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An analogue of the Lyapunov inequality for an ordinary second-order differential equation with a fractional derivative and a variable coefficient

This paper studies an ordinary second-order differential equation with a fractional differentiation operator in the sense of Riemann-Liouville with a variable coefficient. We use the Green's function's method to find a representation of the solution of the Dirichlet problem for the equation under consideration when the solvability condition is satisfied. Green's function to the problem is constructed in terms of the fundamental solution of the equation under study and its properties are proved. The necessary integral condition for the existence of a nontrivial solution to the homogeneous Dirichlet problem, called an analogue of the Lyapunov inequality, is found.

Keywords: fractional Riemann–Liouville integral, fractional Riemann–Liouville derivative, Gerasimov–Caputo fractional derivative, Dirichlet problem, Green's function, analogue of Lyapunov inequality.

Introduction

In the interval $0 < x < l$, consider the equation

$$\mathbf{L}u \equiv u''(x) + q(x)D_{0x}^{\alpha}u(x) = f(x), \qquad 0 < \alpha < 1, \tag{1}$$

where

$$D_{0x}^{\alpha}u(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int\limits_0^x \frac{u(t)dt}{(x-t)^{\alpha}}$$

is the operator of fractional (in the sense of Riemann-Liouville) differentiation of order α [1], $\Gamma(z)$ is the Euler gamma function, $q(x)$ and $f(x)$ are given functions, $u(x)$ is the desired function.

In [2] (see Theorem 3), the unconditional and unambiguous solvability of the Dirichlet problem $u(0) = 0$, $u(l) = 0$ for the equation (1) is proved for $q(x) \leq 0$. Also in [2], for $q(x) = \lambda$, where $\lambda = \text{const}$, the question of the spectrum of the homogeneous Dirichlet problem for the homogeneous equation (1) is investigated, in particular, it is shown that the numbers $\lambda \leq 0$ cannot be eigenvalues of the operator $\mathbf{L}$, and the number $\lambda > 0$ is an eigenvalue of the operator $\mathbf{L}$ if and only if $E_{2-\alpha,2}(-\lambda) = 0$, where

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\rho k + \mu)}$$

– Mittag-Leffler type function [3; 117].

Lyapunov's inequality plays an important role in the study of spectral properties of ordinary differential equations. More detailed information can be found in [4–6]. Here we give the classical Lyapunov inequality.

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If $u(x)$ is a nontrivial solution to the problem

$$u''(x) + q(x)u(x) = 0, \qquad u(a) = u(b) = 0,$$

where $q(x)$ is a real, continuous function, then it holds true the inequality

$$\int_a^b |q(x)|dx > \frac{4}{b-a}. \tag{2}$$

There are works where various generalizations of the Lyapunov inequality (2) are constructed. For example, in [7], for an ordinary fractional differential equation containing a composition of fractional derivatives with different beginnings, the necessary integral condition for the existence of a nontrivial solution to the homogeneous Dirichlet problem is found, namely: *if $u(x)$ is a nontrivial solution to the problem*

$$D_{0x}^{\alpha}\partial_{1x}^{\alpha}u(x) - q(x)u(x) = 0, \qquad u(0) = u(1) = 0, \quad \frac{1}{2} < \alpha < 1,$$

where $q(x)$ is a real continuous function, then the following inequality is true:

$$\int_0^1 |q(x)|dx > (2\alpha - 1)\frac{\Gamma^2(\alpha)}{h}, \qquad h = \sup_{0<x<1}\left[(1-x)^{2\alpha-1} - (1-x^{2\alpha-1})^2\right].$$

The work [8] shows that for the existence of a nontrivial solution to the homogeneous Dirichlet problem for an ordinary second-order differential equation with a distributed integration operator

$$u''(x) + q(x)\int_0^{\beta} \mu(\alpha)D_{0x}^{-\alpha}u(x)d\alpha = 0, \qquad u(0) = u(l) = 0,$$

the condition must be fulfilled

$$\int_0^{\beta} |\mu(\alpha)|\frac{l^{\alpha}d\alpha}{\Gamma(\alpha+1)}\int_0^l |q(x)|dx \geq \frac{4}{l},$$

which is an analogue of the Lyapunov inequality.

