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D. Dauitbek[1,2], Y. Nessipbayev[1,2,*], K. Tulenov[1,2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan*
(E-mail: dauitbek@math.kz, yerlan.nessipbayev@gmail.com, tulenov@math.kz)

Examples of weakly compact sets in Orlicz spaces

This paper provides a number of examples of relatively weakly compact sets in Orlicz spaces. We show some results arising from these examples. Particularly, we provide a criterion which ensures that some Orlicz function is increasing more rapidly than another (in a sense of T. Ando). In addition, we point out that if a bounded subset K of the Orlicz space L_Φ is not bounded by the modular $\boldsymbol{\Phi}$, then it is possible for a set K to remain unbounded under any modular Ψ increasing more rapidly than Φ.

Keywords: conjugate (complementary) functions, relative weak compactness, Orlicz spaces, $N-$functions.

Introduction

We provide a number of examples of relatively weakly compact sets in Orlicz spaces based mainly on criteria obtained by a classical work of T. Ando from 1962 (see [1]). It should be noted that there is a shortage of such examples in the literature. Some (maybe the most important) examples may be found in the classical book by M.M. Rao and Z.D. Ren [2]. Another paper, devoted to the study of weak compactness in Orlicz spaces that we use extensively in this paper is by J. Alexopoulos [3].

On the contrary, weak compactness criteria in both Orlicz function and sequence spaces have been stidied by many researchers, see, for example [1–14], and references therein.

In particular, T. Ando see [1] obtained weak compactness criteria in Orlicz (function) spaces from the perspective of Köthe duality. The study results of T. Ando were extended (with some restrictive condition) from the setting of finite measure spaces to the setting of σ-finite measure spaces in the work of M. Nowak in 1986 [11]. The objective of this paper is to study such criteria and provide examples that satisfy these criteria. We also prove some related propositions (see Propositions 2.13 and 2.17).

1 *Preliminaries*

Initially, the study provides the definition of an N-function (as in [1]), which will be used throughout the text.

Definition 1.1. A convex function $\Phi : [0,\infty) \to [0,\infty)$ is called an N-function if
(i) $\Phi(0) = 0$,
(ii) $\frac{\Phi(\lambda)}{\lambda} \to \infty$ as $\lambda \to \infty$.

We note that in the above definition by T. Ando, it is not necessarily true that $\frac{\Phi(\lambda)}{\lambda} \to 0$ as $\lambda \to 0+$, unlike in many other classical works (e.g., [15, formulae 1.12 and 1.15], [2; 13], [3, Proposition 1.1]).

The following two definitions specify some important classes of N-functions.

Definition 1.2. ([2, Definition 1] and [3, Definition 1.5]) An $N-$function Φ is said to satisfy the $\triangle_2$ condition ($\Phi \in \triangle_2$) if $\limsup_{x\to\infty} \frac{\Phi(2x)}{\Phi(x)} < \infty$. That is, there is a $K > 0$ so that $\Phi(2x) \le K \cdot \Phi(x)$ for large values of x.

Definition 1.3. ([2, Definition 2] and [3, Definition 1.8]) An $N-$function Φ is said to satisfy the ∇_2 condition ($\Phi \in \nabla_2$) if there is a $K > 0$ so that $(\Phi(x))^2 \le \Phi(Kx)$ for large values of x.

[*]Corresponding author.
E-mail: yerlan.nessipbayev@gmail.com

1.1 Decreasing rearrangement

Let (I, m) denote the measure space, where $I = (0, \infty)$ (resp. $(0, 1)$), equipped with Lebesgue measure m. Let $L(I, m)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure m. Define $S(I, m)$ to be the subset of $L(I, m)$, which consists of all functions f such that $m(\{t : |f(t)| > s\}) < \infty$ for some $s > 0$. Note that if $I = (0, 1)$, then $S(I, m) = L(I, m)$.

For $f \in S(I, m)$, we denote by $\mu(f)$ the decreasing rearrangement of the function $|f|$. That is,

$$\mu(t, f) = \inf\{s \geq 0 : \ m(\{|f| > s\}) \leq t\}, \quad t > 0.$$

1.2 Orlicz spaces

Definition 1.4. A function $G : [0, \infty) \to [0, \infty]$ is said to be an Orlicz function if [9; 258]
(i) $G(0) = 0$,
(ii) G is not identically equal to zero,
(iii) G is convex,
(iv) G is continuous at zero.

It follows from the definitions that not every $N-$function is an Orlicz function (e.g., an N-function may be discontinuous at zero). The converse also does not hold. For example, the function $G(t) = t$ is an Orlicz function but not an $N-$function. For an Orlicz function (or N-function) G we shall consider an (extended) real-valued function $\mathbf{G}(f)$ (also called the modular defined by an N-function G) defined on the class of all measurable functions f on I, by

$$\mathbf{G}(f) = \int_I G(|f(t)|)dt.$$

The set

$$L_G = \{f \in S(I, m) : \quad \|f\|_{L_G} < \infty\},$$

where

$$\|f\|_{L_G} = \inf\left\{c > 0 : \quad \int_I G\left(\frac{|f|}{c}\right) dm \leq 1\right\}$$

is called an Orlicz space defined by the Orlicz function (or N-function) G (equipped with Orlicz norm).

