(s) \leq \\ \leq M\left(\int\limits_0^s [a_1(w_h,\, w_h)(t)dt + \int\limits_0^s [a_3(\dot{e}_h, \dot{e}_h) + a_2(e_h, e_h)\ + a_1(e_h, e_h)](t)dt\right), \end{gathered} \tag{19}$$

where $M - const$. Applying the Gronwall lemma for inequality (19), we obtain the error estimate

$$\begin{gathered} E_h(s) + \int\limits_0^s [a_3(\ddot{\xi}_h, \ddot{\xi}_h) + a_2(\dot{\xi}_h, \dot{\xi}_h)\,](t)dt + a_1(w_h,\, w_h)(s) \leq \\ \leq \int\limits_0^s [a_3(\dot{e}_h, \dot{e}_h) + a_2(e_h, e_h) + a_1(e_h, e_h)](t)dt, \end{gathered}$$

It is evident that $k_0 \left\|w_h(s)\right\|_1^2 \leq a(w_h, w_h)(s) \leq k_1 \left\|w_h(s)\right\|_1^2$, $a(\xi_h, \xi_h)(s) = \left\|\xi_h(s)\right\|_1^2$, $a(\dot{\xi}_h, \dot{\xi}_h)(s) = \left\|\dot{\xi}_h(s)\right\|_1^2$, $a(e_h, e_h)(s) = \left\|e_h(s)\right\|_1^2$, $a(\dot{e}_h, \dot{e}_h)(s) = \left\|\dot{e}_h(s)\right\|_1^2$, so for the error we have the final estimate:

$$\left\|\dot{\xi}_h(s)\right\|_1^2 + \left\|\xi_h(s)\right\|_1^2 + \int\limits_0^s \left[\left\|\ddot{\xi}_h(t)\right\|_1^2 + \left\|\dot{\xi}_h(t)\right\|_1^2\right] dt \leq M \int\limits_0^s \left[\left\|\dot{e}_h(t)\right\|_1^2 + \left\|e_h(t)\right\|_1^2\right] dt. \tag{20}$$

The following estimates hold for solution $u(x,t) \in W_2^{k+1}(\Omega),\ \forall t \in [0, T]$ [18], [19]:

$$\left\|\dot{e}_h(t)\right\|_1 \leq Mh^k \left\|\dot{u}(t)\right\|_{k+1}, \left\|e_h(t)\right\|_1 \leq Mh^k \left\|u(t)\right\|_{k+1}.$$

Therefore, based on (20) and triangle inequality $\|z_h\| \leq \|e_h\| + \|\xi_h\|$, the assertion of the theorem holds.

5 Estimation of accuracy in time

Let us now proceed to estimate the discretization error of problem (9) with respect to time. To approximate problem (9), scheme (12) is used, and to estimate the accuracy with respect to time variable, the Bramble-Hilbert lemma is used. Note that solution $u_h(t)$ of the semidiscrete problem (9) for each t is an element of the discrete subspace $u_h(t) \in H_h$.

Let us denote subspace H_τ of functions of argument t, which are Hermitian splines of the form (11) on interval $[t_n, t_{n+1}]$, $n = 0, 1, 2, ...$ Solution of scheme (12) is $y(t) \in H_\tau$. $y(t)$ is an element of subspace H_h for each t simultaneously. Actually $y(x,t) \in H_h^\tau = H_h \otimes H_\tau$.

The following theorem holds.

Theorem 2. Let $D^* = D > 0$, $\ B^* = B \geq 0$, $A^* = A > 0$. In addition, let the approximation conditions (13) and stability conditions be met

$$D - \mu\tau^2 A \geq \varepsilon D, \quad \forall\varepsilon \in (0,1), \quad \mu = \max\{\alpha, \beta, \gamma, \eta\}. \tag{21}$$

Then, for the solution of scheme (12) approximating the solution to problem (9) such that $\frac{d^4u_h}{dt^4}(t) \in C[0,T]$, the following accuracy estimate holds

$$\|\dot{u}_h(t) - \dot{y}(t)\|_1 + \|u_h(t) - y(t)\|_1 + \int_0^s \|\ddot{u}_h(t) - \ddot{y}(t)\|_1^2 \, dt +$$

$$+ \int_0^s \|\dot{u}_h(t) - \dot{y}(t)\|_1^2 \, dt \leq M\tau^3 \sqrt{\int_0^t \left\| \frac{d^4u_h}{dt^4}(t') \right\|_1^2 dt'}, \ M - const.$$