In this paper, a representation of the solution to the Dirichlet problem for the equation (1), using the Green function, is found in the case when $q(x) \leq 0$, and an analogue of the Lyapunov inequality is proved.

Problem statement

We call a regular solution a function $u(x)$ that belongs to the class $C[0,l] \cap C^2]0,l[$ and satisfies the equation (1) for all $x \in]0,l[$.

Problem. Find a regular solution $u(x)$ to the equation (1) in the interval $]0,l[$ satisfying the conditions

$$u(0) = u_0, \quad u(l) = u_l, \tag{3}$$

where u_0, u_l are the specified constants.

Supporting statements

Let $q(x)$ be absolutely continuous on the segment $[0,l]$. Consider two functions defined in the compact $\overline{\Omega} = [0,l] \times [0,l]$

$$W(x,t) = x - t - \int\limits_t^x (x-s)R(s,t)ds, \tag{4}$$

$$G(x,t) = H(x-t)W(x,t) - \frac{1}{W(l,0)}\Big[W(x,0)W(l,t)\Big]. \tag{5}$$

Here

$$R(x,t) = \sum_{n=1}^{\infty}(-1)^{n+1}K_n(x,t), \qquad K_1(x,t) = \partial_{xt}^{\alpha}[(x-t)q(t)],$$

$$K_{n+1}(x,t) = \int\limits_t^x K_n(x,s)K_1(s,t)dt, \ \ n \in \mathbb{N}, \qquad \partial_{0x}^{\alpha}u(x) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^x \frac{u'(t)dt}{(x-t)^{\alpha}}, \ \ \alpha \in]0,1[$$

– the operator of fractional (in the sense of Gerasimov–Caputo) differentiation of order α, $H(x)$ – Heaviside function.

Lemma 1. The function $W(x,t)$ with respect to the variable x is the solution of the problem

$$W_{xx}(x,t) + q(x)D_{tx}^{\alpha}W(x,t) = 0, \tag{6}$$

$$W(t,t) = 0, \qquad W_x(t,t) = 1, \qquad \forall t \in [0,x], \tag{7}$$

and according to the variable t is the solution of the problem

$$W_{tt}(x,t) + \partial_{xt}^{\alpha}[q(t)W(x,t)] = 0, \tag{8}$$

$$W(x,x) = 0, \qquad W_t(x,x) = -1, \qquad \forall x \in [0,l]. \tag{9}$$

Lemma 1 is proved by directly substituting formula (4) into the equalities (6)–(9).

Definition. The Green function of the Dirichlet problem (3) for equation (1) is called the function $v(x,t)$, having the following properties:
1. $v(x,t)$ is continuous in $\overline{\Omega}$.
2. $v(x,t)$ as a function of the variable x is the solution of the problem

$$v_{xx}(x,t) + q(x)D_{0x}^{\alpha}v(x,t) = 0, \qquad v(0,t) = 0, \quad v(l,t) = 0, \tag{10}$$

by the variable t is the solution of the problem

$$v_{tt}(x,t) + \partial_{lt}^{\alpha}[q(t)v(x,t)] = 0, \qquad v(x,0) = 0, \quad v(x,l) = 0. \tag{11}$$

3. For $t = x$, the derivatives $v_x(x,t)$ and $v_t(x,t)$ have a jump equal to one, that is

$$v_x(x,x+0) - v_x(x,x-0) = -1, \tag{12}$$

$$v_t(x,x+0) - v_t(x,x-0) = 1. \tag{13}$$

Lemma 2. Let the condition $W(l,0) \neq 0$ be fulfilled. Then the function $G(x,t)$, defined by formula (5), is the Green function of the Dirichlet problem (3) for equation (1).

Proof. The continuity of the Green function $G(x,t)$ in the compact $\overline{\Omega}$ follows from the continuity of the function $W(x,t)$ in this compact $\overline{\Omega}$.