In fact, we have the following ([2, Chapter 3.5, Theorem 1]):

Proposition 1.5. If an N-function $\Phi \in \triangle_2$, then L_Φ is separable (provided the measure space is separable).

It should be stated that notions of N-functions and Orlicz functions used interchangeably in many situations. However, in this text we will denote N-functions by Greek letters Φ, Ψ and Orlicz functions by Latin letters G, F to distinguish between them.

Using various (partial) order relations on Orlicz functions one may define the corresponding relations in the Orlicz spaces. We also note that since (in this paper) Orlicz (function) spaces are defined on finite measure spaces we only need local order relations. However, some results will be also stated for Orlicz spaces on positive half-line (with small differences on local relations).

We define the notion of majorization for Orlicz functions (for σ-finite measure spaces). Let G_1 and G_2 be two Orlicz functions.

Definition 1.6. (e.g., [16, Definition 16.1.1]) We say that
(1) G_1 majorises G_2 at 0 ($G_1 \succ_0 G_2$) if there exist positive numbers a, b, x_0 such that

$$G_2(x) \leq bG_1(ax) \quad \text{for all} \quad 0 \leq x \leq x_0.$$

(2) G_1 majorises G_2 at ∞ ($G_1 \succ_\infty G_2$) if there exist positive numbers a, b, x_0 such that

$$G_2(x) \leq bG_1(ax) \quad \text{for all} \quad x \geq x_0.$$

(3) G_1 majorises G_2 ($G_1 \succ G_2$) if $G_1 \succ_0 G_2$ and $G_1 \succ_\infty G_2$.

Moreover, one can set $b = 1$ in the above definition (see [16, Proposition 16.1.2]). Also, the condition $G_1 \succ G_2$ may be checked via the following (see [16, Proposition 16.1.3]):

Proposition 1.7. $G_1 \succ G_2$ if and only if

$$G_2(x) \leq bG_1(ax), \quad x \geq 0$$

for some $b > 0$ and $a > 0$.

Also, we provide a definition of equivalent Orlicz functions on $\sigma-$finite measure space (see [16, Definition 16.3.1]):

Definition 1.8. Two Orlicz functions G_1 and G_2 are called equivalent, denoted $G_1 \approx G_2$, if $G_1 \succ G_2$ and $G_2 \succ G_1$.

The following definition for equivalence of N-functions on finite measure space may be found in [3, Definition 1.3]:

Definition 1.9. For $N-$functions Φ_1, Φ_2 we write $\Phi_1 \prec \Phi_2$ if there is a $K > 0$ so that $\Phi_1(x) \leq \Phi_2(Kx)$ for large values of x. If $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$ then we say that Φ_1 and Φ_2 are equivalent.

Note for finite measure space, the notion of majorisation is slightly different as we do not care about majorisation at zero.

We will denote by Ψ the function complementary (or conjugate) to an $N-$function Φ in the sense of Young (with the condition $\frac{\Phi(t)}{t} \to 0+$ as $t \to 0$), defined by ([15; 11])

$$\Psi(t) = \sup\{s|t| - \Phi(s) : \quad s \geq 0\}.$$

We notice that Ψ is again an $N-$function (see [9; 258]).

2 Weakly compact sets in Orlicz spaces

In this section, we recall known criteria of relative weak compactness in Orlicz spaces and provide examples of such sets. We will also state some concluding remarks and prove related propositions.

The following theorem was proved by T. Ando in [1, Theorem 1].

Theorem 2.1. Let Φ be an N-function and let (Ω, Σ, μ) be a finite measure space. A subset K of L_Φ is relatively $\sigma(L_\Phi, L_\Psi)-$compact if and only if

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} \to \alpha \int_\Omega |f(t)| d\mu \quad \text{as} \quad \lambda \downarrow 0$$

uniformly with respect to $f(t) \in K$, where $\alpha = \lim_{\lambda \to 0+} \Phi(\lambda)/\lambda$.

It is worthwhile to note that Theorem 2.1 (unlike many others) is valid for any $N-$function (in the sense of Ando), that is, α is not necessarily zero by definition. We note that the extension of this Theorem to the σ-finite case was given by M. Nowak [14, Theorem 1.1] only in the case $\alpha = 0$.

Below we provide examples of relatively weakly compact sets in $L_\Phi[0,1]$ by using Theorem 2.1. Later we will provide another criteria of weak compactness in Orlicz spaces and apply these criteria to the following examples.

