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U.K. Koilyshov[1,2,*], K.A. Beisenbaeva[3], S.D. Zhapparova[1]

[1]*Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[2]*Institute of Mathematics and Mathematical Modelling, Almaty, Kazakhstan;*
[3]*Academy of Logistics and Transport, Almaty, Kazakhstan*
(E-mail: koylyshov@mail.ru, beysenbaeva56@mail.ru, zhapparova.saule@gmail.com)

A priori estimate of the solution of the Cauchy problem in the Sobolev classes for discontinuous coefficients of degenerate heat equations

Partial differential equations of the parabolic type with discontinuous coefficients and the heat equation degenerating in time, each separately, have been well studied by many authors. Conjugation problems for time-degenerate equations of the parabolic type with discontinuous coefficients are practically not studied. In this work, in an n-dimensional space, a conjugation problem is considered for a heat equation with discontinuous coefficients which degenerates at the initial moment of time. A fundamental solution to the set problem has been constructed and estimates of its derivatives have been found. With the help of these estimates, in the Sobolev classes, the estimate of the solution to the set problem was obtained.

Keywords: conjugation problem, heat equation, degenerating equations, discontinuous coefficients.

Partial differential equations of the parabolic type with discontinuous coefficients are studied in the works [1–8]. Time-degenerate equations of heat conduction are studied in the works [9, 10]. The conjugation problems for the periodic equations of the parabolic type with discontinuous coefficients are slightly studied. We consider the Cauchy problem for a degenerating equation with discontinuous coefficients: find functions $u_1(x,t), u_2(x,t)$ that satisfy the equations

$$t^p \frac{\partial u_1}{\partial t} = a_1^2 \Delta u_1 + f_1(x,t), \quad (x,t) \in D_{n+1}^- = \{(x,t), x' \in R^{n-1}, x_n < 0, t > 0\}, \tag{1}$$

$$t^p \frac{\partial u_2}{\partial t} = a_2^2 \Delta u_2 + f_2(x,t), \quad (x,t) \in D_{n+1}^+ = \{(x,t), x' \in R^{n-1}, x_n > 0, t > 0\}, \tag{2}$$

with initial conditions

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \tag{3}$$

and with conjugation conditions

$$u_1\Big|_{x_n=-0} = u_2\Big|_{x_n=+0}, \tag{4}$$

$$k_1 \frac{\partial u_1}{\partial x_n}\Big|_{x_n=-0} = k_2 \frac{\partial u_2}{\partial x_n}\Big|_{x_n=+0}, \tag{5}$$

where $x' = (x_1, x_2, ..., x_{n-1})$,
$k_i > 0$, $p < 1$, $(i = 1, 2)$.

The feature of the problem is that equations (1) and (2) with discontinuous coefficients degenerate at the initial moment $t = 0$.

*Corresponding author.
E-mail: koylyshov@mail.ru

Method of solving.

To solve problems (1)–(5) let us consider an auxiliary problem A: in the domain $D_{n+1}(x \in R^n$, $t > 0)$, find functions $u_1(x,t), u_2(x,t)$ that satisfy the equations

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + f_1(x,t), \quad (x,t) \in D^-_{n+1} = \{(x,t), x' \in R^{n-1}, x_n < 0, t > 0\}, \tag{6}$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + f_2(x,t), \quad (x,t) \in D^+_{n+1} = \{(x,t), x' \in R^{n-1}, x_n > 0, t > 0\}, \tag{7}$$

with initial conditions

$$u_1(x,0) = \varphi_1(x), \quad u_2(x,0) = \varphi_2(x), \tag{8}$$

and with conjugation conditions

$$u_1\Big|_{x_n=-0} = u_2\Big|_{x_n=+0}, \tag{9}$$

$$k_1 \frac{\partial u_1}{\partial x_n}\Big|_{x_n=-0} = k_2 \frac{\partial u_2}{\partial x_n}\Big|_{x_n=+0}, \tag{10}$$

where $k_i > 0$, $(i = 1,2)$. Applying to problem (6)–(10) the Fourier transform with respect to variables $x' = (x_1, x_2, ..., x_{n-1})$ and the Laplace transform with respect to variable t, we obtain an inhomogeneous second-order differential equation

