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On the nonlocal problems in time for subdiffusion equations with the Riemann-Liouville derivatives

Initial boundary value problems with a time-nonlocal condition for a subdiffusion equation with the Riemann-Liouville time-fractional derivatives are considered. The elliptical part of the equation is the Laplace operator, defined in an arbitrary $N-$ dimensional domain Ω with a sufficiently smooth boundary $\partial\Omega$. The existence and uniqueness of the solution to the considered problems are proved. Inverse problems are studied for determining the right-hand side of the equation and a function in a time-nonlocal condition. The main research tool is the Fourier method, so the obtained results can be extended to subdiffusion equations with a more general elliptic operator.

Keywords: time-nonlocal problems, Riemann-Liouville derivatives, subdiffusion equation, inverse problems.

Introduction

Let $\beta < 0$, $0 < \rho < 1$ and a function $q(t)$ be defined on $[0, \infty)$. Denote by $J_t^\beta q(t)$ and $\partial_t^\rho q(t)$ the fractional integrals and the Riemann-Liouville derivatives, respectively, defined as (see, e.g. [1; 14]):

$$J_t^\beta q(t) = \frac{1}{\Gamma(-\beta)} \int_0^t \frac{q(\xi)}{(t-\xi)^{\beta+1}} d\xi, \qquad \partial_t^\rho q(t) = \frac{d}{dt} J_t^{\rho-1} q(t), \quad t > 0.$$

Let Ω be an arbitrary N – dimensional domain with a sufficiently smooth boundary $\partial\Omega$.

Consider the following time-nonlocal problem:

$$\partial_t^\rho u(x,t) - \triangle u(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \le T; \tag{1}$$

$$u(x,t)\big|_{\partial\Omega} = 0; \tag{2}$$

$$J_t^{\rho-1} u(x,t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1} u(x,t) + \varphi(x), \quad 0 < \xi \le T, \quad x \in \overline{\Omega}, \tag{3}$$

where $f(x,t)$, $\varphi(x)$ are given functions, α is a constant, ξ is a fixed point and $\triangle = \sum\limits_{k=1}^N \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. This problem is also called *the forward problem.*

We note the following property of the Riemann-Liouville integrals, which simplifies the verification of the initial condition (3) (see, e.g. [1; 104]):

$$\lim_{t\to+0} J_t^{\rho-1} u(x,t) = \Gamma(\rho) \lim_{t\to+0} t^{1-\rho} u(x,t).$$

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From here, in particular, it follows that the solution to the forward problem can have a singularity at zero $t = 0$ of order $t^{\rho-1}$.

When solving the forward problem, we will first solve various auxiliary problems for equations with the Riemann-Liouville derivative. We will also consider inverse problems. The definition of a classical solution in all these cases is exactly the same. As an example, we present the definition of the classical solution to the forward problem (1)–(3).

Definition 1. A function $u(x,t)$ with the properties

1 $t^{1-\rho}u(x,t) \in C(\overline{\Omega} \times [0,T])$,

2 $\partial_t^\rho u(x,t),\ \triangle u(x,t) \in C(\overline{\Omega} \times (0,T])$

and satisfying conditions (1)–(3) is called *the solution* to the forward problem.

The main goal of this work is to study the influence of parameter α on the correctness of problem (1)–(3). In this regard, we will apply the Fourier method, which ensures the consideration of the following spectral problem

$$\begin{cases} -\triangle v(x) = \lambda v(x), & x \in \Omega; \\ v(x)\big|_{\partial\Omega} = 0. \end{cases} \tag{4}$$

Since the boundary $\partial\Omega$ is sufficiently smooth, this problem has a complete in $L_2(\Omega)$ set of orthonormal eigenfunctions $\{v_k(x)\}$, $k \geq 1$, and a countable set of positive eigenvalues $\{\lambda_k\}$, (see, e.g., [2–4]). It is convenient to assume that $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$.

We note that the method proposed here, based on the Fourier method, is applicable to equation (1) with an arbitrary elliptic differential operator $A(x,D)$, if only the corresponding spectral problem has a complete system of orthonormal eigenfunctions in $L_2(\Omega)$.

We also note that if $\alpha = 0$, then the considering forward problem passes to *the backward problem*, which is well-studied in work [5]. The backward problem for equation (1) with the Caputo derivative was studied in [6–8]. Therefore, further we assume that $\alpha \neq 0$. About backward problems, we note only the following: These problems are not well-posed in the sense of Hadamard, i.e., a small change in function φ in condition (3) leads to a large change in the solution.

As will be shown below, if $\alpha \notin [0,1)$, then, under standard conditions on the given functions f and φ, problem (1)–(3) is unconditionally solvable and has a unique solution. If $\alpha \in (0,1)$, then the solvability of the problem depends on whether there exists an eigenvalue λ_{k_0} of the spectral problem (4) such that $E_\rho(-\lambda_{k_0} t^\rho) = \alpha$ and what is the multiplicity p_0 of this eigenvalue λ_{k_0} (here E_ρ is the Mittag-Leffler function, see the definition below). If such an eigenvalue exists, then for the solution to the problem to exist, it is necessary that each function f and φ satisfy p_0 additional orthogonality conditions. Moreover, the solution of the problem will not be unique. If there is no eigenvalue λ_{k_0} for which $E_\rho(-\lambda_{k_0} t^\rho) = \alpha$, then problem (1)–(3) is again unconditionally solvable.

