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V. Ala*

Mersin University, Science and Letters Faculty, Mathematics Department, Mersin, Turkey
(E-mail: volkanala@mersin.edu.tr)

New exact solutions of space-time fractional Schrödinger-Hirota equation

In this study, improved Bernoulli sub-equation function method (IBSEFM) is presented to construct the exact solutions of the nonlinear conformable fractional Schrödinger-Hirota equation (FSHE). By using the traveling wave transformation FSHE turns into the ordinary differential equation (ODE) and by the aid of symbolic calculation software, new exact solutions are obtained. 2D, 3D figures and contour surfaces acquired from the values of the solutions are plotted. The results show that the proposed method is powerful, effective and straightforward for formulating new solutions to various types of nonlinear fractional partial differential equations in applied sciences.

Keywords: conformable fractional derivative, Schrödinger-Hirota equation, improved Bernoulli sub-equation function method (IBSEFM).

1 Introduction

Fractional differential equations are the generalization of classical differential equations with integer order. In recent years, fractional differential equations become the field of scientists to investigate the expediency of non-integer order derivatives in different areas of physics and mathematics. These equations have become a useful tool for describing numerous nonlinear phenomena of physics such as heat conduction systems, nonlinear chaotic systems, viscoelasticity, plasma waves, acustic gravity waves, diffusion processes [1–3]. Many numerical and analytical methods have been developed and successfully employed to solve these equations such as modified Kudryashov method [4], homotopy perturbation method [5], new extended direct algebraic method [6], fractional Riccati expansion method [7], modified extended tanh method [8].

During the last few years, a straightforward definition of conformable derivative has been given [9]. The conformable derivative operator which is compatible to many real-world problems provides some properties of classical calculus: derivative of the quotient of two functions, the chain rule, the product of two functions [10]. In addition, many techniques have been applied to find exact solutions for conformable nonlinear partial differential equations [11–16].

In this study, FSHE is considered as follows:

$$iq_t^{(\mu)} + \frac{1}{2}q_{xx} + |q|^2 q + i\lambda q_{xxx} = 0,\ t \geq 0,\ 0 < \mu \leq 1, i = \sqrt{-1}, \tag{1}$$

where λ is a nonlinear dispersion term, q is the function of the independent variables of x and t. The operator $q_t^{(\mu)}$ represents a conformal derivative operator defined only for a positive domain of t [10]. Before beginning the solution procedure, let us give some properties of the conformable derivative:

The conformable derivative of order α with respect to the independent variable t is defined as [9]:

$$D_t^a(y(t)) = \lim_{\tau \to 0} \frac{y(t + \tau t^{1-\alpha}) - y(t)}{\tau},\ t > 0,\ \alpha \in (0,1]\,,$$

for a function $y = y(t) : [0, \infty) \to \mathbb{R}$.

*Corresponding author.
E-mail: volkanala@mersin.edu.tr

Theorem 1. Assume that the order of the derivative $\alpha \in (0,1]$ and suppose that $u = u(t)$ and $v = v(t)$ are $\alpha-$ differentiable for all positive t. Then,

i. $D_t^a(c_1u + c_2v) = c_1D_t^a(u) + c_2D_t^a(v)$, for $\forall\ c_1, c_2 \in \mathbb{R}$.

ii. $D_t^a(t^k) = kt^{k-a}, \forall\ k \in \mathbb{R}$.

iii. $D_t^a(\lambda) = 0$, for all constant function $u(t) = \lambda$.

iv. $D_t^a\,(uv) = uD_t^a(v) + vD_t^a(u)$.

v. $D_t^a\left(\frac{u}{v}\right) = \frac{vD_t^a(u) - uD_t^a(v)}{v^2}$.

vi. $D_t^a\,(u)\,(t) = t^{1-\alpha}\frac{du}{dt}$.

The conformable differential operator satisfies some critical fundamental properties like the chain rule, Taylor series expansion, and Laplace transform.

Theorem 2. Let $u = u(t)$ be an $\alpha-$ conformable differentiable function and assume that v is a differentiable function. Then,

$$D_t^a(u \circ v)(t) = t^{1-\alpha}v'(t)u'(v(t)).$$

The proofs of these properties are given in [17] and [9] respectively.

The rest of the paper is organized as follows: in the second section, descripton of the IBSEFM is given; in the third section, the application of IBSEFM is mentioned; in the last section, this study provides conclusions.