$$\frac{d^2\,\widetilde{\widetilde{u}}_1}{dx_n^2} - \left(p + |s'|^2\right)\widetilde{\widetilde{u}}_1 = -\widetilde{\widetilde{f}}_1(s', x_n, p) - \widetilde{\varphi}_1(s', x_n), \quad x_n < 0, \tag{11}$$

$$\frac{d^2\,\widetilde{\widetilde{u}}_2}{dx_n^2} - \left(p + |s'|^2\right)\widetilde{\widetilde{u}}_2 = -\widetilde{\widetilde{f}}_2(s', x_n, p) - \widetilde{\varphi}_2(s', x_n), \quad x_n > 0, \tag{12}$$

where $s' = (s_1, s_2, ..., s_{n-1})$. Conjugation conditions (9)–(10) take the following form:

$$\widetilde{\widetilde{u}}_1\Big|_{x_n=-0} = \widetilde{\widetilde{u}}_2\Big|_{x_n=+0}, \tag{13}$$

$$k_1 \frac{d\widetilde{\widetilde{u}}_1}{dx_n}\Big|_{x_n=-0} = k_2 \frac{d\widetilde{\widetilde{u}}_2}{dx_n}\Big|_{x_n=+0}, \tag{14}$$

The solutions to equations (11)–(12) have the form:

$$\widetilde{\widetilde{u}}_1(s', x_n, p) = \left(c_1 - \frac{1}{2\sqrt{p+|s'|^2}} \int_0^{x_n} \widetilde{\widetilde{F}}_1(s', \xi_n, p) e^{-\sqrt{p+|s'|^2}\xi_n}\, d\xi_n\right) e^{\sqrt{p+|s'|^2}x_n} +$$
$$+ \left(c_2 + \frac{1}{2\sqrt{p+|s'|^2}} \int_0^{x_n} \widetilde{\widetilde{F}}_1(s', \xi_n, p) e^{\sqrt{p+|s'|^2}\xi_n}\, d\xi_n\right) e^{-\sqrt{p+|s'|^2}x_n}, \quad x_n < 0,$$

$$\widetilde{\widetilde{u}}_2(s', x_n, p) = \left(d_1 - \frac{1}{2\sqrt{p+|s'|^2}} \int_0^{x_n} \widetilde{\widetilde{F}}_2(s', \xi_n, p) e^{-\sqrt{p+|s'|^2}\xi_n}\, d\xi_n\right) e^{\sqrt{p+|s'|^2}x_n} +$$
$$+ \left(d_2 + \frac{1}{2\sqrt{p+|s'|^2}} \int_0^{x_n} \widetilde{\widetilde{F}}_2(s', \xi_n, p) e^{\sqrt{p+|s'|^2}\xi_n}\, d\xi_n\right) e^{-\sqrt{p+|s'|^2}x_n}, \quad x_n > 0,$$

here $\widetilde{\widetilde{F}}_i(s',x_n,p)=\widetilde{\widetilde{f}}_i(s',x_n,p)+\widetilde{\varphi}_i(s',x_n)$, $(i=1,2)$. We obtain a solution to problem (11)-(14):

$$\widetilde{u}_1(s',x_n,p)=\int\limits_{-\infty}^{0}\frac{\widetilde{\widetilde{F}}_1(s',\xi_n,p)}{2\sqrt{p+|s'|^2}}\left(e^{-\sqrt{p+|s'|^2}|x_n-\xi_n|}+\lambda e^{\sqrt{p+|s'|^2}(x_n+\xi_n)}\right)d\xi_n+$$
$$+\mu_2\int\limits_{0}^{+\infty}\frac{\widetilde{\widetilde{F}}_2(s',\xi_n,p)}{2\sqrt{p+|s'|^2}}e^{-\sqrt{p+|s'|^2}(\xi_n-x_n)}\,d\xi_n,\quad x_n<0,$$

$$\widetilde{u}_2(s',x_n,p)=\int\limits_{0}^{+\infty}\frac{\widetilde{\widetilde{F}}_2(s',\xi_n,p)}{2\sqrt{p+|s'|^2}}\left(e^{-\sqrt{p+|s'|^2}|x_n-\xi_n|}-\lambda e^{-\sqrt{p+|s'|^2}(x_n+\xi_n)}\right)d\xi_n+$$
$$+\mu_1\int\limits_{-\infty}^{0}\frac{\widetilde{\widetilde{F}}_1(s',\xi_n,p)}{2\sqrt{p+|s'|^2}}e^{-\sqrt{p+|s'|^2}(x_n-\xi_n)}\,d\xi_n,\quad x_n>0,$$

here $\lambda=\frac{k_1-k_2}{k_1+k_2},\quad \mu_i=\frac{2k_i}{k_1+k_2},\quad (i=1,2)$.

