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S.N. Askhabov*

Chechen State Pedagogical University, Kadyrov Chechen State University, Grozny, Russia
(E-mail: askhabov@yandex.ru)

On a second-order integro-differential equation with difference kernels and power nonlinearity

The article studies a second-order integro-differential equation with difference kernels and power nonlinearity. A connection is established between this equation and an integral equation of the convolution type, which arises when describing the processes of liquid infiltration from a cylindrical reservoir into an isotropic homogeneous porous medium, the propagation of shock waves in pipes filled with gas and others. Since non-negative continuous solutions of this integral equation are of particular interest from an applied point of view, solutions of the corresponding integro-differential equation are sought in the cone of the space of continuously differentiable functions. Two-sided a priori estimates are obtained for any solution of the indicated integral equation, based on which the global theorem of existence and uniqueness of the solution is proved by the method of weighted metrics. It is shown that any solution of this integro-differential equation is simultaneously a solution of the integral equation and vice versa, under the additional condition on the kernel that any solution of this integral equation is a solution of this integro-differential equation. Using these results, a global theorem on the existence, uniqueness and method of finding a solution to an integro-differential equation is proved. It is shown that this solution can be found by the method of successive approximations of the Picard type and an estimate for the rate of their convergence is established. Examples are given to illustrate the obtained results.

Keywords: integro-differential equation, power nonlinearity, difference kernels, weight metrics method.

Introduction

In this paper, we study the second-order nonlinear integro-differential equation

$$u^{\alpha}(x)=\int_{0}^{x} h(x-t)u'(t)\,dt+\int_{0}^{x} k(x-t)u''(t)\,dt, \quad x>0, \quad \alpha>1, \tag{1}$$

with initial conditions:

$$u\,(0)=0, \quad u'(0)=0.$$

On the kernels $h(x)$ and $k(x)$ of equation (1) the conditions:

$$h\in C^2\,[0,\infty)\,,\ h''(x) \text{ does not decrease on } [0,\infty)\,,\ h(0)=h'(0)=0 \text{ and } h''(0)\geq 0, \tag{2}$$

$$k\in C^3\,[0,\infty)\,,\ k'''(x) \text{ does not decrease on } [0,\infty)\,,\ k(0)=k'(0)=k''(0)=0 \text{ and } k'''(0)>0 \tag{3}$$

are imposed.

The integro-differential equation (1) is closely related to the convolution type nonlinear integral equation

$$u^{\alpha}(x)=\int_{0}^{x} K\,(x-t)\,u\,(t)\,dt, \quad x>0, \quad \alpha>1, \tag{4}$$

*Corresponding author.
E-mail: askhabov@yandex.ru

at $K(x) = h'(x) + k''(x)$, arising when describing the processes of fluid infiltration from a cylindrical reservoir into an isotropic homogeneous porous medium [1, 2], the shock wave propagation in pipes filled with gas [3, 4] and others (see [5–7]).

Equations (1) and (4) have the trivial solution $u(x) \equiv 0$. From the theoretical and applied points of view, nontrivial nonnegative continuous solutions of these equations are of special interest. Since, for $0 < \alpha \leq 1$, equations (1) and (4) can only have the trivial solution $u(x) \equiv 0$ in the cone Q of the space $C[0,\infty)$ consisting of all nonnegative continuous functions on $[0,\infty)$, then it makes sense to study them only for $\alpha > 1$. Any solution of equations (1) and (4) in the cone Q, including nontrivial ones, satisfies the condition $u(0) = 0$. In addition, if $u(x) \in Q$ is a nontrivial solution to equation (1) or equation (4), then for any $\delta > 0$ its shifts:

$$u_\delta(x) = u\left(x-\delta\right) \text{ at } x > \delta; \quad u_\delta(x) = 0 \text{ at } x \leq \delta, \text{ and } u_{-\delta}(x) = u\left(x+\delta\right) \text{ at } x > 0$$