We will also study two inverse problems for determining the right-hand side of the equation and function φ in the nonlocal condition (3), respectively. In this case, for both inverse problems, as an additional condition, we take the condition

$$u(x,\theta) = \Psi(x), \quad 0 < \theta \leq T, \quad \theta \neq \xi, \quad x \in \overline{\Omega}. \tag{5}$$

Here, to avoid the uniqueness problem, we will assume that $\alpha \geq 1$. In the case of the inverse problem of determining the right-hand side of the equation, we will assume that f depends only on the spatial variables x: $f = f(x)$.

Note that all these problems for equation (1) with the fractional Caputo derivative were considered in [9]. However, in this work, the existence of a generalized solution to the problems is proved. The convenience of studying the generalized solution by the Fourier method lies in the fact that when proving the convergence of the corresponding series, one can use the Parseval equality and reduce the question of the convergence of functional series to the study of the convergence of numerical series. When proving uniform convergence, this approach does not work. Therefore, in the present paper, we apply the lemma of Krasnoselskii et al. [10], which reduces the study of uniform convergence to the study of convergence in $L_2(\Omega)$.

Usually, to determine the solution of non-stationary differential equations uniquely, an initial condition is specified. However, in some cases, non-local conditions are used, for example, in the form of an integral over time (see [11], in the case of diffusion equations, [12] for fractional-order equations), or in the form of a relationship between the value of the solution at the initial and final times (see [13], [14]). We also note papers [15], [16], where boundary value problems given with fractional derivatives are studied.

As for the inverse problem of determining the function φ, we point that such a problem was studied only in the work [17] (with the exception of work [9], which was mentioned above). The authors of [17] considered this problem for the subdiffusion equation, which includes the fractional Caputo derivative, the elliptic part of which is a differential expression with two variables and constant coefficients.

The inverse problems of determining the right-hand side (the heat source density) of various subdiffusion equations have been considered by many researchers (see, e.g., [18]). We note that the inverse problem of determining the right-hand side of the equation given in an abstract form $f(x,t)$ has not yet been studied. The obtained results deal with the separated source term $s(t)f(x)$. The appropriate choice of the overdetermination depends on the choice whether the unknown is $s(t)$ or $f(x)$. It should be noted that studies of inverse problems, where the function $s(t)$ is the unknown, are relatively few (see, e.g., [18] in the case of fractional order equations and [19]–[21] in the case of equations of integer order).

Many authors have considered an equation in which $s(t) \equiv 1$ and $f(x)$ is unknown (see, e.g., [22]–[40]). Let us mention just a few of these works. The case of subdiffusion equations whose elliptic part is an ordinary differential expression is considered in [22]–[28]. The authors of the papers [29]–[33] studied subdiffusion equations in which the elliptic part is either a Laplace operator or a second-order operator. The article [34] examined the inverse problem for an abstract subdiffusion equation with the Cauchy condition. In article [34] and in most other articles, including [29]–[32], the Caputo derivative is used as a fractional derivative. Recent articles [35]–[36] are devoted to the inverse problem for the subdiffusion equation with the Riemann-Liouville derivative.

In [33], [38], [39], non-self-adjoint differential operators (with non-local boundary conditions) were taken as the elliptical part of a subdiffusion equation, and solutions to the inverse problem were found in the form of bioorthagonal series.

In our previous work [40], we examined the inverse problem for the simultaneous determination of the order of the Riemann-Liouville fractional derivative and the source function in the subdiffusion equations. Using the classical Fourier method, the authors proved the uniqueness and existence of a solution to this inverse problem.

We also note works [41]–[44] close to the given topic in which inverse problems of determining boundary functions in problems of control of heat propagation processes are studied.

1 Preliminaries

In this section, we formulate the lemma noted above from the study by Krasnoselskii et al. [10], the fundamental result of V.A. Il'in [3] on the convergence of the Fourier coefficients and recall some properties of the Mittag-Leffler function.

Let A stand for the operator acting in $L_2(\Omega)$ as $Ag(x) = -\triangle g(x)$ with the domain of definition $D(A) = \{g \in C^2(\overline{\Omega}) : g(x) = 0, x \in \partial\Omega\}$. We denote the self-adjoint extension of A in $L_2(\Omega)$ by $\hat{A}$.

To formulate the indicated lemma, it is necessary to introduce the power of operator $\hat{A}$.

Let σ be an arbitrary real number. The power of operator A, acting in $L_2(\Omega)$ is defined as:

$$\hat{A}^\sigma g(x) = \sum_{k=1}^{\infty} \lambda_k^\sigma \, g_k \, v_k(x), \quad g_k = (g, v_k),$$

and the domain of definition has the form

$$D(\hat{A}^\sigma) = \{g \in L_2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 < \infty\}.$$

For elements of $D(\hat{A}^\sigma)$ we introduce the norm

$$||g||_\sigma^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} |g_k|^2 = ||\hat{A}^\sigma g||^2.$$

The following lemma plays an essential role in our reasoning (see, e.g., [10; 453]).

Lemma 1 . Let $\sigma > \frac{N}{4}$. Then operator $\hat{A}^{-\sigma}$ continuously maps the space $L_2(\Omega)$ into $C(\overline{\Omega})$, and moreover, the following estimate holds

$$\|\hat{A}^{-\sigma} g\|_{C(\Omega)} \leq C \|g\|_{L_2(\Omega)}.$$

When proving the existence of solutions to forward and inverse problems, it is necessary to study the convergence of series of the form

$$\sum_{k=1}^{\infty} \lambda_k^\tau |h_k|^2, \ \tau > \frac{N}{2}, \tag{6}$$

where h_k is the Fourier coefficient of function $h(x)$. In the case of integers τ, the conditions for the convergence of such series in terms of the membership of the function $h(x)$ in classical Sobolev spaces $W_2^k(\Omega)$ were obtained in the work of V.A. Il'in [3]. To formulate these conditions, we introduce the class $\dot{W}_2^1(\Omega)$ as the closure in the $W_2^1(\Omega)$ norm of the set of all functions that are continuously differentiable in Ω and vanish near the boundary of Ω.