The solutions to equations (6)–(10) have the form:

$$\begin{aligned}
u_1(x,t)&=\int\limits_{R^{n-1}}\int\limits_{-\infty}^{0}\left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}+\lambda\frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}\right]\varphi_1(\xi',\xi_n)\,d\xi'd\xi_n+\\
&+\mu_2\int\limits_{R^{n-1}}\int\limits_{0}^{+\infty}\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}\varphi_2(\xi',\xi_n)\,d\xi'd\xi_n+\\
&+\int\limits_0^t d\tau\int\limits_{R^{n-1}}\int\limits_{-\infty}^{0}\left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}+\lambda\frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}\right]f_1(\xi',\xi_n,\tau)\,d\xi'd\xi_n+\\
&+\mu_2\int\limits_0^t d\tau\int\limits_{R^{n-1}}\int\limits_{0}^{+\infty}\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}f_2(\xi',\xi_n,\tau)\,d\xi'd\xi_n,\quad D_n^-,\\
u_2(x,t)&=\int\limits_{R^{n-1}}\int\limits_{0}^{+\infty}\left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}-\lambda\frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}\right]\varphi_2(\xi',\xi_n)\,d\xi'd\xi_n+\\
&+\mu_1\int\limits_{R^{n-1}}\int\limits_{-\infty}^{0}\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}\varphi_1(\xi',\xi_n)\,d\xi'd\xi_n+\\
&+\int\limits_0^t d\tau\int\limits_{R^{n-1}}\int\limits_{0}^{+\infty}\left[\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}-\lambda\frac{e^{-\frac{|x'-\xi'|^2+(x_n+\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}\right]f_2(\xi',\xi_n,\tau)\,d\xi'd\xi_n+\\
&+\mu_1\int\limits_0^t d\tau\int\limits_{R^{n-1}}\int\limits_{-\infty}^{0}\frac{e^{-\frac{|x'-\xi'|^2+(x_n-\xi_n)^2}{4(t-\tau)}}}{\left(2\sqrt{\pi(t-\tau)}\right)^n}f_1(\xi',\xi_n,\tau)\,d\xi'd\xi_n,\quad D_n^+,
\end{aligned}\tag{15}$$

where $d\xi'=d\xi_1d\xi_2\cdot\ldots\cdot d\xi_{n-1}$, $|x'-\xi'|=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+\ldots+(x_{n-1}-\xi_{n-1})^2}$. We introduce the notation $G(x'-\xi',x_n\pm\xi_n,t)=\frac{e^{-\frac{|x'-\xi'|^2+(x_n\pm\xi_n)^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}$. Then

$$
\begin{aligned}
u_1(x,t) = &\int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G(x'-\xi', x_n-\xi_n, t) + \lambda\, G(x'-\xi', x_n+\xi_n, t)\Big] \varphi_1(\xi', \xi_n)\, d\xi' d\xi_n + \\
&+ \mu_2 \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G(x'-\xi', x_n-\xi_n, t)\varphi_2(\xi', \xi_n)\, d\xi' d\xi_n + \\
&+ \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G(x'-\xi', x_n-\xi_n, t-\tau) + \lambda\, G(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \\
&+ \mu_2 \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G(x'-\xi', x_n-\xi_n, t-\tau) f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^-,
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
u_2(x,t) = &\int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G(x'-\xi', x_n-\xi_n, t) - \lambda\, G(x'-\xi', x_n+\xi_n, t)\Big] \varphi_2(\xi', \xi_n)\, d\xi' d\xi_n + \\
&+ \mu_1 \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G(x'-\xi', x_n-\xi_n, t)\varphi_1(\xi', \xi_n)\, d\xi' d\xi_n + \\
&+ \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G(x'-\xi', x_n-\xi_n, t-\tau) - \lambda\, G(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_2(\xi', \xi_n, \tau)\, d\xi' d\xi_n + \\
&+ \mu_1 \int\limits_0^t d\tau \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G(x'-\xi', x_n-\xi_n, t-\tau) f_1(\xi', \xi_n, \tau)\, d\xi' d\xi_n, \quad D_n^+.
\end{aligned} \tag{17}
$$

We have obtained the solution to auxiliary problem (6)–(10) in the form (16)–(17).