Proof. Difference scheme (12) corresponds to the weak statement

$$\begin{gathered}\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{y}(t), \ddot{\vartheta}_\tau) - a_2(\dot{y}(t), \dot{\vartheta}_\tau) + a_1(y(t), \vartheta_\tau)\Big] dt + a_3(\dddot{y}(t), \vartheta_\tau)\big|_{t_n}^{t_{n+1}} - a_3(\ddot{y}(t), \dot{\vartheta}_h)\Big|_{t_n}^{t_{n+1}} + \\ + a_2(\dot{y}(t), \vartheta_\tau)\big|_{t_n}^{t_{n+1}} = \int_{t_n}^{t_{n+1}} (f(t), \vartheta_\tau) dt, \ \forall\vartheta_\tau(x) \in H_h^\tau,\end{gathered} \tag{22}$$

where $y(t)$ is the Hermitian spline (11). Choosing $\vartheta = \vartheta_\tau$ in (14) and subtracting the identity (22), we have the following identity for error: $\zeta_\tau(t) = u_h(t) - y(t)$:

$$\begin{gathered}\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{\zeta}_\tau, \ddot{\vartheta}_\tau) - a_2(\dot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_1(\zeta_\tau, \vartheta_\tau)\Big](t) dt + a_3\left(\dddot{\zeta}_\tau(t), \vartheta_\tau\right)\big|_{t_n}^{t_{n+1}} - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau)\Big|_{t_n}^{t_{n+1}} + \\ + a_2(\dot{\zeta}_\tau, \vartheta_\tau)\Big|_{t_n}^{t_{n+1}} = 0, \ \forall\vartheta_\tau \in H_h^\tau.\end{gathered} \tag{23}$$

Let us represent $\zeta_\tau(t)$ as $\zeta_\tau(t) = u_h(t) - y(t) = u_h(t) - u_I^\tau(t) + u_I^\tau(t) - y(t)$, where $u_I^\tau(t)$ is interpolant $u_h(t)$, i.e. $u_I^\tau(t)$, as well as $y(t)$, is the Hermitian spline, such that $u_I^\tau(t_n) = u_h(t_n)$, $\dot{u}_I^\tau(t_n) = \dot{u}_h(t_n)$, $n = 0, 1, ...$ The scheme error is $\zeta_\tau(t) = \xi_\tau(t) + e_\tau(t)$, where $e_\tau = u_h - u_I^\tau$, $\ \xi_\tau = u_I^\tau - y$. We choose test function $\vartheta_\tau(t) = -\int_t^s \xi_\tau(t)dt'$, $\ t < s$; $\ \vartheta_\tau(t) = 0$, $\ t \geq s$. Then identity (23) can be written as:

$$\int_{t_n}^{t_{n+1}} \Big[a_3(\ddot{\xi}_\tau, \dot{\xi}_\tau) + a_2(\dot{\xi}_\tau, \xi_\tau) + a_1(\vartheta_\tau, \vartheta_\tau)\Big] dt + [a_3(\dddot{\xi}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)]\Big|_{t_n}^{t_{n+1}} =$$

$$= -\int\limits_{t_n}^{t_{n+1}} [a_3(\ddot{e}_\tau, \xi_\tau) + a_2(\dot{e}_\tau, \xi_\tau) + a_1(e_\tau, \vartheta_\tau)]dt.$$

Hence, given the following relations:

$$a_3(\ddot{\xi}_\tau,\, \dot{\xi}_\tau) = \frac{1}{2}\frac{d}{dt}a_3(\dot{\xi}_\tau,\, \dot{\xi}_\tau),\; a_2(\dot{\xi}_\tau,\, \xi_\tau) = \frac{1}{2}\frac{d}{dt}a_2(\xi_\tau,\, \xi_\tau),\; a_1(\dot{\vartheta}_\tau,\, \vartheta_\tau) = \frac{1}{2}\frac{d}{dt}a_1(\vartheta_\tau,\, \vartheta_\tau),$$

$$a_3(\ddot{e}_\tau,\, \dot{\xi}_\tau) = \frac{d}{dt}a_3(\dot{e}_\tau,\, \dot{\xi}_\tau) - a_3(\dot{e}_\tau,\, \ddot{\xi}_\tau),\; a_2(\dot{e}_\tau,\, \xi_\tau) = \frac{d}{dt}a_2(e_\tau,\, \xi_\tau) - a_2(e_\tau,\, \dot{\xi}_\tau),$$

from the last identity we obtain

$$E_\tau(t_{n+1}) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(t_{n+1}) + [a_3(\dddot{\zeta}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)]\Big|_{t_n}^{t_{n+1}} =$$

$$= E_\tau(t_n) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(t_n) - [a_3(\dot{e}_\tau, \dot{\xi}_\tau)(t_{n+1}) - a_3(\dot{e}_\tau, \dot{\xi}_\tau)(t_n) + a_2(e_\tau, \xi_\tau)(t_{n+1}) - a_2(e_\tau, \xi_\tau)(t_n)] +$$

$$+\int\limits_{t_n}^{t_{n+1}} \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt,$$

where $E_\tau(t) = 0.5[a_3(\dot{\xi}_\tau, \dot{\xi}_\tau) + a_2(\xi_\tau, \xi_\tau)]$. Now let us sum this equation over $n = \overline{1, m-1}$, where m corresponds to the time point $s = m\tau$:

$$\begin{gathered} E_\tau(s) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(s) + [a_3(\dddot{\zeta}_\tau, \dot{\vartheta}_\tau) - a_3(\ddot{\zeta}_\tau, \dot{\vartheta}_\tau) + a_2(\dot{\zeta}_\tau, \vartheta_\tau)]\,|_0^s = \\ = E_\tau(0) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(0) - [a_3(\dot{e}_\tau, \dot{\xi}_\tau)(s) - a_3(\dot{e}_\tau, \dot{\xi}_\tau)(0) + a_2(e_\tau, \xi_\tau)(s) - a_2(e_\tau, \xi_\tau)(0)] + \\ +\int\limits_0^s \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt, \end{gathered} \tag{24}$$