So, if function $h(x)$ satisfies the conditions

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+1}(\Omega), \quad \text{and} \quad h(x), \triangle h(x), \cdots, \triangle^{\left[\frac{N}{4}\right]} h(x) \in \dot{W}_2^1(\Omega), \tag{7}$$

then the number series (6) (we can take $\tau = \frac{N}{2} + 1$ if N is even, and $\tau = \frac{N+1}{2}$ if N is odd) converges.

Similarly, if in (6) we replace τ by $\tau + 2$, then the convergence conditions will have the form:

$$h(x) \in W_2^{\left[\frac{N}{2}\right]+3}(\Omega), \quad \text{and} \quad h(x), \triangle h(x), \cdots, \triangle^{\left[\frac{N}{4}\right]+1} h(x) \in \dot{W}_2^1(\Omega). \tag{8}$$

Next, let us remind some properties of the Mittag-Leffler functions. For $0 < \rho < 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function with two parameters (see, e.g. [1; 12]):

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}.$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

In what follows, we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The estimate has the form (see, e.g. [45; 136])

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0, \tag{9}$$

where μ is an arbitrary complex number. This estimate essentially follows from the following asymptotic estimate (see, e.g. [45; 134]):

$$E_{\rho,\mu}(-t) = \frac{t^{-1}}{\Gamma(\mu-\rho)} + O\big(t^{-2}\big). \tag{10}$$

We will also use a coarser estimate with a positive number λ and $0 < \varepsilon < 1$:

$$|t^{\rho-1} E_{\rho,\rho}(-\lambda t^\rho)| \leq \frac{Ct^{\rho-1}}{1+\lambda t^\rho} \leq C\lambda^{\varepsilon-1} t^{\varepsilon\rho-1}, \quad t > 0, \tag{11}$$

which is easy to verify. Indeed, let $t^\rho\lambda < 1$, then $t < \lambda^{-1/\rho}$ and

$$t^{\rho-1} = t^{\rho-\varepsilon\rho} t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$

If $t^\rho\lambda \geq 1$, then $\lambda^{-\varepsilon} \leq t^{\varepsilon\rho}$ and

$$\lambda^{-1} t^{-1} = \lambda^{-1+\varepsilon}\lambda^{-\varepsilon} t^{-1} \leq \lambda^{\varepsilon-1} t^{\varepsilon\rho-1}.$$

Proposition 1. The Mittag-Leffler function of negative argument $E_\rho(-x)$ is monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_\rho(-x) < 1. \tag{12}$$

Proof of this proposition can be found, for example, in [9].

Proposition 2. Let $\rho > 0$ and $\lambda \in \mathbb{C}$. Then for all positive t one has

$$\int\limits_0^t \eta^{\rho-1} E_{\rho,\rho}(\lambda\eta^\rho) d\eta = t^\rho E_{\rho,\rho+1}(\lambda t^\rho), \tag{13}$$

and

$$J_t^{\rho-1}\Big(t^{\rho-1} E_{\rho,\rho}(\lambda t^\rho)\Big) = E_\rho(\lambda t^\rho). \tag{14}$$

Proof of this proposition can be found, for example, in [45; 120].

2 Well-posedness of the forward problem

First, we consider the problem for the homogeneous equation:

$$\begin{cases} \partial_t^\rho w(x,t) - \triangle w(x,t) = 0, \quad x \in \Omega \quad 0 < t \leq T; \\ w(x,t)\big|_{\partial\Omega} = 0; \\ J_t^{\rho-1} w(x,t)\big|_{t=\xi} = \alpha \lim\limits_{t\to 0} J_t^{\rho-1} w(x,t) + \psi(x), \quad 0 < \xi \leq T, \quad x \in \overline{\Omega}, \end{cases} \tag{15}$$

where $\psi(x)$ is a given function.

Theorem 1. Let function $\psi(x)$ satisfy conditions (7).

If $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $E_\rho(-\lambda_k\xi^\rho) \neq \alpha$ for all $k \geq 1$, then problem (15) has a unique solution, which has the form

$$w(x,t) = \sum_{k=1}^{\infty} \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x), \tag{16}$$

where ψ_k is the Fourier coefficient of function $\psi(x)$.

If $\alpha \in (0,1)$ and $E_\rho(-\lambda_{k_0}\xi^\rho) = \alpha$ for some eigenvalue λ_{k_0} with the multiplicity p_0, then we assume that the orthogonality conditions

$$\psi_k = (\psi, v_k) = 0, \ \ k \in K_0 = \{k_0, k_0+1, ..., k_0+p_0-1\} \tag{17}$$

are satisfied. Then solutions to problem (15) have the form

$$w(x,t) = \sum_{k\notin K_0} \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}\, t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)v_k(x) + \sum_{k\in K_0} b_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)v_k(x), \tag{18}$$

with arbitrary coefficients b_k, $k \in K_0$.

Proof. In accordance with the Fourier method, we will look for a solution to problem (15) in the form of a series:

$$w(x,t) = \sum_{k=1}^{\infty} T_k(t)v_k(x),$$

where $T_k(t)$, $k \geq 1$, are solutions to the nonlocal problems:

$$\partial_t^\rho T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \leq T, \tag{19}$$

$$J_t^{\rho-1}T_k(t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1}T_k(t) + \psi_k. \tag{20}$$

Let us denote

$$\lim_{t\to 0} J_t^{\rho-1}T_k(t) = b_k. \tag{21}$$

Then, the unique solution of the equation (19), that satisfies the condition (21) has the form $T_k(t) = b_k t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)$ (see, e.g. [46; 173], [1; 16], and [47]).