Using [11], for the function $\Gamma(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{\left(2\sqrt{\pi t}\right)^n}$, we obtain an estimate:

$$|D_x^k D_t^m \Gamma(x,t)| \le \frac{Ce^{-\delta\frac{|x|^2}{t}}}{t^{\frac{n+k}{2}+m}}.$$

This estimate is valid from [12]. Here $\delta < \frac{1}{4}$.

For the function $G(x'-\xi', x_{n-1}-\xi_{n-1}, t)$ the same estimate can be given:

$$|D_x^k D_t^m G(x'-\xi', x_{n-1}-\xi_{n-1}, t)| \le \frac{Ce^{-\delta\frac{|x-\xi|^2}{t}}}{t^{\frac{n+k}{2}+m}}.$$

Now consider the auxiliary problem B. Consider the Cauchy problem for a degenerate heat equation: in the domain $D_{n+1}^+ = \{(x,t), x \in R^n, t > 0\}$ to find a function $u(x,t)$ that satisfies the equation

$$t^p \frac{\partial u}{\partial t} = \Delta u + f(x,t), \quad (x,t) \in D_{n+1} = \{(x,t), x \in R^n, t > 0\}, \tag{18}$$

with initial condition

$$u(x,0) = \varphi(x). \tag{19}$$

By applying the Fourier transform in variables $x = (x_1, ..., x_n)$ to equation (18)

$$t^p \frac{\partial \widetilde{u}}{\partial t} + |s|^2 \widetilde{u} = \widetilde{f}(s,t), \tag{20}$$

we obtain a non-homogeneous differential equation of the first order. Here $s = (s_1, s_2, ..., s_n)$, $|s| = \sqrt{s_1^2 + s_2^2 + ... + s_n^2}$, $\quad p < 1$. The initial condition (19) takes the following form:

$$\widetilde{u}(s,0) = \widetilde{\varphi}(s). \tag{21}$$

Taking into account the initial condition (21), the solution to equation (20) has the form:

$$\widetilde{u}(s,t) = \widetilde{\varphi}(s) e^{-\frac{t^q}{q}|s|^2} + \int\limits_0^t \frac{\widetilde{f}(s,\tau)}{\tau^p} e^{-\frac{(t^q-\tau^q)}{q}|s|^2}\, d\tau, \tag{22}$$

here $q = 1 - p$.
Applying the inverse Fourier transform to equality (22), using the convolution formula, formulas [13] and (15), we obtain a solution to problem (18)–(19):

$$u(x,t) = \int\limits_{R^n} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x-\xi|^2}{4t^q}} \varphi(\xi)\, d\xi + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^n} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi\left(t^q - \tau^q\right)}\right)^n} e^{-\frac{q|x-\xi|^2}{4(t^q-\tau^q)}} f(\xi,\tau)\, d\xi. \tag{23}$$

If we introduce the notation $\Gamma_q(x,t) = \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x|^2}{4t^q}}$, then formula (23) can be written in the form:

$$u(x,t) = \int\limits_{R^n} \Gamma_q(x-\xi,t)\varphi(\xi)\, d\xi + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^n} \Gamma_q(x-\xi,t-\tau) f(\xi,\tau)\, d\xi. \tag{24}$$

In [14], the function $\Gamma_q(x,t)$ was constructed in one-dimensional space. As shown in [12], for this function we can accept the following estimate:

$$|D_x^k D_t^m \Gamma_q(x,t)| \leq \frac{C e^{-\delta \frac{|x|^2}{t^q}}}{t^{\frac{q(n+k)}{2}+m}}, \tag{25}$$

where $\delta < \frac{1}{4}$.

The results of research.