Taking into account the properties of functions $\vartheta_\tau(t)$ and initial conditions $\zeta_\tau(0) = \dot{\zeta}_\tau(0) = \ddot{\zeta}_\tau(0) = \dddot{\zeta}_\tau(0) = 0$, $\xi_\tau(0) = \dot{\xi}_\tau(0) = \ddot{\xi}_\tau(0) = \dddot{\xi}_\tau(0) = 0$, we obtain from (24)

$$E_\tau(s) + 0.5a_1(\vartheta_\tau, \vartheta_\tau)(0) = \int\limits_{t_n}^{t_{n+1}} \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, \vartheta_\tau)\Big](t)dt. \tag{25}$$

We introduce one more function

$$w_\tau(t) = \int\limits_0^t \xi_\tau(t')dt' \in H_\tau,\;\; t < s;\;\; w_\tau(t) = 0,\;\; t \geq s.$$

Then, $\vartheta_\tau(t) = w_\tau(t) - w_\tau(s)$ and finally, from (25) we have the energy identity:

$$E_\tau(s) + 0.5a_1(w_\tau, w_\tau)(s) = \int\limits_0^s [a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau) + a_1(e_\tau, w_\tau(t) - w_\tau(s))](t)dt. \tag{26}$$

Let us estimate the terms on the right-hand side of (26):

$$\int\limits_0^s a_3(\dot{e}_\tau, \ddot{\xi}_\tau)dt \leq \varepsilon_1 \int\limits_0^s a_3\,(\ddot{\xi}_\tau, \ddot{\xi}_\tau)dt + \frac{1}{4\varepsilon_1}\int\limits_0^s a_3(\dot{e}_\tau, \dot{e}_\tau)dt,$$

$$\int_0^s a_2(e_\tau, \dot{\xi}_\tau)dt \leq \varepsilon_2 \int_0^s a_2(\dot{\xi}_\tau, \dot{\xi}_\tau)dt + \frac{1}{4\varepsilon_2} \int_0^s a_2(e_\tau, e_\tau)dt,$$

$$\int_0^s a_1(e_\tau, w_\tau(t) - w_\tau(s))dt \leq$$

$$\leq \varepsilon_3 \int_0^s a_1(w_\tau(t), w_\tau(t))dt + s\varepsilon_3 a_1(w_\tau(s), w_\tau(s)) + \frac{1}{2\varepsilon_3} \int_0^s a_1(e_\tau, e_\tau)dt.$$

Choosing $\varepsilon_1 = \ \varepsilon_2 = 1/2$, and ε_3 from condition $\frac{\varepsilon_1}{2} + \varepsilon_3 T \leq \frac{3}{4}$, we have the following estimate from (26):

$$E_\tau(s) + \int_0^s \Big[a_3(\ddot{\xi}_\tau, \ddot{\xi}_\tau) + a_2(\dot{\xi}_\tau, \dot{\xi}_\tau)\Big](t)dt + a_1(w_\tau,\ w_\tau)(s) \leq$$

$$\leq M \left(\int_0^s [a_1(w_\tau,\ w_\tau)(t)dt + \int_0^s [a_3(\dot{e}_\tau, \dot{e}_\tau) + a_2(e_\tau, e_\tau) + a_1(e_\tau, e_\tau)](t)dt \right),$$

where $M - const$. Hence, applying the Gronwall lemma, we obtain the error estimate

$$E_\tau(s) + \int_0^s \Big[a_3(\dot{e}_\tau, \ddot{\xi}_\tau) + a_2(e_\tau, \dot{\xi}_\tau)\Big](t)dt + a_1(w_\tau,\ w_\tau)(s) \leq$$

$$\leq \int_0^s [a_3(\dot{e}_\tau, \dot{e}_\tau) + a_2(e_\tau, e_\tau) + a_1(e_\tau, e_\tau)]dt.$$

Obviously, $k_0 \left\|w_\tau(s)\right\|_1^2 \leq a(w_\tau, w_\tau)(s) \leq k_1 \left\|w_\tau(s)\right\|_1^2$, $a(\xi_\tau, \xi_\tau)(s) = \|\xi_\tau(s)\|_1^2$, $a(\dot{\xi}_\tau, \dot{\xi}_\tau)(s) = \|\dot{\xi}_\tau(s)\|_1^2$, $a(e_\tau, e_\tau)(s) = \|e_\tau(s)\|_1^2$, $a(\dot{e}_\tau, \dot{e}_\tau)(s) = \|\dot{e}_\tau(s)\|_1^2$, so, we have the final estimate for the error:

$$\left\|\dot{\xi}_\tau(s)\right\|_1^2 + \|\xi_\tau(s)\|_1^2 + \int_0^s \left[\left\|\ddot{\xi}_\tau(t)\right\|_1^2 + \left\|\dot{\xi}_\tau(t)\right\|_1^2\right] dt \leq M \left(\int_0^s [\|\dot{e}_\tau(t)\|_1^2 + \|e_\tau(t)\|_1^2] dt \right). \tag{27}$$