Equality (14) implies

$$J_t^{\rho-1}T_k(t)\bigg|_{t=\xi} = b_k E_\rho(-\lambda_k\xi^\rho).$$

Therefore, from the nonlocal condition (20) we obtain

$$b_k\big(E_\rho(-\lambda_k\xi^\rho) - \alpha\big) = \psi_k. \tag{22}$$

By virtue of property (12) of the Mittag-Leffler function, $E_\rho(-\lambda_k\xi^\rho) \neq \alpha$ for all $\alpha \geq 1$ and $\alpha < 0$ (note, $\xi > 0$ and $\lambda_k > 0$). Therefore, from (22) we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha|\psi_k|, \quad k \geq 1, \tag{23}$$

where C_α is a constant.

Let $0 < \alpha < 1$. Then according to Proposition 1, there is a unique $\lambda_0 > 0$ such that $E_\rho(-\lambda_0\xi^\rho) = \alpha$. If there is no eigenvalue equal to λ_0 , then the estimate in (23) holds with some constant $C_\alpha > 0$.

Thus, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then the formal solution of problem (15) has the form (16).

Finally, let $0 < \alpha < 1$ and there is an eigenvalue equal to λ_0, having the multiplicity p_0: $\lambda_k = \lambda_0$ for $k = k_0, k_0+1, ..., k_0+p_0-1$. Then the nonlocal problem (19), (20) has a solution if the boundary function $\psi(x)$ satisfies the orthogonality conditions (17). Since $\psi_k = 0$, then arbitrary numbers b_k are solutions of equation (22). For all other k we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha|\psi_k|, \quad k \notin K_0.$$

Thus, the formal solution to problem (15) in this case has the form (18).

Let us show that the operators $A = -\triangle$ and ∂_t^ρ can be applied term-by-term to series (16) and the resulting series converges uniformly in $(x,t) \in \overline{\Omega} \times (0,T]$; for series (18), this question is considered in a similar way.

Let $S_j(x,t)$ be the partial sum of series (16). Then

$$-\triangle S_j(x,t) = \sum_{k=1}^{j} \lambda_k \frac{\psi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x).$$

Using the equality

$$\hat{A}^{-\sigma} v_k(x) = \lambda_k^{-\sigma} v_k(x),$$

with $\sigma > \frac{N}{4}$ and applying Lemma 1 for $g(x) = -\triangle S_j(x,t)$, we have

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \le C \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \frac{\psi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) \right|^2, \ t > 0.$$

Here, to estimate the $L_2(\Omega)$ norm, we applied the Parseval's equality.

Apply estimates (9) and (23) to obtain

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \le C_\alpha t^{2\rho-2} \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \frac{\psi_k}{1+\lambda_k t^\rho} \right|^2 \le C_\alpha t^{-2} \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ \tau = 2\sigma > \frac{N}{2}.$$

Therefore, if $\psi(x)$ satisfies conditions (7), then $-\triangle u(x,t) \in C(\overline{\Omega} \times (0,T])$. From equation (15) one has $\partial_t^\rho u(x,t) = \triangle u(x,t)$, $t > 0$, and the above estimates imply

$$||\partial_t^\rho w(x,t)||^2_{C(\Omega)} \le C_\alpha t^{-2} \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ t > 0,$$

which means $\partial_t^\rho w(x,t) \in C(\overline{\Omega} \times (0,T])$.

For $S_j(x,t)$, taking into account estimate (9), we obtain

$$||t^{1-\rho} S_j(x,t)||^2_{C(\Omega)} \le C_\alpha \sum_{k=1}^{j} \lambda_k^\tau |\psi_k|^2, \ \tau > \frac{N}{2}.$$

Hence $t^{1-\rho} w(x,t) \in C(\overline{\Omega} \times [0,T])$, which was required by the definition of the solution to problem (15).

The uniqueness of the solution to problem (15) is proved in exactly the same way as in work [9]. For the convenience of the reader, we present this proof.

It is sufficient to show that the solution to the problem:

$$\begin{cases} \partial_t^\rho w(x,t) - \triangle w(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T; \\ w(x,t)\big|_{\partial\Omega} = 0; \\ J_t^{\rho-1} w(x,t)\big|_{t=\xi} = \alpha \lim\limits_{t\to 0} J_t^{\rho-1} w(x,t), \quad 0 < \xi \le T, \quad x \in \overline{\Omega}, \end{cases}$$

is identically equal to zero.

Let $w_k(t) = (w(x,t), v_k(x))$. Since operator $A = -\triangle$ is self-adjoint, one has

$$\partial_t^\rho w_k(t) = (\partial_t^\rho w(x,t), v_k(x)) = (Aw(x,t), v_k(x)) = (w(x,t), Av_k(x)) = -\lambda_k w_k(t)$$

or

$$\partial_t^\rho w_k(t) = -\lambda_k w_k(t) \tag{24}$$

and the nonlocal condition implies

$$J_t^{\rho-1} w_k(t)\big|_{t=\xi} = \alpha \lim_{t\to 0} J_t^{\rho-1} w_k(t). \tag{25}$$

Let us denote $\lim\limits_{t\to 0} J_t^{\rho-1} w_k(t) = b_k$. Then the unique solution to the differential equation (24) with this initial condition has the form $w_k(t) = b_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)$ (see, e.g.[46; 174]). From equality (14) and the nonlocal conditions of (25) we obtain the following equation to find the unknown numbers b_k:

$$b_k\big(E_\rho(-\lambda_k \xi^\rho) - \alpha\big) = 0. \tag{26}$$

If $\alpha \notin [0,1)$, then by virtue of the Proposition 1 we obtain $b_k = 0$ for all $k \geq 1$. If $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all k, then $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ and therefore $b_k = 0$. Hence, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all k, we have all b_k are equal to zero, therefore $w_k(t) = 0$. By virtue of completeness of the set of eigenfunctions $\{v_k(x)\}$, we conclude that $w(x,t) \equiv 0$. Thus, problem (15) in this case has a unique solution.