are also solutions to these equations, which is verified by direct substitution. Consequently, equations (1) and (4) can have a continuum of nontrivial solutions in the cone Q. Therefore, to make the problem of finding non-trivial solutions of equations (1) and (4) correct and since continuous positive solutions for $x > 0$ are of interest from the applied point of view, we will look for solutions to the integro-differential equation (1) in the cone

$$Q_0^2 = \left\{u(x): \quad u(x) \in C\left[0,\infty\right) \cap C^2\left(0,\infty\right), \quad u(0) = u'(0) = 0 \text{ and } u(x) > 0 \text{ at } x > 0\right\},$$

and solutions of the integral equation (4) will be sought in the cone

$$Q_0 = \{u(x): \quad u(x) \in C\left[0,\infty\right), \quad u\left(0\right) = 0 \text{ and } u(x) > 0 \text{ at } x > 0\}.$$

Conditions (2), (3) imply that the kernel $K(x) = h'(x) + k''(x)$ of equation (4) satisfies the condition:

$$K(x) \in C^1\left[0,\infty\right), \quad K'(x) \text{ does not decrease on } [0,\infty), \quad K(0) = 0 \text{ and } K'(0) > 0. \tag{5}$$

We consider equation (4) based on the two-sided a priori estimates and the weighted metrics method, an analogue of the Bielecki method (see [8; 218]).

In contrast to the Bielecki's method, during the construction of the metric, this study uses an exact a priori estimate from below of the solution to equation (4) as a weight function, which allows us to prove the global existence and uniqueness theorem for the solution to equation (4) without restrictions on the domain of its existence.

For the first time, in works [1, 2], the method of weight metrics was applied to equation (4) under the condition that $K(0) > 0$. In addition, in [1, 2], when constructing the metric, the role of the weight function is played by the difference between the upper and lower a priori estimates, and for the correctness of this metric (so that the denominator does not vanish), a specially overestimated a priori estimate from above is used. As a result, such a metric led to additional restrictions and rather cumbersome calculations in [1, 2].

This paper shows that any solution to equation (1) in Q_0^2 under conditions (2), (3) is simultaneously a solution of equation (4) and vice versa, under the additional condition imposed on the kernel $K(x) = h'(x) + k''(x)$ any solution of equation (4) from Q_0 belongs to the class Q_0^2 and is a solution of equation (1). The main result of the paper is that, using the above relationship between equations (1) and (4), the global existence, uniqueness theorem is proved and the method for solving equation (1) is found. The Picard successive approximation method is applied to solve the considered equation. The convergence rate estimates are established. Examples are provided to illustrate the obtained results.

Main part

Before proceeding to the study of equation (1), we first consider equation (4). The next two lemmas contain information on the properties of non-negative solutions (if they exist) for equation (4).

Lemma 1. Let the condition (5) hold. If $u \in Q_0$ is a solution to the integral equation (4), then the function $u(x)$ does not decrease on $[0, \infty)$ and is twice continuously differentiable for $x > 0$ i.e., $u \in C^2(0, \infty)$.

Proof. Let us first prove that the function $u(x)$ does not decrease on the entire semiaxis $[0, \infty)$, if $u \in Q_0$ and is a solution to equation (4). Let $x_1, x_2 \in [0, \infty)$ be any number, and $x_1 < x_2$. Since by virtue of condition (5), $K'(x) \geq K'(0) > 0$ for any $x \in [0, \infty)$, i.e., the kernel $K(x)$ increases on $[0, \infty)$, then

$$u^{\alpha}(x_2) - u^{\alpha}(x_1) \equiv \int_0^{x_1} [K(x_2 - t) - K(x_1 - t)]\, u(t)\, dt + \int_{x_1}^{x_2} K(x_2 - t)\, u(t)\, dt \geq 0$$

consequently, $u(x_2) \geq u(x_1)$, which is required.

Finally, we prove that $u \in C^2(0, \infty)$. Once both parts of identity (4) have been differentiated taking into account $K(0) = 0$, obtain

$$u'(x) = \frac{1}{\alpha} u^{1-\alpha}(x) \int_0^x K'(x - t) u(t)\, dt. \tag{6}$$

This means that $u'(x)$ is continuous at $x > 0$. However, then $u''(x)$ exists and is continuous as the product of two continuously differentiable functions for any $x > 0$. Accordingly, $u \in C^2(0, \infty)$ and the lemma is completely proved.