Example 2.2. (i) Let $\Phi(x) = e^x - 1$. Then the subset $K = \{f_p(x) = x^p : \quad p \geq 1\}$ of $L_\Phi[0,1]$ is relatively weakly compact. Indeed, K is bounded and since $\alpha = \lim_{\lambda\to 0} \frac{e^\lambda - 1}{\lambda} = 1$, it is enough to show (by Theorem 2.1) that

$$\frac{\int_0^1 e^{\lambda t^p} dt - 1}{\lambda} \to \int_0^1 t^p dt = \frac{1}{p+1}$$

uniformly with respect to $p \geq 1$ as $\lambda \downarrow 0$.

Applying the L'Hopital's rule, we get

$$0 \leq \lim_{\lambda\downarrow 0} \frac{\int_0^1 \left(e^{\lambda t^p} - \lambda t^p - 1\right) dt}{\lambda} = \lim_{\lambda\downarrow 0} \frac{\int_0^1 \left(t^p e^{\lambda t^p} - t^p\right) dt}{1} =$$

$$\leq \lim_{\lambda\downarrow 0} \int_0^1 \left(e^{\lambda t^p} - 1\right) dt \leq \lim_{\lambda\downarrow 0} \int_0^1 \left(e^{\lambda t} - 1\right) dt = \frac{e^\lambda - 1}{\lambda} - 1 \to 0$$

as $\lambda \to 0$ uniformly with respect to $f \in K$ (independent of p).

For example, when $p = 1$, then

$$\frac{\int_0^1 e^{\lambda t} dt - 1}{\lambda} = \frac{e^\lambda - 1 - \lambda}{\lambda^2} \to \frac{1}{2}$$

as $\lambda \downarrow 0$ as desired.

However, when $p > 1$, the integral $\int_0^1 e^{\lambda t^p} dt$ is not expressed in terms of elementary functions.

(ii) Let $\Phi(x) = e^x - x - 1$ and the subset K as in (i). Note in this case $\alpha = 0$ (so Φ is an $N-$function). Obviously

$$\frac{\int_0^1 \left(e^{\lambda t^p} - \lambda t^p - 1\right) dt}{\lambda} \to 0$$

uniformly with respect to p as $\lambda \downarrow 0$ as it is reduced to case (i). For example, when $p = 1$

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \frac{\int_0^1 (e^{\lambda t} - \lambda \cdot t - 1) dt}{\lambda} = \frac{2e^\lambda - 2 - \lambda^2 - 2\lambda}{2\lambda^2} \to 0$$

as $\lambda \downarrow 0$. As in (i), The uniform convergence holds for every $p > 1$, however, as in example (i), the integral is not expressed in terms of elementary functions either.

We note that in the statement of Theorem 2.1 the uniform convergence is crucial as the following example illustrates, i.e., pointwise convergent is not sufficient.

Example 2.3. Let $\Phi(x) = x \cdot \ln(x+1)$ and $K = \{f_p(x) = e^{px} : \quad p > 0\}$ be a subset of $L_\Phi[0,1]$. Note that Φ is an $N-$function with $\alpha = 0$. Now we check the condition of uniform convergence as in Theorem 2.1:

$$\frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \frac{\int_0^1 \lambda e^{px} \cdot \ln(\lambda e^{px} + 1) dx}{\lambda} =$$

$$= \frac{\lambda e^p \cdot \ln(\lambda e^p + 1) - \lambda \cdot \ln(\lambda + 1) - \lambda e^p + \lambda + \ln(\lambda e^p + 1) - \ln(\lambda + 1)}{p \cdot \lambda}$$

$$\approx \frac{(\lambda e^p)^2 - \lambda^2}{p \cdot \lambda} = \frac{\lambda(e^{2p} - 1)}{p} \to 0$$

as $\lambda \downarrow 0$. However, it is easy to see that the convergence is not uniform with respect to p since

$$\sup_{p>0} \frac{\mathbf{\Phi}(\lambda f)}{\lambda} = \infty$$

for all $\lambda > 0$. Hence, by Theorem 2.1, the subset K is not relatively weakly compact in $L_\Phi[0,1]$.

Note, however, if $0 < p \leq 1$, that is $K = \{f_p(x) = e^{px} : \ 0 < p \leq 1\}$, then K is relatively weakly compact in $L_\Phi[0,1]$. We also note that the set K in Example 2.3 is not bounded (in norm) in $L_\Phi[0,1]$. Hence, this fact clearly implies that K is not relatively weakly compact. In general, norm boundedness does not imply weak compactness.

Also note that $L_\Phi[0,1]$ is separable since $\Phi \in \triangle_2$ by Proposition 1.5. Indeed, $\Phi(2x) \leq K \cdot \Phi(x)$ for large x since $\ln(2x+1) \leq 3\ln(x+1) = \ln(x+1)^3$ or equivalently, $2x+1 \leq (x+1)^3$ for large x.

Below we state another two criteria of weak compactness criteria in Orlicz spaces due to T. Ando.