Now, suppose that $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$. Then $E_\rho(-\lambda_k \xi^\rho) = \alpha$, $k \in K_0$ and therefore equation (26) has the following solution: $b_k = 0$ if $k \notin K_0$ and b_k is an arbitrary number for $k \in K_0$. Thus, in this case, there is no uniqueness of the solution to problem (15). Theorem 1 is completely proved.

Now consider the following auxiliary initial-boundary value problem:

$$\begin{cases} \partial_t^\rho \omega(x,t) - \triangle \omega(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T; \\ \omega(x,t)\big|_{\partial\Omega} = 0; \\ \lim\limits_{t\to 0} J_t^{\rho-1} \omega(x,t) = 0, \quad x \in \overline{\Omega}. \end{cases} \tag{27}$$

We have the following theorem for this problem:

Theorem 2. Let $t^{1-\rho} f(x,t)$ as a function of x satisfy conditions (7) for all $t \in [0,T]$. Then problem (27) has a unique solution and this solution has the representation

$$\omega(x,t) = \sum_{k=1}^{\infty} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] v_k(x), \tag{28}$$

where $f_k(t)$ are the Fourier coefficients of function $f(x,t)$: $f_k(t) = (f(\cdot,t), v_k)$.

Proof. It is known that the formal solution of the problem (27) has the form (28) (see, e.g. [46; 173], [47]). In order to prove that function (28) is actually a solution to the problem, it remains to substantiate this formal statement, i.e., to show that the operators $A = -\triangle$ and ∂_t^ρ can be applied term-by-term to series (28) and the resulting series converges uniformly in $(x,t) \in \overline{\Omega} \times (0;T]$.

Let $S_j(x,t)$ be the partial sum of series (28). Then

$$-\triangle S_j(x,t) = \sum_{k=1}^{j} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] \lambda_k v_k(x).$$

Let $\sigma > \frac{N}{4}$. Repeating the above reasoning based on Lemma 1, we arrive at

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \leq \left\| \hat{A}^{-\sigma} \sum_{k=1}^{j} \lambda_k^{\sigma+1} v_k(x) \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right\|^2_{C(\Omega)} \leq$$

$$\leq \left\| \sum_{k=1}^{j} \lambda_k^{\sigma+1} v_k(x) \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right\|^2_{L_2(\Omega)} \leq$$

(apply Parseval's equality to obtain)

$$\leq C \sum_{k=1}^{j} \lambda_k^{2(\sigma+1)} \left| \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right|^2, \ t > 0.$$

Then, by inequality (11) with $0 < \varepsilon < 1$ one has

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \leq C \sum_{k=1}^{j} \left[\int_0^t \eta^{\varepsilon\rho-1} (t-\eta)^{\rho-1} \lambda_k^{\sigma+\varepsilon} |(t-\eta)^{1-\rho} f_k(t-\eta)| d\eta \right]^2,$$

or, by the generalized Minkowski inequality,

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \leq C \left[\int_0^t \eta^{\varepsilon\rho-1} (t-\eta)^{\rho-1} \left(\sum_{k=1}^{j} |\lambda_k^\tau (t-\eta)^{1-\rho} f_k(t-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2, \ \tau = \sigma + \varepsilon > \frac{N}{2}.$$

Since $t^{1-\rho} f(x,t)$ as a function of x satisfies conditions (7) for all $t \in [0,T]$, then

$$|| -\triangle S_j(x,t)||^2_{C(\Omega)} \leq C, \ t \geq 0.$$

Hence $-\triangle \omega(x,t) \in C\big(\overline{\Omega} \times [0,T]\big)$ and in particular $\omega(x,t) \in C\big(\overline{\Omega} \times [0,T]\big)$.

Further, from equation (1) one has $\partial_t^\rho S_j(t) = \triangle S_j(x,t) + \sum\limits_{k=1}^{j} f_k(t) v_k(x)$, $t > 0$. Therefore, from the above reasoning, we have $\partial_t^\rho \omega(x,t) \in C\big(\overline{\Omega} \times (0,T]\big)$.

The uniqueness of the solution can be proved by the standard technique based on completeness in $L_2(\Omega)$ of the set of eigenfunctions $\{v_k(x)\}$ (see, e.g. [5]).

Theorem 2 is completely proved.

Now let us move on to solving the main problem (1)–(3). Let $\varphi(x)$ and $t^{1-\rho} f(x,t)$ (for all $t \in [0,T]$) satisfy conditions (7). If we put $\psi(x) = \varphi(x) - J_t^{\rho-1} \omega(x,t)\Big|_{t=\xi}$ and $\omega(x,t)$ and $w(x,t)$ are the solutions of problems (27) and (15) correspondingly, then function $u(x,t) = \omega(x,t) + w(x,t)$ is a solution to problem (1)–(3). Therefore, we can use the already proven assertions.