2 Description of the IBSEFM

In this section, we give the fundamental properties of the IBSEFM. This method is direct, significant, advanced algebraic method for establishing reliable exact solutions for both nonlinear and nonlinear fractional partial differential equations [11, 12, 18–21]. We present five main steps of the IBSEFM as follows:

Step 1: Let us take account of the following conformable partial differential equation of the style

$$P(v, D_t^{(\alpha)}v, D_x^{(\alpha)}v, D_{xt}^{(2\alpha)}v, ...) = 0, \tag{2}$$

where $D_t^{(\alpha)}$ is the conformable fractional derivate operator, $v(x,t)$ is an unknown function, P is a polynomial and its partial derivatives contain fractional derivatives. The aim is to convert conformable nonlinear partial differential equation with a suitable fractional transformation into the ordinary differantial equation. The wave transformation as

$$v(x,t) = V\,(\xi)\,, \quad \xi = \xi\,(x, t^\alpha)\,. \tag{3}$$

Using (3) and the properties of conformable fractional derivate, it enables us to convert (2) into an ODE in the form

$$N(V, V', V'', ...) = 0. \tag{4}$$

If we integrate (4) term to term, we obtain integration constants which can be determined later.

Step 2: Hypothesize the solution of (4) can be presented as follows:

$$V(\xi) = \frac{\sum\limits_{i=0}^{n} a_iF^i(\xi)}{\sum\limits_{j=0}^{m} b_jF^j(\xi)} = \frac{a_0 + a_1F(\xi) + a_2F^2(\xi) + ...a_nF^n(\xi)}{b_0 + b_1F(\xi) + b_2F^2(\xi) + ...b_mF^m(\xi)}, \tag{5}$$

where $a_0, a_1, ..., a_n$ and $b_0, b_1, ..., b_m$ are chosen arbitrary constants of the balance principle and the form of Bernoulli differential equation is as follows:

$$F'(\xi) = \sigma F(\xi) + dF^M(\xi), \; d \neq 0, \sigma \neq 0, \; M \in \mathbb{R}/\left\{0,1,2\right\}, \tag{6}$$

where $F(\xi)$ is a polynomial.

Step 3: The positive integers m, n, M are found by balance principle that is both nonlinear term and the highest order derivative term of (4).

Substituting (5) and (6) into (2) it gives us an equation of polynomial $\Theta(F)$ of F as follows;

$$\Theta(F(\xi)) = \rho_s F(\xi)^s + ... + \rho_1 F(\xi) + \rho_0 = 0,$$

where $\rho_i, i = 0, ..., s$ are to be determined later.

Step 4: Equating all the coefficients of $\Theta(F(\xi))$ which yields us an algebraic equation system;

$$\rho_i = 0, i = 0, .., s.$$

Step 5: When we solve (4), we get the following two cases with respect to σ and d,

$$F(\xi) = \left[\frac{-de^{\sigma(\epsilon-1)} + \epsilon\sigma}{\sigma e^{\sigma(\epsilon-1)\xi}}\right]^{\frac{1}{1-\epsilon}}, \quad d \neq \sigma, \tag{7}$$

$$F(\xi) = \left[\frac{(\epsilon-1) + (\epsilon+1)\tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}{1 - \tanh\left(\sigma(1-\epsilon)\frac{\xi}{2}\right)}\right], d = \sigma, \; \epsilon \in \mathbb{R}.$$

Using a complete discrimination system of $F(\xi)$, we obtain the analytical solutions of (4) via Wolfram Mathematica and categorize the exact solutions of (4). To achieve better results, we can plot two and three-dimensional figures of analytical solutions by considering proper values of parameters.

3 Application of the IBSEFM

In this section, we will applicate the IBSEFM to obtain the exact solutions to space-time fractional Schrödinger-Hirota equation. Let us consider the following wave transform:

$$q(x,t) = U(\xi)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right), \quad \xi = x - 2\omega\frac{t^\mu}{\mu}, \tag{8}$$

where the coefficient η and ω are constants that represent soliton frequency and soliton wave number respectively. Introducing (8) we get

$$q_t^{(\mu)} = \left(-2\omega U' + i\eta U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right), \tag{9}$$

$$q_{xx} = \left(U'' + 2i\omega U' - \omega^2 U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right), \tag{10}$$

$$q_{xxx} = \left(U''' + 3i\omega U'' - 3\omega^2 U' - i\omega^3 U\right)\exp\left(i\left(\omega x + \eta\frac{t^\mu}{\mu}\right)\right). \tag{11}$$