Linear bounded functionals $e_\tau(t)$, $\dot{e}_\tau(t)$ vanish on polynomials up to the third degree inclusive with respect to variable t. Then, based on the Bramble-Hilbert lemma, the following estimate holds [10], [13]:

$$\int_0^s \left\|e_\tau(t')\right\|_1^2 dt' = \overline{M}^2 \tau^8 \int_0^s \left\|\frac{d^4 u_h}{dt^4}(t)\right\|_1^2 dt, \int_0^s \|\dot{e}_\tau(t)\|_1^2 dt \leq \overline{M}^2 \tau^6 \int_0^s \left\|\frac{d^4 u_h}{dt^4}(t)\right\|_1^2 dt. \tag{28}$$

Consequently, estimates (27), (28) imply the assertion of the theorem.

6 On convergence of the scheme

Note that in the estimate of Theorem 2, the error depends on solution $u_h(t)$ of the semidiscrete problem (9), while it is desirable to have smoothness conditions for the solution of original problem (2)–(4). To do this, we use the following estimate [18], [19]:

$$\|u_h\|_k = \|u - u + u_h\|_k \leq \|u\|_k + \|u - u_h\|_k \leq \|u\|_k + Ch|u|_{k+1} \leq \bar{C}\|u\|_{k+1}, \ k = 0, 1.$$

Constant $\bar{C}$ does not depend on h.

Consequently, the estimate in Theorem 2 takes the following form

$$\left\|\dot{\xi}_\tau(s)\right\|_1^2 + \|\xi_\tau(s)\|_1^2 + \int\limits_0^s \left[\left\|\ddot{\xi}_\tau(t)\right\|_1^2 + \left\|\dot{\xi}_\tau(t)\right\|_1^2\right] dt \leq M\tau^3 \sqrt{\int\limits_0^t \left\|\frac{\partial^4 u}{\partial t^4}(x, t')\right\|_2^2 dt'}.$$

On the basis of Theorems 1 and 2, the following assertion holds.

Theorem 3. Let the conditions of Theorem 2 be satisfied. Then for the solution of scheme (12) approximating the solution of problem (2)–(4) such that $u(x,t)$, $\frac{\partial u}{\partial t}(x,t) \in L_2\left\{[0,T];\ W_2^{k+1}(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)\right\}$, $\frac{\partial^4 u}{\partial t^4}(x,t) \in C\left\{[0,T]\ ;\ W_2^2(\Omega)\right\}$, the following accuracy estimate is true:

$$\left\|\frac{\partial u}{\partial t}(x,t) - \frac{\partial u_h}{\partial t}(x,t)\right\|_1 + \|u(x,t) - u_h(x,t)\|_1 + \int\limits_0^t \left\|\frac{\partial^2 u}{\partial t^2}(x,t') - \frac{\partial^2 u_h}{\partial t^2}(x,t')\right\|_1 dt' +$$

$$+ \int\limits_0^t \left\|\frac{\partial u}{\partial t}(x,t') - \frac{\partial u_h}{\partial t}(x,t')\right\|_1 dt' \leq M\left\{h^k\left(\sqrt{\int\limits_0^t \|u(x,t')\|_{k+1}^2 dt'} + \sqrt{\int\limits_0^t \left\|\frac{\partial u}{\partial t}(x,t')\right\|_{k+1}^2 dt'}\right) +\right.$$

$$\left. +\tau^3\sqrt{\int\limits_0^t \left\|\frac{\partial^4 u}{\partial t^4}(x,t')\right\|_2^2 dt'}\right\}, \ \forall t \in [0,T], M = M(r_D,\, \omega) > 0.$$

When choosing a degree $k = 3$ polynomial on each finite element in space, we have the third-order accuracy in space steps h.

Let us verify the stability condition (21). We represent the operators of scheme (12) in the following form

$$D = A_1 + A_2 + A_3 - r_D^{-2}E,\ A = (\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3,$$

where operators $A_k \geq 0$ correspond to stiffness matrices $A_k = (b_k(\varphi_l, \varphi_m))_{l,m=1}^M$ with bilinear form $b_k(u,\vartheta) = \int\limits_\Omega (u_{x_k}\vartheta_{x_k})\, dx$. Condition (21) takes the following form

$$(1-\varepsilon)(A_1 + A_2 + A_3 - r_D^{-2})E \geq \mu\tau^2(\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3,$$

or with

$$\left\|A_1 + A_2 + A_3 - r_D^{-2}E\right\| / \left\|(\omega_{B_i}^2 + \omega_{p_i}^2)(A_1 + A_2 + A_3) - \omega_{B_i}^2 r_D^{-2}E + \omega_{p_i}^2\omega_{B_i}^2 A_3\right\| < 1,$$

we obtain $\tau^2 \leq \frac{1-\varepsilon}{\mu}$, where $0 < \varepsilon < 1$.