Lemma 2. Let the condition (5) hold. If a function $u \in Q_0$ and is a solution to the integral equation (4), then for any $x \geq 0$ the inequalities

$$F(x) \equiv c(\alpha) \cdot x^{2/(\alpha-1)} \leq u(x) \leq \left(\frac{\alpha - 1}{\alpha} \int_0^x K(t)\, dt \right)^{1/(\alpha-1)} \equiv G(x), \tag{7}$$

where

$$c(\alpha) = \left(\frac{K'(0) \cdot (\alpha - 1)^2}{2\alpha \cdot (\alpha + 1)} \right)^{1/(\alpha-1)},$$

are valid.

Proof. Let $u(x) \in Q_0$ be a solution of equation (4). Lemma 1 implies that the function $u(x)$ does not decrease on $[0, \infty)$ and $u \in C^2(0, \infty)$.

Prove the estimate $F(x) \leq u(x)$. By differentiating identity (4) twice, in view of condition (5), obtain:

$$(u^{\alpha}(x))'' = \int_0^x K''(x - t) u(t) dt + K'(0) u(x) \geq K'(0) u(x).$$

Introduce the new function $v(x)$, denoting $u^{\alpha}(x) = v(x)$. The result is the second-order non-linear differential inequality $v'' \geq K'(0) v^{1/\alpha}$ that does not contain an explicitly independent variable x. By substituting this inequality $v' = p$, $p = p(v)$ (then $v'' = p \cdot p'$) we get $p \cdot p' \geq K'(0) v^{1/\alpha}$. Since

$$v(x) \equiv \int_0^x K(x - t)\, v^{1/\alpha}(t)\, dt \quad \textit{and} \quad K(0) = 0,$$

then

$$v'(x) \equiv \int\limits_0^x K'(x-t)\, v^{1/\alpha}(t)\, dt.$$

Hence, $v(0) = v'(0) = 0$ and $v'(x) \geq 0$. Therefore, writing the previous inequality as $pdp \geq K'(0)v^{1/\alpha}dv$ and integrating from 0 to x, we obtain

$$\frac{[v'(x)]^2}{2} \geq \frac{K'(0))\cdot\alpha}{\alpha+1}[v(x)]^{(\alpha+1)/\alpha} \quad or \quad v'(x) \geq \sqrt{\frac{2K'(0)\cdot\alpha}{\alpha+1}}\cdot[v(x)]^{(\alpha+1)/(2\alpha)}.$$

Now separate the variables and integrate again from 0 to x and obtain

$$\frac{2\alpha}{\alpha-1}\cdot[v(x)]^{(\alpha-1)/(2\alpha)} \geq \sqrt{\frac{2K'(0)\cdot\alpha}{\alpha+1}}\cdot x$$

or

$$[v(x)]^{1/\alpha} \geq \left[\left(\frac{\alpha-1}{2\alpha}\right)^2\cdot\frac{2K'(0)\cdot\alpha}{\alpha+1}\cdot x^2\right]^{1/(\alpha-1)} \equiv F(x).$$

Recalling that $u^\alpha(x) = v(x)$, from the last inequality we obtain the provable lower bound: $u(x) \geq F(x)$.

It remains to prove the upper estimate, i.e. $u(x) \leq G(x)$. Since $K(x)$ and $u(x)$ are nondecreasing functions, by applying the Chebyshev inequality (17.6) [6] in (4) obtain:

$$u(x) \leq \left(\int\limits_0^x K(t)\, u(t)\, dt\right)^{\frac{1}{\alpha}} \qquad for\ any\ \ x > 0. \tag{8}$$

Hence,

$$K(x)u(x)\left(\int\limits_0^x K(t)\, u(t)\, dt\right)^{-1/\alpha} \leq K(x).$$

Therefore, once the integration has taken place, get:

$$\left(\int\limits_0^x K(t)\, u(t)\, dt\right)^{1/\alpha} \leq \left(\frac{\alpha-1}{\alpha}\right)^{1/(\alpha-1)}\left(\int\limits_0^x K(t)\, dt\right)^{1/(\alpha-1)} \equiv G(x). \tag{9}$$

Using estimate (9) by inequality (8) obtain: $u(x) \leq G(x)$, which is required.