Thus, if $\alpha \notin [0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then

$$u(x,t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) + \omega_k(t) \right] v_k(x), \tag{29}$$

where

$$\omega_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta.$$

The uniqueness of the function $u(x,t)$ follows from the uniqueness of the solutions $\omega(x,t)$ and $w(x,t)$.

If $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, then

$$u(x,t) = \sum_{k \notin K_0} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) + \omega_k(t) \right] v_k(x) + \sum_{k \in K_0} b_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k(x). \tag{30}$$

The orthogonality conditions (17) have the form

$$(\varphi, v_k) = (\, J_t^{\rho-1}\omega(x,t)\Big|_{t=\xi}, v_k),\ k\in K_0;\ K_0=\{k_0, k_0+1, ..., k_0+p_0-1\}.$$

Instead of these conditions, we can take orthogonality conditions that is easy to verify:

$$(\varphi, v_k) = 0,\ (f(\cdot,t), v_k)=0,\ \text{for all } t\in[0,T],\ k\in K_0;\ K_0=\{k_0, k_0+1, ..., k_0+p_0-1\}. \tag{31}$$

Thus, we have proved the main result of this section:

Theorem 3. Let $\varphi(x)$ and $t^{1-\rho}f(x,t)$ (for all $t\in[0,T]$) satisfy conditions (7). If $\alpha\notin[0,1)$ or $\alpha\in(0,1)$, but $\lambda_k\neq\lambda_0$ for all $k\geq 1$, then problem (1)–(3) has a unique solution and this solution has the form (29).

If $\alpha\in(0,1)$ and $\lambda_k=\lambda_0$, $k\in K_0$, then we assume that the orthogonality conditions (31) are satisfied. The solution of problem (1)–(3) has the form (30) with arbitrary coefficients b_k, $k\in K_0$.

3 Inverse problem of determining the right-hand side of the equation

Let us consider *the inverse problem*

$$\begin{cases} \partial_t^\rho u(x,t) - \triangle u(x,t) = f(x), \quad 0<t\leq T; \quad x\in\Omega; \\ u(x,t)\big|_{\partial\Omega}=0; \\ J_t^{\rho-1}u(x,t)\big|_{t=\xi} = \alpha \lim\limits_{t\to 0} J_t^{\rho-1}u(x,t)+\varphi(x), \quad 0<\xi\leq T, \quad x\in\overline{\Omega}, \end{cases} \tag{32}$$

with the additional condition

$$u(x,\theta)=\Psi(x), \quad 0<\theta\leq T, \quad \theta\neq\xi, \quad x\in\overline{\Omega}, \tag{33}$$

where the unknown function $f(x)$, characterizing the action of heat sources, does not depend on t and $\Psi(x), \varphi(x)$ are given functions, $\alpha\geq 1$, ξ and θ are fixed points of $(0,T]$.

Note that if $\theta=\xi$, then the nonlocal condition in (32) coincides with the Cauchy condition $\lim\limits_{t\to 0} J_t^{\rho-1}u(x,t)=\varphi_1$ with some φ_1. In this case, this inverse problem was studied in [35].

Theorem 4. Let functions $\varphi(x), \Psi(x)$ satisfy conditions (8). Then the inverse problem (32), (33) has a unique solution $\{u(x,t), f(x)\}$ and this solution has the following form

$$f(x)=\sum_{k=1}^{\infty}\Bigg[\frac{\alpha-E_\rho(-\lambda_k\xi^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha-E_\rho(-\lambda_k\xi^\rho)]}\,\Psi_k+$$

$$+\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)+\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha-E_\rho(-\lambda_k\xi^\rho)]}\,\varphi_k\Bigg]v_k(x), \tag{34}$$

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{E_{\rho,\rho}(-\lambda_k t^\rho)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,t^{\rho-1}[\varphi_k-f_k\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)]+f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)\right]v_k(x). \tag{35}$$

Proof. Let us first show that the series (34) and (35) are formal solutions to the inverse problem. Then we show the uniform convergence and differentiability of these series.

Suppose $f(x)$ is known. Then the unique solution to problem (32) has the form (29). Since $f(x)$ does not depend on t, then, owing to formulas

$$\omega_k(t)=f_k\int\limits_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)d\eta$$

and (13), it is easy to verify that the formal solution of problem (32) has the form of (35).

Due to the additional condition (33) and completeness of the system $\{v_k(x)\}$ we obtain:

$$\frac{E_{\rho,\rho}(-\lambda_k\theta^\rho)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}[\varphi_k - f_k\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)] + f_k\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho) = \Psi_k.$$

After simple calculations, we get

$$f_k = \frac{\alpha - E_\rho(-\lambda_k\xi^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho) + \theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\,\Psi_k +$$

$$+\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho) + \theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\,\varphi_k \equiv f_{k,1} + f_{k,2}.$$

Therefore, series (34) is a formal solution of the inverse problem.

Let us prove the convergence of this series uniformly in $x \in \overline{\Omega}$.