This condition is interesting because the time step is not related to the space step and is determined by the scheme parameters. For the parameters of scheme (12), for example, for $\alpha = 1/10$, $\beta = 1/60$, $\gamma = 1/40$, $\eta = 1/12$ we have $\mu = 1/10$. So finally $\tau \leq \sqrt{10(1-\varepsilon)}$.

7 *Algorithm for implementing the scheme*

Consider one of the possible algorithms for implementing scheme (12). We rewrite it in the following form

$$\begin{cases} m_{11}\hat{y} + m_{12}\hat{\dot{y}} + m_{13}\hat{\ddot{y}} = \Phi_1, \\ m_{21}\hat{y} + m_{22}\hat{\dot{y}} + m_{23}\hat{\ddot{y}} = \Phi_2, \\ m_{31}\hat{y} + m_{32}\hat{\dot{y}} + m_{33}\hat{\ddot{y}} = \Phi_3. \end{cases} \tag{29}$$

Here

$$m_{11} = -\eta\frac{\tau^3}{2}A,\ m_{12} = D_\eta,\ m_{13} = -\frac{\tau}{2}D,\ m_{21} = D_\gamma,\ m_{22} = -\frac{\tau}{2}D_\gamma,\ m_{23} = \eta\tau^2 D,$$

$$m_{31} = -\eta\frac{\tau^3}{2}A,\ m_{32} = D_\alpha,\ m_{33} = -\frac{\tau}{2}D_\beta,\ \Phi_1 = \tau\varphi_1 + \eta\frac{\tau^3}{2}Ay + D_\eta\dot{y} + \frac{\tau}{2}D\ddot{y},$$

$$\Phi_2 = \tau\varphi_2 + D_\gamma y + \frac{\tau}{2}D_\gamma\dot{y} + \eta\tau^2 D\ddot{y},\ \Phi_3 = \tau\varphi_3 + \eta\frac{\tau^3}{2}Ay + D_\alpha\ddot{y} + \frac{\tau}{2}D_\beta\ddot{y}.$$

Assuming the mutual commutability of operators D, B and A, we exclude $\hat{\ddot{y}}$ from the system of equations (29). As a result, we obtain the following system of equations

$$\begin{cases} g_{11}\hat{y} + g_{12}\hat{\dot{y}} = \widetilde{\Phi}_1, \\ g_{21}\hat{y} + g_{22}\hat{\dot{y}} = \widetilde{\Phi}_2, \end{cases} \tag{30}$$

where

$$g_{11} = m_{23}m_{11} - m_{13}m_{21},\ g_{12} = m_{23}m_{12} - m_{13}m_{22},\ g_{21} = m_{33}m_{11} - m_{13}m_{31},$$

$$g_{22} = m_{33}m_{12} - m_{13}m_{32},\ \tilde{\Phi}_1 = m_{23}\Phi_1 - m_{13}\Phi_2,\ \tilde{\Phi}_2 = m_{33}\Phi_1 - m_{13}\Phi_3.$$

Further, excluding $\hat{\dot{y}}$ from (30), we obtain

$$C\hat{y} = F \tag{31}$$

where $C = g_{22}g_{11} - g_{12}g_{21}$, $F = g_{21}\tilde{\Phi}_1 - g_{12}\tilde{\Phi}_2$.

After determining $\hat{y}$ from (31), we find $\hat{\dot{y}}$ from one of equations (30), for example, from the first equation

$$C_1\hat{\dot{y}} = F_1,$$

where $C_1 = g_{22}g_{12}$, $F_1 = g_{22}\tilde{\Phi}_1 - g_{22}g_{11}\hat{y}$. Then, the value of $\hat{\ddot{y}}$ is found from system (29), for example, also from the first equation $C_2\hat{\ddot{y}} = F_2$, where $C_2 = m_{13}$, $F_2 = \Phi_1 - m_{11}\hat{y} - m_{12}\hat{\dot{y}}$.

As is known, problems (5), (6) were obtained as a result of approximation of space variables, so, the matrices corresponding to operators $D,\ B,\ A$ are ill-conditioned and sparse. Then, the conditionality of matrix C also worsens. Therefore, the implementation of the scheme by directly solving equation (31) is not desirable, so, in the numerical modeling of problems with specific data, it is better to factorize operator C. In addition, operators $D,\ B,\ A$ may turn out to be degenerate. Then, to eliminate the problem of operator degeneracy, the regularization principle is applied, which allows applying the spectrum of shift-operators: $\widetilde{D} = D + \varepsilon E,\ \widetilde{B} = B + \varepsilon E,\ \widetilde{A} = A + \varepsilon E$ for self-adjoint operators. Here, $\varepsilon > 0$ is a small parameter setting the value of the spectrum of shift-operators. As a result, scheme (12) is replaced by a regularized scheme with operators $\widetilde{D},\ \widetilde{B},\ \widetilde{A}$.