Example 1. The function

$$u^*(x) = \left(\frac{(\alpha-1)^2}{2\alpha\cdot(\alpha+1)}\right)^{1/(\alpha-1)} x^{2/(\alpha-1)}$$

is a solution to the equation (4) for $K(x) = x$.

Example 1 shows that $F(x) \equiv u^*(x)$ at $K(x) = x$, i.e., a priori lower bound of the solution to the equation (4) is unimprovable.

Obviously, Lemma 2 implies that the solution to equation (4) should be sought in the class

$$P = \{u(x):\ \ u(x) \in C\,[0,\infty) \ \ and \ \ F(x) \leq u(x) \leq G(x)\},$$

as $F(0) = G(0) = 0$ and $F(x) > 0$ at $x > 0$.

Now consider the operator T:

$$(Tu)(x) = \left(\int_0^x K(x-t)u(t)dt \right)^{1/\alpha}, \qquad x > 0.$$

Lemma 3. The operator T transforms the class P into itself.

Proof. Assume $u(x) \in P$ is an arbitrary function. Consequently, we have to prove that $(Tu)(x) \in P$. By Theorem 17.9 [6] $(Tu)(x) \in C[0,\infty)$. It remains to prove that $F(x) \le (Tu)(x) \le G(x)$. As $u(x) \ge F(x)$ and by condition (5) $K'(x) \ge K'(0) > 0$, $K(0) = 0$, then

$$[(Tu)(x)]^\alpha \ge \int_0^x K(x-t)c(\alpha)t^{2/(\alpha-1)}dt = c(\alpha)\frac{\alpha-1}{\alpha+1}\int_0^x K(x-t)dt^{(\alpha+1)/(\alpha-1)} =$$

$$= c(\alpha)\frac{\alpha-1}{\alpha+1}\int_0^x K'(x-t)t^{(\alpha+1)/(\alpha-1)}dt \ge c(\alpha)\frac{\alpha-1}{\alpha+1}K'(0)\int_0^x t^{(\alpha+1)/(\alpha-1)}dt \equiv [F(x)]^\alpha,$$

that is $(Tu)(x) \ge F(x)$.

On the other hand, since $u(x) \le G(x)$ then taking into account condition (6) and the Chebyshev integral inequality (17.6) [6], where the role of function $u(x)$ is already played by the function $G(x)$, which is non-decreasing either (see the proof of Theorem 17.12 [6]) we get:

$$[(Tu)(x)]^\alpha \le \int_0^x K(t)\,G(t) \equiv [G(x)]^\alpha, \qquad i.e. \qquad (Tu)(x) \le G(x).$$

Lemma 3 is proved.

Now consider the class

$$P_b = \{u(x): \quad u(x) \in C[0,b] \ \text{ and } \ F(x) \le u(x) \le G(x)\},$$

where $b > 0$ is any number, and introduce the metric in it ρ_b by imposing $\forall u(x), v(x) \in P_b$:

$$\rho_b(u,v) = \sup_{0<x\le b} \frac{|u(x)-v(x)|}{x^{2/(\alpha-1)}e^{\beta x}}, \quad \text{where} \quad \beta > 0 \ \text{ is any number.}$$

It is proved directly in view of the equalities $F(0) = G(0) = 0$ and the complete metric space $C[0,b]$ with Chebyshev metric that the pair (P_b, ρ_b) forms a complete metric space (see Theorem 17.13 [6]).