Substituting (9)–(11) into (1) and detaching the real and imaginary parts yield $\omega = -\frac{1}{3\lambda}$ and $U(\xi)$ satisfy the following ordinary differential equation:

$$-\left(\frac{5}{54\lambda^2}+\eta\right)U+\frac{3}{2}U''+U^3=0, \tag{12}$$

where $U''=\frac{d^2U}{d\xi^2}$. When we reconsider (12) for balance principle, considering among U'' and U^3, the relationship as follow:

$$M=n-m+1. \tag{13}$$

(13) shows us the different cases of the solutions of (12) and some analytical solutions can be constructed. According to the balance, we consider $M=3, m=1, n=3$ for (12) and (13), the following equations hold:

$$U(\xi)=\frac{a_0+a_1F(\xi)+a_2F^2(\xi)+a_3F^3(\xi)}{b_0+b_1F(\xi)}\equiv\frac{\Upsilon(\xi)}{\Psi(\xi)}, \tag{14}$$

$$U'(\xi)\equiv\frac{\Upsilon'(\xi)\Psi(\xi)-\Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)}, \tag{15}$$

and

$$U''(\xi)\equiv\frac{\Upsilon'(\xi)\Psi(\xi)-\Upsilon(\xi)\Psi'(\xi)}{\Psi^2(\xi)}-\frac{[\Upsilon(\xi)\Psi'(\xi)]'\,\Psi^2(\xi)-2\Upsilon(\xi)[\Psi'(\xi)]^2\Psi(\xi)}{\Psi^4(\xi)}, \tag{16}$$

where $F'=\sigma F+dF^3$, $a_3\neq 0$, $b_1\neq 0$, $\sigma\neq 0$, $d\neq 0$. Using (14)-(16) in (13), obtained from coefficients of polynomial of F we get:

$F^0: a_0^3-\eta a_0b_0^2-\frac{5a_0b_0^2}{54\lambda^2}=0,$

$F: 3a_0^2a_1-\eta a_1b_0^2-\frac{5a_1b_0^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_1b_0^2-2\eta a_0b_0b_1-\frac{5a_0b_0b_1}{27\lambda^2}-\frac{3}{2}\sigma^2a_0b_0b_1=0,$

$F^2: 3a_0a_1^2+3a_0^2a_2-\eta a_2b_0^2-\frac{5a_2b_0^2}{54\lambda^2}+6\sigma^2a_2b_0^2-2\eta a_1b_0b_1-\frac{5a_1b_0b_1}{27\lambda^2}-\frac{3}{2}\sigma^2a_1b_0b_1-\eta a_0b_1^2-\frac{5a_0b_1^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_0b_1^2=0,$

$F^3: a_1^3+6a_0a_1a_2+3a_0^2a_3+6d\sigma a_1b_0^2-\eta a_3b_0^2-\frac{5a_3b_0^2}{54\lambda^2}+\frac{27}{2}\sigma^2a_3b_0^2-6d\sigma a_0b_0b_1-2\eta a_2b_0b_1-\frac{5a_2b_0b_1}{27\lambda^2}+\frac{9}{2}\sigma^2a_2b_0b_1-\eta a_1b_1^2-\frac{5a_1b_1^2}{54\lambda^2}=0,$

$F^4: 3a_1^2a_2+3a_0a_2^2+6a_0a_1a_3+18d\sigma a_2b_0^2-2\eta a_3b_0b_1-\frac{5a_3b_0b_1}{27\lambda^2}+\frac{33}{2}\sigma^2a_3b_0b_1-\eta a_2b_1^2-\frac{5a_2b_1^2}{54\lambda^2}+\frac{3}{2}\sigma^2a_2b_1^2=0,$

$F^5: 3a_1a_2^2+3a_1^2a_3+6a_0a_2a_3+\frac{9}{2}d^2a_1b_0^2+36d\sigma a_3b_0^2-\frac{9}{2}d^2a_0b_0b_1+18d\sigma a_2b_0b_1-\eta a_3b_1^2-\frac{5a_3b_1^2}{54\lambda^2}+6\sigma^2a_3b_1^2=0,$

$F^6: a_2^3+6a_1a_2a_3+3a_0a_3^2+12d^2a_2b_0^2+\frac{3}{2}d^2a_1b_0b_1+48d\sigma a_3b_0b_1-\frac{3}{2}d^2a_0b_1^2+6d\sigma a_2b_1^2=0,$

$F^7: 3a_2a_3^2+\frac{63}{2}d^2a_3b_0b_1+\frac{9}{2}d^2a_2b_1^2=0,$

$F^8: a_3^3+12d^2a_3b_1^2=0,$

$F^9: 3a_2^2a_3+3a_1a_3^2+\frac{45}{2}d^2a_3b_0^2+\frac{27}{2}d^2a_2b_0b_1+18d\sigma a_3b_1^2=0.$