The second property is proved by direct substitution of equality (5) in formulas (10), (11) and taking into account the relations (6) – (9)

$$G_{xx}(x,t)+q(x)D^{\alpha}_{0x}G(x,t)=H(x-t)\Big[W_{xx}(x,t)+q(x)D^{\alpha}_{tx}W(x,t)\Big]-$$
$$-\frac{W(l,t)}{W(l,0)}\Big[W_{xx}(x,0)+q(x)D^{\alpha}_{0x}W(x,0)\Big]=0, \tag{14}$$
$$G_{tt}(x,t)+\partial^{\alpha}_{lt}[q(t)G(x,t)]=H(x-t)\Big[W_{tt}(x,t)+\partial^{\alpha}_{xt}[q(t)W(x,t)]\Big]+$$
$$+\frac{W(x,0)}{W(l,0)}\Big[W_{tt}(l,t)+\partial^{\alpha}_{lt}[q(t)W(l,t)]\Big]=0. \tag{15}$$

From the representation (5), by virtue of the relations $W(0,0)=0$, $W(l,l)=0$, we have the equality

$$G(0,t)=0,\quad G(l,t)=0,\quad G(x,0)=0,\quad G(x,l)=0. \tag{16}$$

Differentiating equality (5) by x and by t

$$G_x(x,t)=H(x-t)W_x(x,t)-\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,t)\Big], \tag{17}$$
$$G_t(x,t)=H(x-t)W_t(x,t)-\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,t)\Big], \tag{18}$$

and substituting formulas (17), (18) into relations (12), (13), taking into account that $W(x)$ is a continuous function, $H(x)$ is a function discontinuous at zero, and taking into account the equalities $W_x(x,x)=1$, $W_t(x,x)=-1$, we get

$$G_x(x,x+0)-G_x(x,x-0)=\lim_{\varepsilon\to-0}H(\varepsilon)W_x(x,x)-\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,x)\Big]-$$
$$-\lim_{\varepsilon\to+0}H(\varepsilon)W_x(x,x)+\frac{1}{W(l,0)}\Big[W_x(x,0)W(l,x)\Big]=-1, \tag{19}$$
$$G_t(x,x+0)-G_t(x,x-0)=\lim_{\varepsilon\to-0}H(\varepsilon)W_t(x,x)-\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,x)\Big]-$$
$$-\lim_{\varepsilon\to+0}H(\varepsilon)W_t(x,x)+\frac{1}{W(l,0)}\Big[W(x,0)W_t(l,x)\Big]=1, \tag{20}$$

which proves the validity of formulas (12) and (13). Lemma 2 is proved.

It follows from the relations (17), (18), by virtue of formulas (7) and (9), the equalities

$$G_x(x,0)=0,\qquad G_x(x,l)=0, \tag{21}$$
$$G_t(x,0)=W_t(x,0)-\frac{W(x,0)W_t(l,0)}{W(l,0)},\qquad G_t(x,l)=\frac{W(x,0)}{W(l,0)}. \tag{22}$$

Presentation of the solution

At this point, we will find a representation of the solution to the problem (1), (3).

Theorem 1. Let $q(x)$ be absolutely continuous on the segment $[0,l]$, and $q(x)\le 0$, $f(x)\in L[0,l]\cap C]0,l[$. Then if the condition $W(l,0)\ne 0$ is satisfied, there is a unique regular solution to the problem (1), (3). The solution has the form

$$u(x)=-u_0G_t(x,0)+u_lG_t(x,l)+\int_0^l G(x,t)f(t)dt. \tag{23}$$

Proof. Let $u(x)$ be a regular solution of the equation (1). Multiply both sides of the equation (1) by the function $G(x,t)$ after changing the variable x into t, and integrate the resulting equality by t in the range from 0 to l. Then we have

$$\int_0^l G(x,t)u''(t)dt + \int_0^l G(x,t)q(t)D_{0t}^{\alpha}u(t)dt = \int_0^l G(x,t)f(t)dt. \tag{24}$$

Integrating in parts the first term of the left side of equality (24), given that the function $G_t(x,t)$ has a jump, having previously split the integration interval into two intervals from 0 to x and from x to l, we will have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{l-\varepsilon} G(x,t)u''(t)dt = \lim_{\varepsilon\to 0}\Bigg[u'(l-\varepsilon)G(x,l-\varepsilon) - u'(\varepsilon)G(x,\varepsilon) - \int_{\varepsilon}^{x} G_t(x,t)u'(t)dt -$$

$$-\int_{x}^{l-\varepsilon} G_t(x,t)u'(t)dt\Bigg] = u'(l)G(x,l) - u'(0)G(x,0) - u(l)G_t(x,l) + u(0)G_t(x,0) +$$

$$+u(x)\big[G_t(x,x+0) - G_t(x,x-0)\big] + \int_0^l G_{tt}(x,t)u(t)dt. \tag{25}$$