Choose a number $\mu \in (0,b)$ so that condition

$$K'(\mu) < \alpha \cdot K'(0) \tag{10}$$

is satisfied and sets

$$\beta = \frac{1}{K'(0)} \sup_{\mu \le x \le b} \frac{K'(x) - K'(0)}{x}.$$

Then, by lemma 18.5 [6], we obtain that the inequality

$$K(x)e^{-\beta x} \le x \cdot K'(\mu) \tag{11}$$

holds.

Theorem 1. If the kernel $K(x)$ satisfies condition (6), then equation (4) has a unique solution $u^*(x)$ in the cone Q_0 (and in P_b for any $b > 0$). This solution can be found by the successive approximations method using the formula $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, which converge to it according to the metric ρ_b at any $b < \infty$, and the convergence rate estimate

$$\rho_b\left(u_n, u^*\right) \leq \frac{\gamma^n}{1-\gamma}\rho_b\left(Tu_0, u_0\right) \tag{12}$$

is valid, where $\gamma = K'(\mu)/\left[\alpha K'(0)\right] < 1$, and $u_0(x) \in P_b$ the initial approximation (on arbitrary function).

Proof. Write equation (4) as the operator equation $u = Tu$. First, show that the operator T acting according to Lemma 3 from P_b to P_b is contractive.

Let $u, v \in P_b$ be arbitrary functions. It is clear that

$$|u(x) - v(x)| \leq x^{2/(\alpha-1)}e^{\beta x}\rho_b\left(u, v\right).$$

Therefore, using inequality (11) get

$$\left|\int\limits_0^x K(x-t)\left[u\left(t\right) - v(t)\right]dt\right| \leq \rho_b(u,v)\int\limits_0^x K(x-t)e^{-\beta(x-t)}e^{\beta x}t^{2/(\alpha-1)}dt \leq$$

$$\leq e^{\beta x}K'(\mu)\rho_b(u,v)\int\limits_0^x\left(x-t\right)t^{2/(\alpha-1)}dt = \frac{(\alpha-1)^2K'(\mu)}{2\alpha\left(\alpha+1\right)}e^{\beta x}\rho_b(u,v)x^{2\alpha/(\alpha-1)}.$$

Next employing the Lagrange theorem (finite-increments formula) in view of the later estimate (see the proof of Theorem 17.14 [6]) obtain

$$\left|\left(Tu\right)(x) - \left(Tv\right)(x)\right| \leq \frac{1}{\alpha}\frac{\left|\int\limits_0^x K(x-t)\left[u\left(t\right) - v\left(t\right)\right]dt\right|}{\left[F(x)\right]^{\alpha-1}} \leq \frac{K'(\mu)}{\alpha K'(0)}e^{\beta x}x^{2/(\alpha-1)}\rho_b\left(u,v\right),$$

whence

$$\rho_b\left(Tu, Tv\right) \leq \frac{K'(\mu)}{\alpha\cdot K'(0)}\cdot\rho_b\left(u,v\right), \tag{13}$$

i.e., the operator T, by condition (10) is a contractive operator. Hence, based on the contraction mapping principle, the equation $u = Tu$ has the unique solution $u^*(x) \in P_b$, which can be found by the formula $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, and the estimate (12) is valid.

The only thing left to show is that equation (4) has a unique solution in the cone Q_0. Suppose $P_\infty = \cup_{b>0}P_b$. Since equation (4) has the unique solution in P_b at any $b > 0$ and the contraction coefficient in (13) does not depend on b, equation (4) has the unique solution $u^*(x)$ in P_∞. Since any solution of equation (4) in Q_0 satisfies a priori estimates (7), this solution will also be the only one in Q_0.

Theorem 1 is proved.

Let us finally proceed to the study of integro-differential equation (1).

The following lemma establishes the relationship between integro-differential equation (1) and integral equation (4).

Lemma 4. Let conditions (2) and (3) be satisfied. Then any solution of equation (1) in the cone Q_0^2 is a solution to integral equation (4) in the cone Q_0. Conversely, if conditions (2), (3), and the additional condition

$$\lim_{x\to 0}\frac{\int\limits_0^x K'(x-t)\left[\int\limits_0^t K\left(s\right)ds\right]^{1/(\alpha-1)}dt}{\left[\int\limits_0^x K\left(x-t\right)\cdot t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}=0, \tag{14}$$

are satisfied, then any solution of integral equation (4) in the cone Q_0 belongs to the cone Q_0^2 and is a solution of equation (1).