When we solve above the system of the equations of F using Wolfram Mathematica, the coefficients are obtained as:

Case 1. For $\sigma\neq d$,

$$a_0=\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}\,b_0}{3\lambda};\;\; a_1=\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}\,b_1}{3\lambda}; a_2=2i\sqrt{3}db_0;\;\; a_3=2i\sqrt{3}db_1;\;\; \sigma=-\frac{i\sqrt{\frac{-5}{2}-27\eta\lambda^2}}{9\lambda}.$$

Substituting these coefficients along with (7) in (14), we obtain the following solution to (1) as follows:

$$q_1(x,t)=\left(\frac{i\sqrt{\frac{-5}{6}-9\eta\lambda^2}}{3\lambda}-\frac{2i\sqrt{3}d}{e^{\frac{2\sqrt{\frac{-5}{2}-27\eta\lambda^2}\left(x-2\omega\frac{t^\mu}{\mu}\right)}{9\lambda}}\epsilon+\frac{9d\lambda}{\sqrt{\frac{-5}{2}-27\eta\lambda^2}}}\right)e^{i\left(\omega x+\eta\frac{t^\mu}{\mu}\right)}.$$

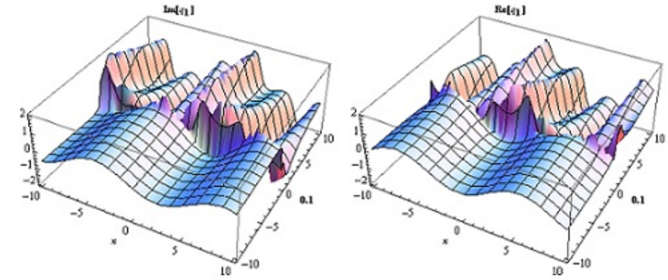

Figure 1. 3D- plots of $q_1(x,t)$ for the values $d=0.4$; $\omega=0.5$; $\mu=0.3$; $\epsilon=0.2$; $\lambda=0.3$; $\eta=0.1$; $\lambda=0.3$; $t=0.1$; $-10<x<10$, $-10<t<10$.

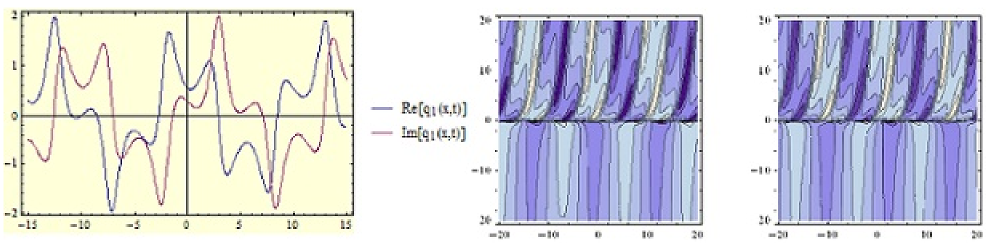

Figure 2. 2D- plots and contour surfaces of $q_1(x,t)$.

Case 2. For $\sigma\neq d$, $a_0=i\sqrt{3}\;\sigma b_0$; $a_1=i\sqrt{3}\;\sigma b_1$; $a_2=2i\sqrt{3}\;db_0$; $a_3=2i\sqrt{3}\;db_1$; $\lambda=-\frac{i\sqrt{\frac{5}{6}}}{3\sqrt{-\eta-3\sigma^2}}$.

Substituting these coefficients along with (7) in (14), we obtain the following solution to (1) as follows:

$$q_2(x,t)=i\sqrt{3}\left(\frac{2d}{e^{-2x\sigma+\frac{4t^\mu\sigma\omega}{\mu}}\epsilon-\frac{d}{\sigma}}+\sigma\right)e^{i\left(\omega x+\eta\frac{t^\mu}{\mu}\right)}.$$

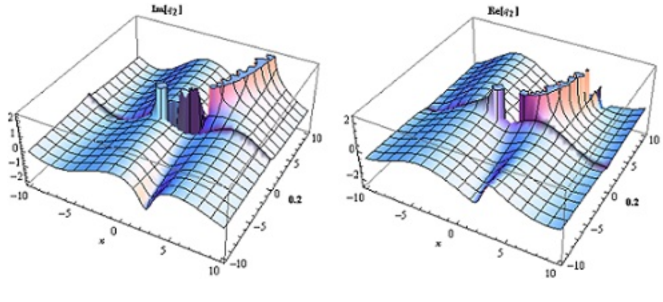

Figure 3. 3D- plots of $q_2(x,t)$ for the values $d=0.3$; $\sigma=0.5$; $\mu=0.3$; $\epsilon=0.2$; $\omega=0.5$; $\eta=0.1$; $t=0.2$;$-10<x<10$, $-10<t<10$.