If $F_j(x)$ is the partial sums of series (34), then by applying Lemma 1 as above, we have

$$||F_j(x)||^2_{C(\Omega)} \le \sum_{k=1}^{j} \lambda_k^{2\sigma}[f_{k,1} + f_{k,2}]^2 \le 2\sum_{k=1}^{j} \lambda_k^{2\sigma} f_{k,1}^2 + 2\sum_{k=1}^{j} \lambda_k^{2\sigma} f_{k,2}^2 \equiv 2I_{1,j} + 2I_{2,j}, \tag{36}$$

where $\sigma > \frac{N}{4}$. Since $\xi > 0$, then $\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho) > 0$. Therefore,

$$I_{1,j} \le \sum_{k=1}^{j} \left|\frac{\alpha - E_\rho(-\lambda_k\xi^\rho)}{\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)]}\right|^2 \lambda_k^{2\sigma}|\Psi_k|^2 = \sum_{k=1}^{j} \frac{\lambda_k^{2\sigma}|\Psi_k|^2}{|\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)|^2}.$$

Apply the asymptotic estimate (10) to get

$$I_{1,j} \le \sum_{k=1}^{j} \frac{\lambda_k^{2(\sigma+1)}|\Psi_k|^2}{\big(1 + O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2} \le C\sum_{k=1}^{j} \lambda_k^{\tau+2}|\Psi_k|^2,\ \tau = 2\sigma > \frac{N}{2}.$$

Since $\theta > 0$ and $\alpha \ge 1$, then $\theta^\rho E_{\rho,\rho+1}(-\lambda_k\theta^\rho)[\alpha - E_\rho(-\lambda_k\xi^\rho)] > 0$. Therefore,

$$I_{2,j} \le \sum_{k=1}^{j} \left|\frac{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)}\right|^2 \lambda_k^{2\sigma}|\varphi_k|^2 = \sum_{k=1}^{j} \frac{\lambda_k^{2(\sigma+1)}|\varphi_k|^2}{|\xi^\rho E_{\rho,\rho+1}(-\lambda_k\xi^\rho)|^2}.$$

By virtue of (10),

$$I_{2,j} \le \sum_{k=1}^{j} \frac{\lambda_k^{2(\sigma+1)}|\varphi_k|^2}{\big(1 + O\big((-\lambda_k\xi^\rho)^{-1}\big)\big)^2} \le C\sum_{k=1}^{j} \lambda_k^{\tau+2}|\varphi_k|^2,\ \tau > \frac{N}{2}.$$

Thus, if $\varphi(x), \Psi(x)$ satisfy conditions (8), then from estimates of $I_{i,j}$ and (36) we obtain $f(x) \in C(\overline{\Omega})$.

Further, the fact that function $u(x,t)$ given by the series (35) is a solution to the inverse problem is proved exactly as in Theorem 1.

The uniqueness of the solution follows from the completeness of the systems of eigenfunctions $\{v_k(x)\}$ (see [9]).

4 The inverse problem of determining function φ from the nonlocal condition

Let us assume that in forward problem (1)–(3) not only function $u(x,t)$, but also function $\varphi(x)$ from nonlocal condition (3) is unknown. As an additional condition for this inverse problem, we again take condition (5). We note that if $\theta=\xi$ in this condition, then the nonlocal condition $J_t^{\rho-1}u(x,t)\big|_{t=\xi} = \alpha \lim\limits_{t\to 0} J_t^{\rho-1}u(x,t)+\varphi(x)$ passes to the Cauchy condition $\lim\limits_{t\to 0} J_t^{\rho-1}u(x,t)=\varphi_1(x)$ (with some $\varphi_1(x)$), which is investigated, for instance, in [35].

Theorem 5. Let $t^{1-\rho}f(x,t)$ as a function of x satisfy conditions (7) for all $t\in[0,T]$ and let function $\Psi(x)$ satisfy conditions (8). Then the inverse problem (1)–(3), (5) has a unique solution $\{u(x,t),\varphi(x)\}$ and this solution has the form

$$\varphi(x)=\sum_{k=1}^{\infty}\left[\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}[\Psi_k-\omega_k(\theta)]+\omega_k(\xi)\right]v_k(x), \tag{37}$$

$$u(x,t)=\sum_{k=1}^{\infty}\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,t^{\rho-1}E_{\rho,\rho}(-\lambda_k t^\rho)+\omega_k(t)\right]v_k(x), \tag{38}$$

where

$$\omega_k(t)=\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta.$$

Proof. The solution to problem (1)–(3) has the form (38) (see Theorem 3). Therefore, condition (5) implies:

$$u(x,\theta)=\sum_{k=1}^{\infty}\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)+\omega_k(\theta)\right]v_k(x)=\Psi(x).$$

Passing to the Fourier coefficients, we have

$$\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)+\omega_k(\theta)=\Psi_k,\ k\geq 1,$$

or

$$\varphi_k=\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}[\Psi_k-\omega_k(\theta)]+\omega_k(\xi).$$

Thus, equality (37) is formally established. Now, we show that series (37) converges uniformly in $x\in\overline{\Omega}$.

Let $\Phi_j(x)$ be the partial sum of series (37). Then applying Lemma 1 as above, we arrive at

$$||\Phi_j(x)||^2_{C(\Omega)}\leq\sum_{k=1}^{j}\lambda_k^{2\sigma}\left|\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}\Big[\Psi_k-\omega_k(\theta)\Big]+\omega_k(\xi)\right|^2\leq$$

$$\leq 3\sum_{k=1}^{j}\lambda_k^{2\sigma}\left[\left|\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{\theta^{\rho-1}E_{\rho,\rho}(-\lambda_k\theta^\rho)}\right|^2\Big[|\Psi_k|^2+|\omega_k(\theta)|^2\Big]+|\omega_k(\xi)|^2\right]\equiv\Phi_j^1+\Phi_j^2+\Phi_j^3, \tag{39}$$

where $\sigma>\frac{N}{4}$. Since $|E_\rho(-\lambda_k\xi^\rho)-\alpha|\leq C$, then by virtue of the asymptotic estimate (10) we obtain

$$\Phi_j^1\leq C\sum_{k=1}^{j}\frac{\lambda_k^{2(\sigma+1)}\theta^2\Gamma^2(1-\rho)}{\big(1+O\big((-\lambda_k\theta^\rho)^{-1}\big)\big)^2}|\Psi_k|^2\leq C_1\sum_{k=1}^{j}\lambda_k^{\tau+2}|\Psi_k|^2,\ \tau=2\sigma>\frac{N}{2}.$$