Proof. Initially, prove the first part of the lemma. Assume $u(x)\in Q_0^2$ and is a solution to equation (1). Then, applying the integration by parts formula twice by identity (1) taking into account conditions (2) and (3), obtain:

$$u^{\alpha}(x)=\int\limits_0^x h(x-t)\,du(t)+\int\limits_0^x k(x-t)\,du'(t)=\int\limits_0^x u(t)h'(x-t)dt+\int\limits_0^x u'(t)k'(x-t)dt=$$

$$=\int\limits_0^x h'(x-t)u(t)dt+\int\limits_0^x u(t)k''(x-t)dt=\int\limits_0^x K(x-t)u(t)dt,$$

i.e. $u(x)\in Q_0$ and is a solution of integral equation (4).

Next, prove the second part of the lemma. Let $u(x)\in Q_0$ be the solution of integral equation (4). Therefore by lemma 1, $u(x)$ does not decrease on $[0,\infty)$ and is twice continuously differentiable on $(0,\infty)$, i.e. $u\in C^2\,(0,\infty)$ and satisfies the inequalities $F(x)\le u(x)\le G(x)$. Prove that $u'(0)=0$. By identity (4) in view of condition (5) get

$$\alpha u^{\alpha-1}(x)u'(x)=\int\limits_0^x K'(x-t)u(t)dt+K(0)u(x)=\int\limits_0^x K'(x-t)u(t)\,dt,$$

whence

$$u'(x)=\frac{\int\limits_0^x K'(x-t)u(t)\,dt}{\alpha\cdot\left[u^{\alpha}(x)\right]^{(\alpha-1)/\alpha}}=\frac{\int\limits_0^x K'(x-t)u(t)\,dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)u(t)dt\right]^{(\alpha-1)/\alpha}}\ge 0. \tag{15}$$

Employing a priori estimates (7), by (15) obtain:

$$0\le u'(x)\le\frac{\int\limits_0^x K'(x-t)G(t)\,dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)F(t)\,dt\right]^{(\alpha-1)/\alpha}}=\frac{\int\limits_0^x K'(x-t)\left[\frac{\alpha-1}{\alpha}\cdot\int\limits_0^t K(s)\,ds\right]^{1/(\alpha-1)}dt}{\alpha\cdot\left[\int\limits_0^x K(x-t)c(\alpha)t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}=$$

$$=\left[\frac{\alpha-1}{\alpha}\right]^{1/(\alpha-1)}\frac{1}{\alpha\cdot[c(\alpha)]^{(\alpha-1)/\alpha}}\cdot\frac{\int\limits_0^x K'(x-t)\left[\int\limits_0^t K(s)\,ds\right]^{1/(\alpha-1)}dt}{\left[\int\limits_0^x K(x-t)\cdot t^{2/(\alpha-1)}dt\right]^{(\alpha-1)/\alpha}}\to 0\quad\text{at}\quad x\to 0,$$

by virtue of condition (14). Therefore $u'(0)=0$.

Thus, $u\in C^2\,[0,\infty)$, $u\,(0)=u'(0)=0$ and $u(x)>0$ at $x>0$, i.e. $u\in Q_0^2$. All that remains is to prove that $u(x)$ is a solution of equation (1). Employing the equality $K(x)=h'(x)+k''(x)$, commutative property of convolution and applying the integration-by-parts formula twice, taking into account conditions (2) and (3), by identity (4) we obtain:

$$u^{\alpha}(x) = \int_0^x [h'(t) + k''(t)]u(x-t)\,dt = \int_0^x u(x-t)dh(t) + \int_0^x u(x-t)dk'(t) =$$

$$= \int_0^x h(t)u'(x-t)dt + \int_0^x k'(t)u'(x-t)\,dt = \int_0^x h(x-t)u'(t)\,dt + \int_0^x u'(x-t)\,dk(t) =$$

$$= \int_0^x h(x-t)u'(t)dt + \int_0^x k(t)u''(x-t)dt = \int_0^x h(x-t)u'(t)\,dt + \int_0^x k(x-t)u''(t)\,dt,$$

i.e. $u(x)$ is a solution of equation (1).