Now let us solve the main problem (1)–(5). Using the solutions to auxiliary problems A and B, the solutions of which have the form (16)–(17) and (24), we can obtain the solution to problem (1)–(5) in the form:

$$
\begin{aligned}
u_1(x,t) = & \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G_q(x'-\xi', x_n-\xi_n, t) + \lambda\, G_q(x'-\xi', x_n+\xi_n, t)\Big]\varphi_1(\xi',\xi_n)\, d\xi' d\xi_n + \\
& + \mu_2 \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x'-\xi', x_n-\xi_n, t)\varphi_2(\xi',\xi_n)\, d\xi' d\xi_n + \\
& + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} \Big[G_q(x'-\xi', x_n-\xi_n, t-\tau) + \lambda\, G_q(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_1(\xi',\xi_n,\tau)\, d\xi' d\xi_n + \\
& + \mu_2 \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x'-\xi', x_n-\xi_n, t-\tau) f_2(\xi',\xi_n,\tau)\, d\xi' d\xi_n, \quad D_n^-,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
u_2(x,t) = & \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G_q(x'-\xi', x_n-\xi_n, t) - \lambda\, G_q(x'-\xi', x_n+\xi_n, t)\Big]\varphi_2(\xi',\xi_n)\, d\xi' d\xi_n + \\
& + \mu_1 \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G_q(x'-\xi', x_n-\xi_n, t)\varphi_1(\xi',\xi_n)\, d\xi' d\xi_n + \\
& + \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \Big[G_q(x'-\xi', x_n-\xi_n, t-\tau) - \lambda\, G_q(x'-\xi', x_n+\xi_n, t-\tau)\Big] f_2(\xi',\xi_n,\tau)\, d\xi' d\xi_n + \\
& + \mu_1 \int\limits_0^t \frac{d\tau}{\tau^p} \int\limits_{R^{n-1}} \int\limits_{-\infty}^{0} G_q(x'-\xi', x_n-\xi_n, t-\tau) f_1(\xi',\xi_n,\tau)\, d\xi' d\xi_n, \quad D_n^+,
\end{aligned}
\tag{27}
$$

where $G_q(x'-\xi', x_n \pm \xi_n, t) = \frac{q^{\frac{n}{2}} e^{-\frac{-q|x'-\xi'|^2 + (x_n \pm \xi_n)^2}{4t^q}}}{\left(2\sqrt{\pi t^q}\right)^n}$. Thus, we have completely solved problem (1)-(5). It is easy to check that the obtained solutions (26)-(27) satisfy equations (1)-(2), initial conditions (3) and conjugation conditions (4)-(5). A similar estimate can be obtained for the function $G_q(x'-\xi', x_n-\xi_n, t)$:

$$
|D_x^k D_t^m G_q(x'-\xi', x_n-\xi_n, t)| \le \frac{Ce^{-\delta\frac{|x-\xi|^2}{t^q}}}{t^{\frac{q(n+k)}{2}+m}}. \tag{28}
$$

The solution to problem (1)–(5) and estimates (25) and (28) can later be used in the study of differential properties and obtaining a priori estimates of initial-boundary value problems in the Sobolev and Holder classes for non-stationary heat equations.

Let us consider the following potential of the initial condition:

$$
h_q(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} \frac{q^{\frac{n}{2}}}{\left(2\sqrt{\pi t^q}\right)^n} e^{-\frac{q|x-\xi|^2}{4t^q}} \varphi(\xi',\xi_n)\, d\xi' d\xi_n,
$$

$$
h_q(x,t) = \int\limits_{R^{n-1}} \int\limits_{0}^{+\infty} G_q(x-\xi,t)\varphi(\xi',\xi_n)\, d\xi' d\xi_n = \int\limits_{R^n} G_q(x-\xi,t)\varphi^*(\xi',\xi_n)\, d\xi' d\xi_n,
$$

here

$$\varphi^*(\xi',\xi_n)=\begin{cases}\varphi(\xi',\xi_n), & \xi_n>0,\\ 0, & \xi_n<0\end{cases},$$

$$h_q(x,t)=\int\limits_{R^n}G_q(x-\xi,t)\varphi^*(\xi',\xi_n)\,d\xi=\Big|x-\xi=y\Big|=\int\limits_{R^n}G_q(y,t)\varphi^*(x-y)\,dy,$$

$$D_th_q(x,t)=\int\limits_{R^n}D_tG_q(y,t)\varphi^*(x-y)\,dy.$$

As $D_tG_q(-y,t)=D_tG_q(y,t)$ is a even function, at the same time, for any $t>0$ $\int\limits_{R^n}D_tG_q(y,t)dy=0$. It can be written as follows: $D_th_q(x,t)=\frac{1}{2}\int\limits_{R^n}D_tG_q(y,t)\Big[\varphi^*(x-y)-2\varphi^*(x)+\varphi^*(x+y)\Big]\,dy$. Using Minkowski's inequality:

$$\left(\int\limits_{R^n}\Big|D_th_q(x,t)\Big|^s dx\right)^{\frac{1}{s}}=\frac{1}{2}\int\limits_{R^n}\Big|D_tG_q(y,t)\Big|\left(\int\limits_{R^n}\Big|\Big[\varphi^*(x-y)-2\varphi^*(x)+\varphi^*(x+y)\Big]\,dy\Big|^s dx\right)^{\frac{1}{s}},$$