8 Conclusions

A boundary value problem for the equation of ion-acoustic waves in a magnetized plasma was considered. On the basis of the finite element method, parametric difference schemes of high-order accuracy were constructed and investigated. A high-order accuracy of the scheme was achieved due to the special discretization of time and space variables. In addition, the presence of parameters in the scheme makes it possible to regularize the schemes in order to optimize the implementation algorithm and the accuracy of the scheme. The corresponding a priori estimates were obtained and, on their basis, theorems on the rate of convergence and accuracy of the constructed algorithms on the smoothness of solutions to the original differential problem were proved under weak assumptions. An algorithm for the implementation of these schemes was proposed. These schemes have certain advantages over other schemes – they are two-layer schemes of high-order accuracy, except the solution itself, its derivative (velocity) is determined with the same accuracy; using the interpolation representation (11), if necessary, a solution can be obtained at any time. In addition, to achieve a certain accuracy, it allows us to select large time steps, etc.

Based on these advantages, it is possible to study other boundary value problems, including nonlocal boundary value problems. Besides, these results can be transferred to loaded equations with local and nonlocal boundary conditions.

References

1 Габов С.А. Новые задачи математической теории волн / С.А. Габов. — М.: Наука, 1998. — 448 с.

2 Zamyshlyaeva A.A. Nonclassical Equations of Mathematical Physics. Linear Sobolev Type Equations of Higher Order / A.A. Zamyshlyaeva, G.A. Sviridyuk // Bulletin of the South Ural State University. Ser. Mathematics. Mechanics. Physics. — 2016. — Vol. 8. — No. 4. — P. 5–16. DOI: 10.14529/mmph160401

3 Свешников А.Г. Линейные и нелинейные уравнения соболевского типа / А.Г. Свешников, А.Б. Альшин, М.О. Корпусов, Ю.Д. Плетнер. — М.: ФИЗМАТЛИТ, 2007. — 736 с.

4 Sutherland B.R. Internal gravity waves / B.R. Sutherland. — Cambridge, Cambridge University Press. — 2010. — 394 p.

5 Булатов В.В. Моделирование волновой динамики стратифицированных сред с учетом нелинейности, вязкости, вращения земли и сжимаемости / В.В. Булатов, Ю.В. Владимиров // Инженерный журн. «Наука и инновации». — 2013. — Вып. 2. — С. 1–25. URL: http://engjournal.ru/appmuth/hidden/606.html

6 Pedlosky J. Waves in the Ocean and Atmosphere / J. Pedlosky. — Springer. — 2013. — 264 p.

7 Калиткин Н.Н. Вычисления на квазиравномерных сетках / Н.Н. Калиткин, А.Б. Альшин, Е.А. Альшина, Б.В. Рогов. — М.: ФИЗМАТЛИТ, 2015. — 224 с.

8 Zamyshlyaeva A.A. Computational Experiment for one Mathematical Model of Ion-Acoustic Waves / A.A. Zamyshlyaeva, A.S. Muravyev // Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programing and Computer Software. — 2015. — Vol. 8. — No. 2. — P. 127–132. DOI: 10.14529/mmp150211.

9 Zamyshlyaeva A.A. Numerical Solution of Optimal Control Problem for the Model of Linear Waves in Plasma / A.A. Zamyshlyaeva, O.N. Tsyplenkova // Journal of Computational and Engineering Mathematics. — 2019. — Vol. 6. — No. 4. — P. 69–78. DOI: 10.14529/jcem190406.

10 Москальков М.Н. Численное моделирование нестационарных процессов механики сплошной среды / М.Н. Москальков, Д. Утебаев. — Ташкент: Фан ва технология, 2012. — 160 с.

11 Utebaev D. On Convergence of Schemes of Finite Element Method of High Accuracy for the Equation of Heat and Moisture Transfer / D. Utebaev, G.Kh. Utepbergenova, K.O. Tleuov // Bulletin of the Karaganda University. Mathematics Series. — 2021. — No. 2. 102. — P. 129–141. DOI: 10.31489/2021M2/129-141

12 Utebaev D. Comparison of Some Numerical Methods Solutions of Wave Equations with Strong Dispersion / D. Utebaev, B. Utebaev // International Uzbekistan-Malaysia Conference on «Computational Models and Technologies» (CMT 2020), Materials AIP Conference Proceedings. — 2021. — Vol. 2365. — P. 020009-1-9. DOI: 10.1063/5.0058159.

13 Gavrilyuk I.P. Exponentially Convergent Algoritms for Abstract Differential Equations / I.P. Gavrilyuk, V.L. Makarov, V. Vasylyk. — Birkh user, Springer Basel (Frontiers in Mathematics). — 2011. — 180 p.