Lemma 4 is proved.

Lemma 4 implies that under conditions (2), (3), and (14) integro-differential equation (1) and integral equation (4) are simultaneously solvable or not, while they have the same solutions. Therefore, based on Theorem 1, the following fundamental theorem is true.

Theorem 2. If conditions (2), (3), and (14) are satisfied, then equation (1) has a unique solution u^* in Q_0^2 (and in the space P_b at any $b > 0$). This solution can be found in the space P_b using the Picard successive approximation method $u_n = Tu_{n-1}$, $n \in \mathbb{N}$, which converge to it according to the metric ρ_b at any $b < \infty$, and estimate convergence rate (12) is valid.

Example 2. When $\alpha > 1$, $h(x) = x^2$ and $k(x) = x^3$, i.e., at $K(x) = 8x$, in the cone Q_0 integral equation (4) has the unique solution

$$u^*(x) = \left[\frac{4(\alpha-1)^2}{\alpha(\alpha+1)}\right]^{1/(\alpha-1)} x^{2/(\alpha-1)}.$$

When $K(x) = 8x$ condition (14) takes the form

$$A(\alpha) \cdot \lim_{x\to 0} x^{(3-\alpha)/(\alpha-1)} = 0, \quad \text{where} \quad A(\alpha) = \frac{8 \cdot 4^{\frac{1}{\alpha-1}}(\alpha-1)}{\alpha+1}\left[\frac{\alpha(\alpha+1)}{4(\alpha-1)^2}\right]^{(\alpha-1)/\alpha}.$$

Hence, for $1 < \alpha < 3$, the function $u^*(x)$ is also the unique solution of integro-differential equation (1) in the cone Q_0^2.

In particular, when $\alpha = 2$, $h(x) = x^2$ and $k(x) = x^3$, equations (1) and (4) have the unique solution $u(x) = \frac{2}{3}x^2$ in the cones Q_0^2 and Q_0, respectively.

Note also that $u^* \in Q_0^2$ only if $1 < \alpha < 3$, since

$$(u^*(x))' = \left[\frac{4(\alpha-1)^2}{\alpha(\alpha+1)}\right]^{1/(\alpha-1)} \frac{2}{\alpha-1} x^{(3-\alpha)/(\alpha-1)}$$

and therefore $u^* \notin Q_0^2$ at $\alpha \geq 3$. This shows that condition (14) is essential to the validity of Lemma 4 and Theorem 2.

Following the monograph [6; 211] it can be proved that for $0 < \alpha < 1$, as in the case of the corresponding linear equations obtained for $\alpha = 1$, equations (1) and (4) have only a trivial solution $u(x) \equiv 0$ in the cone of the space of functions continuous on $C \in [0, \infty)$ consisting of non-negative functions continuous on the half-axis $[0, \infty)$.

Consequently, based on the results obtained non-linear homogeneous integral and integro-differential equations type (1) and (4) except for the trivial solution $u(x) \equiv 0$ can have the non-trivial solution

$u(x) \neq 0$ at $\alpha > 1$ strictly positive at $x > 0$. This is the fundamental difference between the theory of the considered nonlinear equations and the well-developed so far theory of the corresponding linear homogeneous integral and integro-differential equations, which have only the trivial solution $u(x) \equiv 0$. In addition, the theory of nonlinear equations differs from the theory of the corresponding linear equations not only in the obtained results but also in research methods related to the choice of space and nonlinearity properties.

In conclusion, following the works [9–12], it is possible to study integro-differential equations of the form (1) with variable coefficients and inhomogeneities in the linear part, as well as systems of such equations. Other methods for studying nonlinear equations of the convolution type are given in many research works, such as [13], [14].