Lemma 2.4. (see [1; 171]) A subset K of $L_\Phi[0,1]$ is (relatively) weakly compact if and only if it is weakly bounded and equi continuous in the following sense:

$$\sup_{f\in K}\int_E |f(t)\cdot g(t)|d\mu \to 0 \ \text{ as } \ \mu(E)\to 0, \quad E\subset[0,1], \quad g(t)\in L_\Psi[0,1].$$

Lemma 2.5. ([1; 172]) Let B be a σ-algebra of subsets of $(0,1)$. When B is atomless, boundedness by modular $\mathbf{\Phi}(f)$ implies (relative) weak compactness, if and only if $\Phi(\lambda)$ has (∇_2), i.e.

$$\liminf_{\lambda\to\infty}\frac{\Phi(\eta\lambda)}{\Phi(\lambda)} \geq 2\eta \ \text{ for some } \ \eta > 0.$$

Remark. Note that Lemma 2.5 may also be applied to show that the set K in Example 2.2 (both (i) and (ii)) is relatively weakly compact in $L_\Phi[0,1]$. Indeed, as for (i), the boundedness by the modular $\mathbf{\Phi}(f)$ is obvious. Also, (with $\eta = 2$)

$$\liminf_{\lambda\to\infty}\frac{\Phi(2\lambda)}{\Phi(\lambda)} = \liminf_{\lambda\to\infty}\frac{e^{2\lambda}-1}{e^{\lambda}-1} \geq 2\cdot 2 = 4.$$

As for (ii), we have

$$\int_0^1 \Phi(x^p)dx = \int_0^1 (e^{x^p} - x^p - 1)dx \leq \int_0^1 e^{x^p}dx \leq 3, \quad \text{for all } \ p > 0.$$

Taking supremum over all $p > 0$, we obtain boundedness of a set K by modular $\mathbf{\Phi}$. Also, by setting $\eta = 2$, we obtain

$$\liminf_{\lambda\to\infty}\frac{e^{2\lambda}-2\lambda-1}{e^{\lambda}-\lambda-1} \geq 2\cdot 2 = 4.$$

Recall that a set K in Example 2.3 is not relatively weakly compact, which may not be proved via using Lemma 2.5 since a set K is not bounded by the modular $\mathbf{\Phi}(f)$. Indeed,

$$\mathbf{\Phi}(f) = \int_0^1 \Phi(e^{px})dx = \int_0^1 e^{px}\ln(e^{px}+1)dx =$$

$$= \frac{1}{p}[e^p\ln(e^p+1) - 2\ln 2 - e^p + 1 + \ln(e^p+1)] \to \infty \quad \text{as } \ p\to\infty.$$

Though $\Phi(\lambda)$ fails (∇_2),

$$\liminf_{\lambda\to\infty}\frac{\Phi(\eta\lambda)}{\Phi(\lambda)} = \liminf_{\lambda\to\infty}\frac{\eta\lambda\ln(\eta\lambda+1)}{\lambda\ln(\lambda+1)} = \eta < 2\eta \quad \text{for all } \ \eta > 0.$$

Note that in general, if K is not relatively weakly compact, then it is not necessarily true that K is not bounded by the modular $\mathbf{\Phi}(f)$.

Now we provide an example of a set K such that K is bounded by modular $\mathbf{\Phi}$ and $\Phi(\lambda)$ that does not have (∇_2).

Example 2.6. Let $\Phi(x) = x \cdot \ln(x+1)$ and $K = \{f_p(x) = x^p : \quad p > 0\}$. Note K is bounded by the modular $\mathbf{\Phi}(f)$. Indeed,

$$\sup_{p>0} \int_0^1 x^p \cdot \ln(x^p + 1) dx \leq 1.$$

However, since $\Phi(x)$ fails (∇_2), we conclude that K is not relatively weakly compact by Lemma 2.5.

Now we state the relation between conjugate N-functions in terms of $\triangle_2, \nabla_2$ relations (see [2, Chapter 2.3, Theorem 3]).

Remark 2.7. $\Phi(x)$ has ∇_2 if and only if its conjugate $\Psi(x)$ has $\triangle_2$.

For example, let $\Phi(x) = e^x - x - 1,\ x \geq 0$. Then it is easy to see that $\Phi \in \nabla_2$. Its conjugate function $\Psi(x) = x \cdot \ln(x+1) - x + \ln(x+1),\ x \geq 0$ has $\triangle_2$.

The following definition may be found in [17, Definition 53.1], [2, Chapter 1.3].

Definition 2.8. A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called a *Young function* if and only if:
(i) $\Phi(x) = \int_0^{|x|} \phi(s) ds$ for all $x \in \mathbb{R}$;
(ii) $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing;
(iii) $\phi(0) = 0$ and $\phi(s) \to \infty$ as $s \to \infty$.

Throughout this paper, however, we restrict our attention to Young functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. It is noted that every Young function is not an $N-$function (in the sense of Ando). However, in most papers the definition of a Young function coincides with the definition of an $N-$function (with the condition $\lim_{x \to 0+} \frac{\Phi(x)}{x} = 0$), for example, [3, Definition 1.1]. Clearly, not every Orlicz function is a Young (or N-)function. For example, $\Phi(x) = x \arctan x$ is such a function.