$|D_tG_q(y,t)|\le\frac{Ce^{-\frac{|y|^2}{8t^q}}}{t^{\frac{qn}{2}+1}}$, taking into account the inequality, we obtain the following estimate:

$$\|D_th_q(x,t)\|_{s,R^n}\le\frac{C}{t^{\frac{qn}{2}+1}}\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\cdot N(y)dy,\tag{29}$$

where $N(y)=\|\varphi^*(x-y)-2\varphi^*(x)+\varphi^*(x+y)\|_{s,R^n}$. We write inequality (29) as follows:

$$\|D_th_q(x,t)\|_{s,R^n}\le\frac{C}{t^{\frac{qn}{2}+1}}\int\limits_{R^n}e^{-\frac{|y|^2}{8t^qs}}\cdot e^{-\frac{|y|^2}{8t^qs'}}\cdot N(y)dy,$$

where $\frac{1}{s}+\frac{1}{s'}=1$. Then using the Gelder inequality:

$$\|D_th_q(x,t)\|_{s,R^n}\le\frac{C}{t^{\frac{qn}{2}+1}}\left(\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\cdot N^s(y)\,dy\right)^{\frac{1}{s}}\left(\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\,dy\right)^{\frac{1}{s'}},$$

taking into account $\frac{1}{s'}=1-\frac{1}{s}$ we get

$$\left(\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\,dy\right)^{1-\frac{1}{s}}\le C_1t^{\frac{nq}{2}-\frac{nq}{2s}}.$$

So

$$\|D_th_q(x,t)\|_{s,R^n}\le\frac{C_1}{t^{1+\frac{nq}{2s}}}\left(\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\cdot N^s(y)\,dy\right)^{\frac{1}{s}}.$$

Now let us take a norm $\|D_th_q(x,t)\|_{s,D_{n+1}}$. Then from the last inequality we get:

$$\|D_th_q(x,t)\|_{s,D_{n+1}}\le C_1\left(\int\limits_0^{+\infty}\frac{dt}{t^{s+\frac{nq}{2}}}\int\limits_{R^n}e^{-\frac{|y|^2}{8t^q}}\cdot N^s(y)\,dy\right)^{\frac{1}{s}}=C_1\left(\int\limits_{R^n}N^s(y)\,dy\int\limits_0^{+\infty}\frac{e^{-\frac{|y|^2}{8t^q}}}{t^{s+\frac{nq}{2}}}\,dt\right)^{\frac{1}{s}},$$

if we introduce $\frac{|y|^2}{8t}=z$ a replacement:

$$\|D_t h_q(x,t)\|_{s,D_{n+1}} \le C_2\Big(\int\limits_{R^n}\frac{N^s(y)}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dy\Big)^{\frac{1}{s}} = C_2\cdot\Big(\int\limits_{R^n}\int\limits_{R^n}\frac{\Big|\varphi^*(x-y)-2\varphi^*(x)+\varphi^*(x+y)\Big|^s}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dxdy\Big)^{\frac{1}{s}},$$

$$\ll \varphi \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)} = \Big(\int\limits_{R^n}dx\int\limits_{R^n}\frac{\Big|\varphi(x-y)-2\varphi(x)+\varphi(x+y)\Big|^s}{|y|^{\frac{2s}{q}+n-\frac{2}{q}}}\,dy\Big)^{\frac{1}{s}}. \quad (30)$$

Given that (30), then

$$\|D_t h_q(x,t)\|_{s,D_{n+1}} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)}.$$

As estimates $D_x^s h_q(-x,t) = D_x^s h_q(x,t)$ and $D_x^s G_q(x,t)$ are consistent with estimate $D_t G_q(x,t)$, the estimate $\|D_x^2 h_q(x,t)\|_{s,D_{n+1}}$ is also taken similarly. Therefore, the following inequality is obtained:

$$\|D_x^2 h_q(x,t)\|_{s,D_{n+1}} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)}.$$