According to the formula of fractional integration by parts and equalities (16), the second term of the left part of the formula (24) can be rewritten as

$$\int_0^l G(x,t)q(t)D_{0t}^{\alpha}u(t)dt = q(l)G(x,l)D_{0l}^{\alpha-1}u(t) - q(0)G(x,0)D_{00}^{\alpha-1}u(t) -$$

$$-\int_0^l \frac{\partial}{\partial t}[q(t)G(x,t)]D_{0t}^{\alpha-1}u(t)dt = -\int_0^l u(t)D_{lt}^{\alpha-1}\frac{\partial}{\partial t}[q(t)G(x,t)]dt =$$

$$= \int_0^l u(t)\partial_{lt}^{\alpha}[q(t)G(x,t)]dt. \tag{26}$$

Considering formulas (20), (25), and (26) by equality (24) we obtain

$$u(x) + \int_0^l u(t)\Big[G_{tt}(x,t) + \partial_{lt}^{\alpha}[q(t)G(x,t)]\Big]dt =$$

$$= -u'(l)G(x,l) + u'(0)G(x,0) + u(l)G_t(x,l) - u(0)G_t(x,0) + \int_0^l G(x,t)f(t)dt,$$

from which, by virtue of the relations (15) and (16), we obtain the formula (23).

Let us now show that the function $u(x)$, defined by formula (23), is indeed the solution of problem (1), (3). Differentiating equality (23) twice, taking into account the formula (19), we will have

$$u'(x) = -u_0\big[G_t(x,0)\big]' + u_l\big[G_t(x,l)\big]' + \int\limits_0^l G_x(x,t)f(t)dt. \tag{27}$$

From formula (27) we get

$$u''(x) = -u_0\big[G_t(x,0)\big]'' + u_l\big[G_t(x,l)\big]'' + \frac{d}{dx}\left[\int\limits_0^x G_x(x,t)f(t)dt + \int\limits_x^l G_x(x,t)f(t)dt\right] =$$

$$= -u_0\big[G_t(x,0)\big]'' + u_l\big[G_t(x,l)\big]'' + \int\limits_0^l G_{xx}(x,t)f(t)dt + f(x). \tag{28}$$

Further, from formula (23) we have

$$D_{0x}^{\alpha}u(x) = -u_0 D_{0x}^{\alpha}G_t(x,0) + u_l D_{0x}^{\alpha}G_t(x,l) + \int\limits_0^l f(t)D_{0x}^{\alpha}G(x,t)dt. \tag{29}$$

Substituting formulas (28) and (29) into equation (1), by virtue of equality (14), we obtain that the function defined by relation (23) is indeed the solution of equation (1). Taking into account formulas (16), (21), (22), the direct substitution of function (23) into equality (3) gives the correct identities. Theorem 1 is proved.

An analogue of the Lyapunov inequality

At this point, we reduce the homogeneous problem

$$u''(x) + q(x)D_{0x}^{\alpha}u(x) = 0, \qquad u(0) = 0, \quad u(l) = 0 \tag{30}$$

to the Fredholm integral equation of the second kind, with the help of which we obtain an analogue of the Lyapunov inequality.

Since by the condition $u(0) = 0$, then through the property of the fractional differentiation operator we have the equality

$$D_{0x}^{\alpha}u(x) = D_{0x}^{\alpha-1}u'(x).$$

Given the last formula, we will act on both parts of the first equality (30) with the operator D_{0x}^{-1}. Then, with respect to the function $u'(x)$, we obtain the loaded integral equation

$$u'(x) + \int\limits_0^x q(t)D_{0t}^{\alpha-1}u'(t)dt = u'(0). \tag{31}$$

To determine the unknown constant $u'(0)$ in formula (31), we will act on both parts of equality (31) with the operator D_{lx}^{-1}. Then we will have

$$u(l) - u(x) + \int\limits_x^l \int\limits_0^t q(s)D_{0s}^{\alpha-1}u'(s)dsdt = u'(0)(l-x). \tag{32}$$