The notion of Orlicz functions (as well as of $N-$functions or Young functions) is known since 1940s. Nonetheless, for the sake of convenience, we provide a list of Orlicz functions below. We note that some of them are not $N-$functions, and some are not Young functions.

Examples of Orlicz functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ are:

$\Phi(x) = x^p, \quad p \geq 1$ (corresponding to Lebesgue spaces L_p. If $p > 1$, then it is also both an $N-$function and a Young function);

$\Phi(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq 1, \\ \infty & \text{if} \quad x > 1; \end{cases}$ (corresponding to the Lebesgue space L_∞, neither Young nor N-function).

$\Phi(x) = e^x - 1 \quad$ (neither Young nor $N-$function);
$\Phi(x) = e^x - x - 1 \quad$ (both Young and $N-$function);
$\Phi(x) = e^x - \frac{x^2}{2} - x - 1$, and in general $e^x - \sum_{k=0}^n \frac{x^k}{k!}$ for any $n \in \mathbf{N}$;
$\Phi(x) = x \ln(x+1) \quad$ (not Young but $N-$function);
$\Phi(x) = (x+1)\ln(x+1) \quad$ (neither Young nor $N-$function);
$\Phi(x) = x \ln(x^2+1) \quad$ (both Young and $N-$function);
$\Phi(x) = xe^{x^p}$, where $p \geq 1 \quad$ (neither Young nor $N-$function);
$\Phi(x) = (x+e)\ln(x+e) - (x+e) \quad$ (neither Young nor $N-$function);
$\Phi(x) = x \arctan x \quad$ (not Young but $N-$function);

$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1, \\ x^3 & \text{if} \quad x \geq 1; \end{cases}$

$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1/2, \\ x - 1/4 & \text{if} \quad x \geq 1/2; \quad \text{etc.} \end{cases}$

While discussing relative weak compactness in Orlicz space L_{Φ_1}, it is natural to ask whether there is another Orlicz function $\Phi_2 \neq \Phi_1$ such that L_{Φ_2} coincides with L_{Φ_1}. The following theorem shows equivalent conditions when two Orlicz spaces L_{Φ_1} and L_{Φ_2} coincide on $(0, \infty)$ (as sets), see, for example [16, Theorem 16.3.2].

Theorem 2.9. Let Φ_1 and Φ_2 be two Orlicz functions. The following are equivalent:
(1) $\Phi_1 \approx \Phi_2$ (i.e., $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$ as in Definition 1.6) ;
(2) $L_{\Phi_1}(0,\infty) = L_{\Phi_2}(0,\infty)$ as sets;
(3) $||\cdot||_{L_{\Phi_1}}$ and $||\cdot||_{L_{\Phi_2}}$ are equivalent, i.e.,

$$a_1||f||_{L_{\Phi_1}} \leq ||f||_{L_{\Phi_2}} \leq a_2||f||_{L_{\Phi_1}}$$

for all f and some $a_1 > 0,\ a_2 > 0$;
(4) $a_1\varphi_{L_{\Phi_1}}(x) \leq \varphi_{L_{\Phi_2}}(x) \leq a_2\varphi_{L_{\Phi_1}}(x)$ for all $x \geq 0$ and some $a_1 > 0,\ a_2 > 0$;
(5) $\Phi_1(a_1x) \leq \Phi_2(x) \leq \Phi_1(a_2x)$ for all $x \geq 0$ and some $a_1 > 0,\ a_2 > 0$.

In condition (4) above, $\varphi_{L_\Phi}(x)$ stands for the fundamental function of an Orlicz space L_Φ, and is defined as follows:

$$\varphi_{L_\Phi}(x) = \|\mathbf{1_{[0,x]}}\|_{\mathbf{L_\Phi}}.$$

Note that constants a_1 and a_2 in the above conditions (3), (4), and (5) may be chosen the same.

However, if we consider L_Φ on a finite measure space, the notion of equivalent Orlicz functions is slightly different (compare Definitions 1.6 and 1.9), which entails the corresponding changes in Theorem 2.9. For example, let

$$\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1/2, \\ x - 1/4 & \text{if} \quad x \geq 1/2. \end{cases}$$

Then $L_\Phi(0,1) = L_1(0,1)$ while $L_\Phi(0,\infty) \neq L_1(0,\infty)$. Indeed, $\Phi(x) \approx x$ as in Definition 1.9, thus $L_\Phi(0,1) = L_1(0,1)$. To show that $L_\Phi(0,\infty) \neq L_1(0,\infty)$, one may consider a function $f(x) = x^{-1/2}\chi_{(0,1/2)}(x)$, which belongs to $L_1(0,\infty)$ and does not belong to $L_\Phi(0,\infty)$. Or, on the other hand, this is easily checked since one cannot have $x \leq b\Phi(ax)$ for all large x and some positive a and b. Therefore, $\Phi(x)$ is not equivalent to x on $(0,\infty)$.

Now we state Theorem 2.9 for $L_\Phi(0,1)$.