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С.Н. Асхабов

Шешен мемлекеттік педагогикалық университеті;
А.А. Қадыров атындағы Шешен мемлекеттік университеті, Грозный, Ресей

Айырымдық ядролары және дәрежелік сызықтық еместігі бар екінші ретті интегро-дифференциалдық теңдеу туралы

Мақалада айырымдық ядролары және дәрежелік сызықтық еместігі бар екінші ретті интегро-дифференциалдық теңдеу зерттелген. Осы теңдеудің цилиндрлік резервуардан изотропты біртекті кеуекті ортаға сұйықтықтың инфильтрациясы, газ толтырылған құбырлардағы соққы толқындарының таралуы және т.б. процестерін сипаттау кезінде туындайтын үйірткі түріндегі интегралдық теңдеумен байланысы анықталды. Қолданбалы жағынан осы интегралдық теңдеудің теріс емес үзіліссіз шешімдері ерекше қызығушылық тудырады, сондықтан интегро-дифференциалдық теңдеудің сәйкес шешімдері үзіліссіз-дифференциалданатын кеңістік конусында ізделінеді. Көрсетілген интегралдық теңдеудің кез келген шешімі үшін екі жақты априорлы бағалар алынған, оның негізінде шешімнің бар болуы мен бірегейлігінің ғаламдық теоремасы салмақты метрика әдісімен дәлелденген. Берілген интегралдық-дифференциалдық теңдеудің кез келген шешімі бір мезгілде интегралдық теңдеудің шешімі болатыны және керісінше ядроға қосымша шарт қойылған кезде осы интегралдық теңдеудің кез келген шешімі осы интегралдық-дифференциалдық теңдеудің шешімі болатыны көрсетілген. Осы нәтижелерді пайдалана отырып, интегро-дифференциалдық теңдеудің бар болуы, бірегейлігі және шешімін табу әдісі туралы ғаламдық теорема дәлелденді. Бұл шешімді дәйекті Пикард типті жуықтаулар әдісімен табуға болатыны көрсетіліп, олардың жинақтылық жылдамдығына баға белгіленген. Алынған нәтижелерді көрсету үшін мысалдар келтірілген.

Кілт сөздер: интегро-дифференциалдық теңдеу, дәрежелік сызықтық еместік, айырымдық ядролар, салмақты метрика әдісі.

С.Н. Асхабов

Чеченский государственный педагогический университет;
Чеченский государственный университет имени А.А.Кадырова, Грозный, Россия

Об интегро-дифференциальном уравнении второго порядка с разностными ядрами и степенной нелинейностью

В статье рассмотрено интегро-дифференциальное уравнение второго порядка с разностными ядрами и степенной нелинейностью. Установлена связь этого уравнения с интегральным уравнением типа свёртки, возникающим при описании процессов инфильтрации жидкости из цилиндрического резервуара в изотропную однородную пористую среду, распространения ударных волн в трубах, наполненных газом, и других. Поскольку, с прикладной точки зрения, особый интерес представляют неотрицательные непрерывные решения этого интегрального уравнения, решения соответствующего интегро-дифференциального уравнения разыскиваются в конусе пространства непрерывно-дифференцируемых функций. Получены двусторонние априорные оценки для любого решения указанного интегрального уравнения, на основе которых методом весовых метрик доказана глобальная теорема существования и единственности решения. Показано, что любое решение данного интегро-дифференциального уравнения является одновременно и решением интегрального уравнения, и, обратно, при дополнительном условии на ядро, что любое решение этого интегрального уравнения является решением данного интегро-дифференциального уравнения. Используя указанные результаты, доказана глобальная теорема о существовании, единственности и способе нахождения решения интегро-дифференциального уравнения. Показано, что это решение можно найти методом последовательных приближений пикаровского типа, при этом и установлена оценка скорости их сходимости. Приведены примеры, иллюстрирующие полученные результаты.

Ключевые слова: интегро-дифференциальное уравнение, степенная нелинейность, разностные ядра, метод весовых метрик.

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