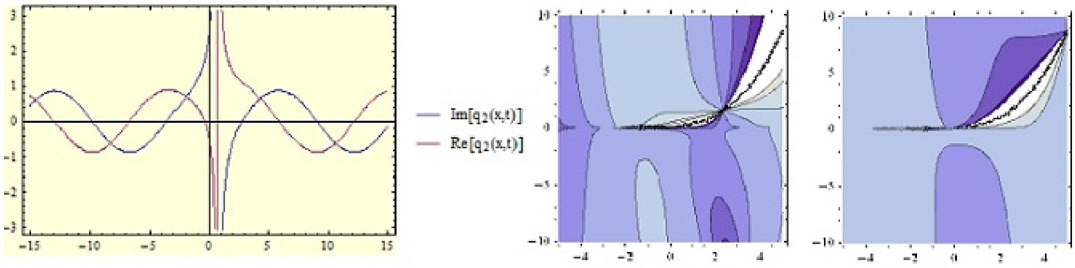

Figure 4. 2D- plots and contour surfaces of $q_2(x,t)$.

4 Conclusions

In this paper, the IBSEFM is applied for FSHE. Using a wave transformation, FSHE has been converted into the ODE which can be solved according to the IBSEFM. By means of this method, exact solutions are obtained. The contourplot surfaces, 3D and 2D figures (Figures 1–4) of all solutions obtained by IBSEFM under the suitable values of parameters are plotted by showing the main characteristic physical properties of the solutions with the help of symbolic software. According to the results, the formats of traveling wave solutions in two and three-dimensional surfaces are similar to the physical meaning of results.

The solutions are solitary wave solutions. It is also clear that the more steps are developed and the better approximations are obtained. The conclusions show that the IBSEFM is simple, effective, and powerful. Thus, in mathematical physics, it is applicable to solve other conformable partial differential equations. In summary, the improved Bernoulli Sub-equation funtion method is influential and suitable for solving other types of nonlinear differential equations in which the balance principle is satisfied.

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В. Ала

Мерсин университеті, ғылым және әдебиет факультеті, математика факультеті, Мерсин, Түркия

Кеңістік пен уақыттан тәуелді бөлшек Шредингер-Хирота теңдеуінің жаңа нақты шешімдері

Мақалада сызықты емес бөлшек тәріздес Шредингер-Хирота теңдеуінің (FSHE) дәл шешімдерін құру үшін жақсартылған Бернулли қосалқы теңдеуі функциясының әдісі (IBSEFM) ұсынылған. Жылжымалы толқын түрлендіруінің көмегімен FSHE кәдімгі дифференциалдық теңдеуге (ODE) түрлендіріледі және символдық есептеуіш бағдарламалық қамтамасыз етудің көмегімен жаңа нақты шешімдер алынады. 2D, 3D фигуралары мен шешімдердің мәндерінен алынған контур беттер салынған.

Нәтижелер көрсеткендей, ұсынылып отырған әдіс қолданбалы ғылымдардағы әртүрлі типті сызықты емес бөлшек дербес туындылы дифференциалдық теңдеулердің жаңа шешімдерін жасау үшін қуатты, тиімді және қарапайым әдіс.

Кілт сөздер: бөлшек тәріздес туынды, Шредингер-Хирота теңдеуі, жақсартылған Бернулли қосалқы теңдеуінің функциясы әдісі (IBSEFM).

В. Ала

Университет Мерсин, Мерсин, Турция

Новые точные решения пространственно-временного дробного уравнения Шрёдингера-Хироты

В статье представлен усовершенствованный метод функций подуравнений Бернулли для построения точных решений нелинейного дробно-подобного уравнения Шрёдингера-Хироты (FSHE). С помощью преобразования бегущей волны FSHE превращается в обыкновенное дифференциальное уравнение, а с использованием программного обеспечения для символьных вычислений получаются новые точные решения. Строятся 2D, 3D фигуры и контурные поверхности, полученные из значений решений. Результаты показывают, что предложенный метод является мощным, эффективным и простым способом для разработки новых решений различных типов нелинейных дробных дифференциальных уравнений в частных производных в прикладных науках.

Ключевые слова: дробно-подобная производная, уравнение Шрёдингера-Хироты, усовершенствованный метод функций подуравнений Бернулли.