Letting x tend to zero in equation (32) and taking into account the equalities $u(0) = 0$, $u(l) = 0$, we get that

$$u'(0) = \frac{1}{l}\int_0^l\int_0^t q(s)D_{0s}^{\alpha-1}u'(s)dsdt. \tag{33}$$

Substituting now formula (33) into equality (31), after simple transformations, we obtain the Fredholm integral equation of the second kind with respect to the function $u'(x)$

$$u'(x) = \int_0^l q(t)\left[\frac{l-t}{l} - H(x-t)\right]D_{0t}^{\alpha-1}u'(t)dt. \tag{34}$$

Theorem 2. Let $q(x)$ be continuous on the segment $[0, l]$, the homogeneous problem (30) has a nontrivial solution $u(x)$. Then there is an inequality

$$\int_0^l |q(x)|dx > \frac{\Gamma(2-\alpha)}{l^{1-\alpha}}. \tag{35}$$

Proof. First, we note that if $u(x)$ is a nontrivial solution of equation (30) satisfying the conditions $u(0) = 0$, $u(l) = 0$, then and the function $u'(x) \neq 0$. It is valid if $u'(x) = 0$, then $u(x) = \text{const}$, and by the condition $u(0) = 0$, therefore, in this case, we have $u(x) = 0$, which contradicts the condition of Theorem 2.

Suppose that

$$\overline{u} = \max_{x\in[0,l]} |u'(x)|.$$

Then from equation (34) we have the inequality

$$\overline{u} \leq \overline{u} \cdot \max_{x\in[0,l]} \int_0^l |q(t)|\left|\frac{l-t}{l} - H(x-t)\right|\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}dt. \tag{36}$$

The function

$$F(x,t) = \left|\frac{l-t}{l} - H(x-t)\right|\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$$

takes the largest value at $x = t = l$, therefore, given the equality

$$F_{\max} = F(l,l) = \frac{l^{1-\alpha}}{\Gamma(2-\alpha)}$$

from the relation (36) we have the inequality

$$\frac{l^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^l |q(t)|dt \geq 1,$$

that is equivalent to (35). Let us call inequality (35) an analogue of Lyapunov's inequality.

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Бөлшек туындылы және айнымалы коэффициентті екінші ретті қарапайым дифференциалдық теңдеу үшін Ляпунов теңсіздігінің аналогы

Мақалада айнымалы коэффициентті Риман–Лиувиль мағынасындағы бөлшек дифференциалдау операторы бар екінші ретті қарапайым дифференциалдық теңдеу зерттелген. Грин функциясының әдісімен қарастырылған теңдеудің шешімділік шартын орындауда Дирихле есебінің шешімінің мәні табылған. Зерттелетін теңдеудің іргелі шешімі бойынша Гриннің тиісті функциясы құрылды және оның қасиеттері дәлелденді. Ляпунов теңсіздігінің аналогы деп аталатын біртекті Дирихле есебінің тривиал емес шешімі болуының қажетті интегралдық шарты табылды.

Кілт сөздер: Риман–Лиувиль бөлшек интегралы, Риман–Лиувиль бөлшек туындысы, Герасимов–Капуто бөлшек туындысы, Дирихле есебі, Грин функциясы, Ляпунов теңсіздігінің аналогы.

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Аналог неравенства Ляпунова для обыкновенного дифференциального уравнения второго порядка с дробной производной и с переменным коэффициентом

В статье исследовано обыкновенное дифференциальное уравнение второго порядка с оператором дробного дифференцирования в смысле Римана–Лиувилля с переменным коэффициентом. Методом функции Грина найдено представление решения задачи Дирихле для рассматриваемого уравнения при выполнении условия разрешимости. Построена соответствующая функция Грина в терминах фундаментального решения исследуемого уравнения и доказаны ее свойства. Найдено необходимое интегральное условие существования нетривиального решения однородной задачи Дирихле, названное аналогом неравенства Ляпунова.

Ключевые слова: дробный интеграл Римана–Лиувилля, дробная производная Римана–Лиувилля, дробная производная Герасимова–Капуто, задача Дирихле, функция Грина, аналог неравенства Ляпунова.

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