14 Samarskii A.A. Difference Schemes with Operator Factors / A.A. Samarskii, P.P. Matus, P.N. Vabishchevich. — Kluwer Academic Publishers. — Boston/Dordrecht/London. — 2002. — 384 p.

15 Aripov M.M. Convergence of high-precision finite element method schemes for the two-temperature plasma equation / M.M. Aripov, D. Utebaev, Zh.A. Nurullaev // International Conference on Analysis and Applied Mathematics (ICAAM 2020), Materials AIP Conference Proceedings. — 2021. — Vol. 2325. — P. 020059-1-6. DOI: 10.1063/12.0003133

16 Samarskii A.A. The Theory of Difference Schemes / A.A. Samarskii. — Vol. 240 of Pure and Applied Mathematics, Marcel Dekker Inc., New York, Basel. — 2001. — 786 p.

17 Треногин В.А. Функциональный анализ / В.А. Треногин. — М.: ФИЗМАТЛИТ, 2002. — 488 с.

18 Zienkiewicz O.C. The Finite Element Method: Fluid Dynamics (Fifth edition) / O.C. Zienkiewicz, R. Taylor. — Butterworth-Heinemann. United Kingdom, Oxford. — 2000. — 352 p.

19 Quarteroni A. Numerical Approximation of Partial Differential Equations / A. Quarteroni, A. Valli. — Berlin: Springer-Verlag. — 1994. — 544 p.

20 Арипов М.М. Разностные схемы для операторного дифференциального уравнения четвертого порядка / М.М. Арипов, Д. Утебаев, Х.Л. Атаджанов, Ж.А. Нуруллаев // ДАН РУз. Сер. Математика, технические науки и естествознание. — 2022. — №3. — С. 11–16.

М.М. Арипов[1], Д. Өтебаев[2], Ж.А. Нуруллаев[1]

[1]*М.Ұлықбек атындағы Өзбекстан ұлттық университеті, Ташкент, Өзбекстан;*
[2]*Бердах атындағы Қарақалпақ мемлекеттік университеті, Нүкіс, Өзбекстан*

Магниттелген плазмадағы ионды-акустикалық толқындардың теңдеуі үшін жоғары дәлдіктегі айырмашылық схемаларының жинақтылығы туралы

Уақыт бойынша төртінші ретті Соболев типті теңдеу үшін дәлдігі жоғары ақырлы элементтер әдісінің көппараметрлі айырымдық схемалары зерттелген. Атап айтқанда, магниттелген плазмадағы ионды-акустикалық толқындардың теңдеуіне арналған бірінші шекаралық есеп қарастырылған. Схеманың жоғары ретті дәлдігі уақыт пен кеңістік айнымалыларының арнайы дискретизациясының арқасында қол жеткізіледі. Схемада параметрлердің болуы схемалардың дәлдігін жоғарғы ретке келтіруге және іске асыру алгоритмін оңтайландыруға мүмкіндік береді. Әлсіз нормадағы априорлық бағалау энергетикалық теңсіздік әдісімен алынады. Осы бағалаудың және Брамбл-Гильберт леммасының негізінде жалпыланған шешімдер кластарында құрастырылған алгоритмдердің жинақтылығы дәлелденді. Айырымдық схеманы жүзеге асыру алгоритмі ұсынылған.

Кілт сөздер: Соболев типті теңдеу, айырымдық схемалар, ақырлы айырымдар әдісі, ақырлы элементтер әдісі, тұрақтылық, жинақтылық, дәлдік.

М.М. Арипов[1], Д. Утебаев[2], Ж.А. Нуруллаев[1]

[1] *Национальный университет Узбекистана имени М. Улугбека, Ташкент, Узбекистан;*
[2] *Каракалпакский государственный университет имени Бердаха, Нукус, Узбекистан*

О сходимости разностных схем повышенной точности для уравнения ионно-звуковых волн в замагниченной плазме

Исследованы многопараметрические разностные схемы метода конечных элементов высокого порядка точности для уравнения соболевского типа четвертого порядка по времени. В частности, рассмотрены первая краевая задача для уравнения ионно-звуковых волн в замагниченной плазме. Высокий порядок точности схемы достигается за счет специальной дискретизации временной и пространственных переменных. Наличие параметров в схеме позволяет произвести регуляризацию точности схем, а также оптимизацию алгоритма реализации. Методом энергетических неравенств получена априорная оценка в некоторой слабой норме. На основе этой оценки и леммы Брэмбла-Гильберта доказана сходимость построенных алгоритмов в классах обобщенных решений. Предложен алгоритм реализации разностной схемы.

Ключевые слова: уравнение соболевского типа, разностные схемы, метод конечных разностей, метод конечных элементов, устойчивость, сходимость, точность.

References

1 Gabov, S.A. (1998). *Novye zadachi matematicheskoi teorii voln [New problems in the mathematical theory of waves].* Moscow: Nauka [in Russian].