Theorem 2.10. Let Φ_1 and Φ_2 be two Orlicz functions. The following are equivalent:
(1) $\Phi_1 \approx \Phi_2$ (i.e., $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$ as in Definition 1.9);
(2) $L_{\Phi_1}(0,1) = L_{\Phi_2}(0,1)$ as sets;
(3) $||\cdot||_{L_{\Phi_1}}$ and $||\cdot||_{L_{\Phi_2}}$ are equivalent, i.e.,

$$a_1||f||_{L_{\Phi_1}} \leq ||f||_{L_{\Phi_2}} \leq a_2||f||_{L_{\Phi_1}}$$

for all f and some $a_1 > 0,\ a_2 > 0$;
(4) $a_1\varphi_{L_{\Phi_1}}(x) \leq \varphi_{L_{\Phi_2}}(x) \leq a_2\varphi_{L_{\Phi_1}}(x)$ for all $x \geq x_0$ and some $a_1 > 0,\ a_2 > 0,\ x_0 > 0$;
(5) $\Phi_1(a_1x) \leq \Phi_2(x) \leq \Phi_1(a_2x)$ for all $x \geq x_0$ and some $a_1 > 0,\ a_2 > 0,\ x_0 > 0$.

The following definition will be needed to state another weak compactness criterion in Orlicz spaces.

Definition 2.11. (see [1; 173]) We say that $\Psi(x)$ is increasing more rapidly than $\Phi(x)$, if for any $\eta > 0$ there exist $\rho, x_0 > 0$ such that

$$\Psi(\rho x) \geq \rho \cdot \eta \cdot \Phi(x) \quad \text{for} \quad x \geq x_0.$$

Sometimes it is convenient to use the following equivalent definition.

Definition 2.12. (see [1; 173]) We say that $\Psi(x)$ is increasing more rapidly than $\Phi(x)$, if for any $\varepsilon > 0$ there exist $\delta, x_0 > 0$ such that

$$\varepsilon\Psi(x) \geq \frac{\Phi(\delta x)}{\delta} \quad \text{for} \quad x \geq x_0.$$

We note that $\Psi(\lambda)$ has (∇_2) if and only if Ψ is increasing more rapidly than itself [1; 173]. If $\Psi(x) \geq \Phi(x)$ for all $x \geq 0$, then it is not necessarily true that Ψ is increasing more rapidly than Φ. Now we prove a result, which allows one to check whether one Orlicz function is increasing more rapidly than another.

Proposition 2.13. Let Φ and Ψ be two Orlicz functions. If $\lim_{x\to\infty} \frac{\Psi(x)}{\Phi(x)} = \infty$, then Ψ is increasing more rapidly than Φ.

Proof. If $\lim_{x\to\infty} \frac{\Psi(x)}{\Phi(x)} = \infty$, then for any $\eta > 1$ there exists $x_1 > 0$ such that $\Psi(x) \geq \eta \cdot \Phi(x)$ for all $x \geq x_1$. Since Ψ is convex there exist $\rho, x_2 > 0$ such that $\Psi(\rho x) \geq \rho \cdot \Psi(x)$ for all $x \geq x_2$. Hence,

$$\Psi(x) \geq \rho \cdot \Psi(x) \geq \rho \cdot \eta \cdot \Phi(x)$$

for all $x \geq x_0$, where $x_0 = \max\{x_1, x_2\}$ (this is even stronger statement than required).

The following theorem is also due to T. Ando [1, Theorem 2].

Theorem 2.14. A subset K of $L_\Phi[0,1]$ is relatively weakly compact if and only if it is bounded by the modular defined by an $N-$function (depending on K) $\Psi(x)$ increasing more rapidly than $\Phi(x)$.

Example 2.15. Let $\Phi(x) = e^x - 1$, then the Orlicz function $\Psi(x) = e^{x^2} - 1$ is increasing more rapidly than $\Phi(x)$.

Indeed, fix any $\varepsilon > 0$ and choose $\delta = 1$. Then we need to show that there exists $x_0 > 0$ such that $\varepsilon \cdot (e^{x^2} - 1) \geq e^x - 1$ for all $x > x_0$. It is obvious that for any $\varepsilon > 0$ one can find such $x_0 > 0$ since $\liminf_{x\to\infty} \frac{e^{x^2}-1}{e^x-1} = \infty$. Note $\Phi(x) = e^x - 1$ is an $N-$function (in the sense of Ando) with $\alpha = \lim_{x\to 0} \frac{\Phi(x)}{x} = 1$, while Ψ is an $N-$function with $\alpha = 0$.

Now using the Theorem 2.14 we prove that a set K in Example 2.2 (both (i) and (ii)) is relatively weakly compact in $L_\Phi[0,1]$.

Example 2.16. Recall in Example 2.2 (i), $\Phi(x) = e^x - 1$. It has been shown that the subset $K = \{f_p(x) = x^p : \ \ p \geq 1\}$ is relatively weakly compact in $L_\Phi[0,1]$.