Theorem 1. The potential of the initial condition satisfies the estimate:

$$\ll h_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} \le C \ll \varphi^* \gg_{W_s^{\frac{2}{q}-\frac{2}{qs}}(R^n)},$$

where

$$\ll h_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} = \|\frac{\partial h_q}{\partial t}\|_{s,D_{n+1}} + \sum_{k,j=1}^{n}\|\frac{\partial^2 h_q}{\partial x_k \partial x_j}\|_{s,D_{n+1}}.$$

This notation $\ll . \gg$ means the main part of the norm in the Sobolev classes.

Consider the following volume potential:

$$g_q(x,t) = \int\limits_0^t \frac{d\tau}{\tau^p}\int\limits_{R^{n-1}}\int\limits_0^{+\infty} G_q(x-\xi, t-\tau) f(\xi', \xi_n, \tau)\, d\xi' d\xi_n.$$

Using the method [15], the following theorem can be proved.

Theorem 2. The following estimates are appropriate for the volume potential:

$$\ll g_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} \le C\|f\|_{W_s^{2,1}(D_{n+1})}, \quad (1<q<\infty),$$

where

$$\ll g_q(x,t) \gg_{W_s^{2,1}(D_{n+1})} = \|\frac{\partial g_q}{\partial t}\|_{s,D_{n+1}} + \sum_{k,j=1}^{n}\|\frac{\partial^2 g_q}{\partial x_k \partial x_j}\|_{s,D_{n+1}}.$$

This notation $\ll . \gg$ means the main part of the norm in the Sobolev classes.

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У.К. Койлышов[1,2], К.А. Бейсенбаева[3], С.Д. Жаппарова[1]

[1]*Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан;*
[2]*ҚР ҒБЖ БжҒМ ҒК Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[3]*Логистика және көлік академиясы, Алматы, Қазақстан*

Коэффициенті үзілісті жылуөткізгіштік теңдеу үшін Коши есебі шешімінің соболев класындағы априорлық бағасы

Коэффициенттері үзілісті параболалық типті дербес туындылы дифференциалдық теңдеулер және уақыт бойынша өзгешеленген жылуөткізгіштік теңдеулердің әрқайсысы жеке-жеке көптеген авторлармен жақсы зерттелген. Коэффициенті үзілісті уақыт бойынша өзгешеленген параболалық типті теңдеулер үшін түйіндес есептер іс жүзінде зерттелмеген. Мақалада n-өлшемді кеңістікте бастапқы уақыт мезетіндегі коэффициенттері үзілісті өзгешеленген жылуөткізгіштік теңдеу үшін бір түйіндес есеп қарастырылған. Қойылған есептің іргелі шешімі табылды және оның туындыларының бағасы алынды. Алынған нәтижені қолдана отырып, берілген есептің шешімінің соболев класындағы бағасы табылды.

Кілт сөздер: түйіндес есеп, жылуөткізгіштік теңдеу, өзгешеленген теңдеу, үзілісті коэффициенттер.

У.К. Койлышов[1,2], К.А. Бейсенбаева[3], С.Д. Жаппарова[1]

[1]*Казахский национальный университет имени Аль-Фараби, Алматы, Казахстан;*
[2]*Институт математики и математического моделирования КН МОН РК, Алматы, Казахстан;*
[3]*Академия логистики и транспорта, Алматы, Казахстан*

Априорная оценка решения задачи Коши для вырождающегося уравнения теплопроводности с разрывными коэффициентами в соболевских классах

Дифференциальные уравнения в частных производных параболического типа с разрывными коэффициентами и вырождающиеся по времени уравнения теплопроводности отдельности хорошо изучены многими авторами. Задачи сопряжения для вырождающегося по времени уравнениям параболического типа с разрывными коэффициентами практически не изучены. В статье рассмотрена одна задача сопряжения для уравнения теплопроводности с разрывными коэффициентами, вырождающегося в начальный момент времени в n-мерном пространстве. Построено фундаментальное решение поставленной задачи, и найдена оценка ее производных. С помощью этих оценок получена оценка решения поставленной задачи в соболевских классах.

Ключевые слова: задача сопряжения, уравнения теплопроводности, вырождающиеся уравнения, разрывные коэффициенты.

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