2 Zamyshlyaeva, A.A., & Sviridyuk, G.A. (2016) Nonclassical Equations of Mathematical Physics. Linear Sobolev Type Equations of Higher Order. *Bulletin of the South Ural State University. Ser. Mathematics. Mechanics. Physics, 4*(8), 5–16. DOI: 10.14529/mmph160401

3 Sveshnikov, A.G., Alshin, A.B., Korpusov, M.O., & Pletner, Iu.D. (2007). *Lineinue i nelineinue uravneniia sobolevskogo tipa [Linear and non-linear equations of the Sobolev type].* Moscow: FIZMATLIT [in Russian].

4 Sutherland, B.R. (2010). *Internal gravity waves.* Cambridge, Cambridge University Press.

5 Bulatov, V.V., & Vladimirov, Iu.V. (2013). Modelirovanie volnovoi dinamiki stratifitsirovanykh sred s uchetom nelineinosti, viazkosti, vrashcheniia zemli i szhimaemosti [Modeling of wave dynamics of stratified media taking into account nonlinearity, viscosity, earth rotation and compressibility]. *Inhzenernyi jurnal «Nauka i innovatsii» — Engineering Journal: Science and Innovation, 2*, 1–25 [in Russian]. URL: http://engjournal.ru/appmuth/hidden/606.html

6 Pedlosky, J. (2013). *Waves in the Ocean and Atmosphere.* Springer.

7 Kalitkin, N.N., Alshin, A.B., Alshina, E.A., & Rogov, B.V. (2015). *Vichisleniia na kvaziravnomernykh setkakh [Computations on quasi-uniform grids].* Moscow: FIZMATLIT [in Russsian].

8 Zamyshlyaeva, A.A., & Muravyev, A.S. (2015). Computational Experiment for one Mathematical Model of Ion-Acoustic Waves. *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programing and Computer Software. 8*(2), 127–132. DOI: 10.14529/mmp150211

9 Zamyshlyaeva, A.A., & Tsyplenkova, O.N. (2019). Numerical Solution of Optimal Control Problem for the Model of Linear Waves in Plasma. *Journal of Computational and Engineering Mathematics, 6*(4), 69–78. DOI: 10.14529/jcem190406

10 Moskalkov, M.N., & Utebaev, D. (2012). *Chislennoe modelirovanie nestatsionarnykh protsessov mexaniki sploshnoi sredy [Numerical modeling of nonstationary processes in continuum mechanics].* Tashkent: Fan va tekhnologiia [in Russian].

11 Utebaev, D., Utepbergenova, G.Kh., & Tleuov, K.O. (2021). On Convergence of Schemes of Finite Element Method of High Accuracy for the Equation of Heat and Moisture Transfer. *Bulletin of Karaganda University. Mathematics Series, 2*(102) 129–141. DOI: 10.31489/2021M2/129-141

12 Utebaev, D., & Utebaev, B. (2021). Comparison of Some Numerical Methods Solutions of Wave Equations with Strong Dispersion. *International Uzbekistan-Malaysia Conference on «Computational Models and Technologies» (CMT 2020), Materials AIP Conference Proceedings*, *2365*, 020009-1-9. DOI: 10.1063/5.0058159

13 Gavrilyuk, I.P., Makarov, V.L., & Vasylyk, V. (2011). *Exponentially Convergent Algoritms for Abstract Differential Equations.* Birkh user, Springer Basel (Frontiers in Mathematics).

14 Samarskii, A.A., Matus, P.P., & Vabishchevich, P.N. (2002). *Difference Schemes with Operator Factors.* Kluwer Academic Publishers, Boston/Dordrecht/London.

15 Aripov, M.M., & Utebaev, D., & Nurullaev, Zh.A. (2021). Convergence of high-precision finite element method schemes for the two-temperature plasma equation. *International Conference on Analysis and Applied Mathematics (ICAAM 2020), Materials AIP Conference Proceedings*, *2325*, 020059-1-6. DOI: 10.1063/12.0003133.

16 Samarskiy, A.A. (2001). *The Theory of Difference Schemes.* Vol. 240 of Pure and Applied Mathematics, Marcel Dekker Inc., New York, Basel.

17 Trenogin, V.A. (2002). *Funktsionalnyi analiz [Functional analysis].* Moscow: FIZMATLIT [in Russian].

18 Zienkiewicz, O.C., & Taylor, R. (2000). *The Finite Element Method: Fluid Dynamics (Fifth edition).* Butterworth-Heinemann. United Kingdom, Oxford.

19 Quarteroni, A., & Valli, A. (1994). *Numerical Approximation of Partial Differential Equations.* Berlin: Springer-Verlag.

20 Aripov, M.M., Utebaev, D., Atadzhanov, Kh.L., & Nurullaev, Zh.A. (2022). Raznostnye skhemy dlia operatornogo differentsialnogo uravneniia chetvertogo poriadka [Difference schemes for a fourth-order operator differential equation]. *DAN RUz., Ser. Matematika, tekhnicheskie nauki i estestvoznanie — Reports of the Academy of Sciences of the Republic of Uzbekistan. Series Mathematics, technical sciences and natural sciences,* 3, 11–16 [in Russian].