Alternatively, by Theorem 2.14 and Example 2.15 it remains to show that a set K is bounded by the modular defined by an $N-$function $\Psi(x) = e^{x^2} - 1$, that is, to show that $\sup_{p\geq 1} \int_0^1 \Psi(x^p)dx < \infty$. Indeed,

$$\int_0^1 (e^{x^{2p}} - 1)dx = \int_0^1 e^{x^{2p}} dx - 1 \leq \int_0^1 e^{x^2} dx - 1 < 1/2.$$

Taking supremum over all $p \geq 1$, we obtain the desired result.

As for (ii), we note that $\Psi(x) = e^{x^2} - 1$ is also increasing more rapidly than $\Phi(x) = e^x - x - 1$, since $e^x - x - 1 \leq e^x - 1$ for all $x \geq 0$. Thus, by the previous argument we may conclude that the set K in Example 2.2 (ii) is also relatively weakly compact.

Now we show the following proposition.

Proposition 2.17. If a set K is not bounded by a modular $\boldsymbol{\Phi}$, defined by an Orlicz function Φ, then it is not necessarily true that K is not bounded by the modular $\boldsymbol{\Psi}$, defined by an Orlicz function Ψ, increasing more rapidly than Φ.

Proof. Indeed, it suffices to find an N-function function Φ, another Orlicz function Ψ, which increases more rapidly than Φ and a function f for which the inequality $\int_0^1 \Phi(f(x))dx \leq \int_0^1 \Psi(f(x))dx$ fails. For such purposes, one may choose $\Phi(x) = x$, $\Psi(x) = x^{100}$ and $f(x) = x$.

Thus, recall that a set K in Example 2.3 was not bounded by the modular $\boldsymbol{\Phi}$, hence by Remark 2.17 it is not necessarily true that K is not bounded by the modular $\boldsymbol{\Psi}$, defined by some Orlicz function Ψ, increasing more rapidly than Φ. However, since a set K is not relatively weakly compact in $L_\Phi[0,1]$ we conclude (by Theorem 2.14) that there is no such function Ψ such that K is bounded by the modular $\boldsymbol{\Psi}$.

Recall that the complementary (or conjugate) function Ψ to Φ in the sense of Young, is defined by (see [15; 11])

$$\Psi(t) = \sup\{s|t| - \Phi(s) : \quad s \geq 0\}. \tag{1}$$

Since in this paper we work on positive half-line $\mathbb{R}_+$ (that is $t \geq 0$), we may omit the modulus sign in the formula (1).

The following constructive way of identifying a conjugate function to a given Young function is given in [2, Theorem 3, Formula (14) and Corollary 2, p. 10].

Theorem 2.18. Let $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a Young function, that is

$$\Phi(x) = \int_0^x \phi(s)ds, \quad x \geq 0,$$

where $\phi(0) = 0$, $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is nondecreasing and left continuous. Let $\psi(\cdot)$ be the (generalized) inverse of ϕ. Then the conjugate function Ψ to Φ may be defined as follows:

$$\Psi(x) = \int_0^x \psi(s)ds, \quad x \geq 0.$$

Now we provide examples of pairs of mutually conjugate Orlicz functions.

Example 2.19. Let $\Phi(x) = e^x - x - 1, \quad x \geq 0$, then it is easy to find its conjugate function (via Theorem 2.18) $\Psi(x) = x \cdot \ln(x+1) - x + \ln(x+1), \quad x \geq 0$, which, by definition, is also an Orlicz (moreover, both of them are $N-$functions) function.

Indeed, $\Phi'(x) = e^x - 1$ whose inverse is $\Psi'(x) = \ln(x+1)$. Thus, integrating by parts we obtain $\Psi(x) = \int_0^x \ln(t+1)dt = x \cdot \ln(x+1) - x + \ln(x+1)$. It is easy to see that this function coincides with the one defined by formula (1).

We note that Φ is not equivalent to Ψ on $(0, \infty)$ (there is no $C > 0$ such that $\Phi(x) \leq \Psi(Cx)$).

Example 2.20. Let $\Phi(x) = \begin{cases} x^2 & \text{if} \quad 0 \leq x < 1, \\ x^3 & \text{if} \quad x \geq 1. \end{cases}$

We note that Φ is an $N-$function and $L_\Phi(0, \infty) \neq L_p(0, \infty)$ for any $p \geq 1$. However, $L_\Phi(0, 1) = L_3(0, 1)$. Its conjugate function is given by

$$\Psi(x) = \begin{cases} \frac{x^2}{4} & \text{if} \quad 0 \leq x < 1, \\ \frac{2}{3\sqrt{3}} x^{3/2} + \frac{1}{4} - \frac{2}{3\sqrt{3}} & \text{if} \quad x \geq 1. \end{cases}$$

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References

1 Ando, T. (1962). Weakly compact sets in Orlicz spaces. *Canad.J.Math., 14,* 170–176.

2 Rao, M.M., & Ren, Z.D. (1991). *The Theory of Orlicz spaces.* New York: Marcel Dekker,

3 Alexopoulos, J. (1994). De La Vallée Poussin's theorem and weakly compact sets in Orlicz spaces. *Quaestiones Mathematicae*, 231–248.

4 Astashkin, S.V. (2019). Rearrangement invariant spaces satisfying Dunford-Pettis criterion of weak compactness. *Contemporary Mathematics, 733*, 45–59.

5 Barcenas, D., & Finol, C.E. (2009). On Vector Measures, Uniform Integrability and Orlicz Spaces. *Operator theory: Advances and Applications, 201*, 51–57.

6 Chong, K.M. (1976). Doubly stochastic operators and rearrangement theorems. *J. Math. Anal. Appl., 56*, 309–316.

7 Chong, K.M. (1974). Spectral orders, uniform integrability and Lebesgue's dominated convergence theorem. *Trans. Amer. Math. Soc., 191*, 395–404.

8 Dodds, P.G., Sukochev, F., & Schlüchtermann, G. (2001). Weak compactness criteria in symmetric spaces of measurable operators. *Math.Proc.Camb.Phil.Soc, 131*, 363–384.

9 Kamińska, A., & Mastylo, M. (2002). The Schur and (weak) Dunford-Pettis properties in Banach Lattices. *J.Austral.Math.Soc., 73*, 251–278.

10 Lefévre, P., Li, D., Queffèlec, H., & Rodriguez-Piazza, L. (2008). Weak compactness and Orlicz spaces. *Colloquium Math., 112*(1), 23–32.

11 Nowak, M. (1986). A characterization of the Mackey topology $\tau(L^{\varphi}, L^{\varphi^*})$ on Orlicz spaces. *Bulletin of the Polish Academy of Sciences, Mathematics, 34*, 9-10, 577–583.

12 Orlicz, W. (1936). Über Raüme (L^M). *Bull. Acad. Polon. Sci. Ser. A*, 93–107.

13 Zhang, C., & Hou, Y.L. (2013). Convergence of weighted averages of martingales. *Sci China Math, 56*, 823–830.

14 Zhang, C., & Hou, Y.L. (2012). Convergence of weighted averages of martingales in noncommutative Banach function spaces. *Acta Mathematica Scientia, 32B*(2), 735–744.

15 Krasnoselskii, M.A., & Rutickii, Ya.B. (1961). *Convex functions and Orlicz spaces*. Translated by Leo F.Boron, Noorhoff Ltd., Groningen.

16 Rubshtein, B.A., Grabarnik, G.Ya., Muratov, M.A., & Pashkova, Y.S. (2016). *Foundations of Symmetric Spaces of Measurable Functions: Lorentz, Marcinkiewicz and Orlicz spaces.* Developments in Mathematics, Volume 45, Springer.

17 Zeidler, E. (1985). *Nonlinear Functional Analysis and its Applications III, Variational methods and Optimization*, Springer-Verlag, New York Inc.

Д. Дәуітбек[1,2], Е. Несіпбаев[1,2], К. Төленов[1,2]

[1]*Математика және математикалық модельдеу институты, Алматы, Қазақстан;*
[2]*Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан*

Орлич кеңістіктеріндегі әлсіз жинақы жиындардың мысалдары

Мақалада Орлич кеңістіктеріндегі салыстырмалы әлсіз жинақы жиындардың кейбір мысалдары келтірілген. Сондай-ақ осы мысалдардан туындайтын кейбір нәтижелер көрсетілген. Атап айтқанда, кейбір Орлич функциясының екіншісіне қарағанда жылдамырақ өсетінін қамтамасыз ететін критерийлер берілген (Т. Андо мағынасында). Сонымен қатар, егер L_Φ Орлич кеңістігінің K шектелген ішкі жиыны модуляр **Φ**-мен шектелмеген болса, онда K жиынының Φ-ға үшін қарағанда жылдам өсетін кез келген Ψ модуляры шекелмеген күйінде қалуы мүмкін екені анықталған.

Кілт сөздер: түйіндес (толықтырғыш) функциялар, салыстырмалы әлсіз жинақылық, Орлич кеңістіктері, $N-$функциялар.

Д. Дауитбек[1,2], Е. Несипбаев[1,2], К. Туленов[1,2]

[1] *Институт математики и математического моделирования, Алматы, Казахстан;*
[2] *Казахский национальный университет имени аль-Фараби, Алматы, Казахстан*

Примеры слабо компактных множеств в пространствах Орлича

В статье мы приводим ряд примеров относительно слабо компактных множеств в пространствах Орлича. Кроме того, получены некоторые результаты, вытекающие из этих примеров. В частности, получен критерий, который гарантирует, что одна функция Орлича возрастает быстрее, чем другая (в смысле Т. Андо). Кроме того, показано, что если ограниченное подмножество K пространства Орлича L_Φ не ограничено модуляром $\mathbf{\Phi}$, то множество K может оставаться неограниченным для любого модуляра Ψ, растущим быстрее, чем Φ.

Ключевые слова: сопряженные (дополнительные) функции, относительная слабая компактность, пространства Орлича, N-функции.