Similarly, by estimates (10) and (11) we have

$$\Phi_j^2 \leq C \sum_{k=1}^{j} \frac{\lambda_k^{2\sigma+2} \theta^2 \Gamma^2(1-\rho)}{\big(1+O\big((-\lambda_k \theta^\rho)^{-1}\big)\big)^2} \left| \int_0^\theta \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(\theta-\eta) d\eta \right|^2 \leq$$

$$\leq \sum_{k=1}^{j} \frac{C_\varepsilon \lambda_k^{2\sigma+2}}{\big(1+O\big((-\lambda_k \theta^\rho)^{-1}\big)\big)^2} \left| \int_0^\theta \eta^{\varepsilon\rho-1} (\theta-\eta)^{\rho-1} \lambda_k^{\varepsilon-1} |(\theta-\eta)^{1-\rho} f_k(\theta-\eta)| d\eta \right|^2$$

(by the generalized Minkowski inequality)

$$\leq C_\varepsilon \left[\int_0^\theta \eta^{\varepsilon\rho-1} (\theta-\eta)^{\rho-1} \left(\sum_{k=1}^{j} \lambda_k^{\tau+2\varepsilon} |(\theta-\eta)^{1-\rho} f_k(\theta-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \leq$$

$$\leq C_\varepsilon \max_{t\in[0,T]} \sum_{k=1}^{j} \lambda_k^{\tau+2\varepsilon} |t^{1-\rho} f_k(t)|^2.$$

For Φ_j^3, one has

$$\Phi_j^3 \leq \sum_{k=1}^{j} \lambda_k^{2\sigma} \left| \int_0^\xi \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(\xi-\eta) d\eta \right|^2 \leq$$

$$\leq C \left[\int_0^\theta \eta^{\rho-1} (\theta-\eta)^{\rho-1} \left(\sum_{k=1}^{j} \lambda_k^{\tau} |(\theta-\eta)^{1-\rho} f_k(\theta-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \leq C \max_{t\in[0,T]} \sum_{k=1}^{j} \lambda_k^{\tau} |t^{1-\rho} f_k(t)|^2.$$

Since functions $\Psi(x)$, $f(x,t)$ satisfy conditions of the theorem, then by virtue of estimate (39), we have $\varphi(x) \in C(\overline{\Omega})$.

The fact that the function defined by equality (38) is a solution to problem (1)–(3) is proved similarly to Theorem 3.

The uniqueness of the solution of the inverse problem follows from the completeness of the system of eigenfunctions $\{v_k(x)\}$ in the space $L_2(\Omega)$ in the standard way.

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References

1 Псху А.В. Уравнения в частных производных дробного порядка / А.В. Псху. — М.: Наука, 2005. — 199 с.

2 Ильин В.А. О фундаментальных системах функций оператора Лапласа в произвольной области и об асимптотической оценке фундаментальных чисел / В.А. Ильин // Докл. АН СССР. — 1967. — 177. — № 1. — С. 25–28.

3 Ильин В.А. О разрешимости смешанных задач для гиперболического и параболического уравнений / В.А. Ильин // Успехи математических наук. —1960. — 15. — № 2(92). — С. 97–154.

4 Ладыженская О.А. Смешанная задача для гиперболического уравнения / О.А. Ладыженская. — М.: Гос. изд-во техн.-теор. лит., 1953. — 280 с.

5 Alimov Sh.A. On the backward problems in time for time-fractional subdiffusion equations / Sh.A. Alimov, R.R. Ashurov // Fractional Differential Calculus. — 2021. — 11. — No. 2. — P. 203–217. DOI: 10.7153/fdc-2021-11-14.

6 Liu J. A backward problem for the time-fractional diffusion equation / J. Liu, M. Yamamoto // Appl. Anal. — 2010. — 89. — P. 1769–1788.

7 Sakamoto K. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems / K. Sakamoto, M. Yamamoto // J. Math. Anal. Appl. — 2011. — 382. — P. 426–447.

8 Floridia G. Well-posedness for the backward problems in time for general time-fractional diffusion equation / G. Floridia, Z. Li, M. Yamamoto // Rend. Lincei Mat. Appl. — 2020. — 31. — P. 593–610.

9 Ashurov R. On the nonlocal problems in time for time-fractional subdiffusion equations / R.R. Ashurov, Yu. Fayziev // Fractal Fract. — 2022. — 6. — No. 41. https://doi.org/10.3390/fractalfract6010041

10 Красносельский М.А. Интегральные операторы в пространствах суммируемых функций / М.А. Красносельский, П.П. Забрейко, Е.И. Пустыльник, П.Е. Соболевский. — М.: Наука, 1966.

11 Pao C.V. Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions/ C.V. Pao // J. Math. Anal. Appl. — 1995. — 195. — P. 702–718.

12 Tuan N.H. On a time fractional diffusion with nonlocal in time conditions / N.H. Tuan, N.A. Triet, N.H. Luc, N.D. Phuong // Advances in Difference Equations. — 2021. — 204. https://doi.org/10.1186/s13662-021-03365-1

13 Ashyralyev A.O. Coercive stability of a multidimensional difference elliptic equation of 2m-th order with variable coefficients / A.O. Ashyralyev, P.E. Sobolevskii // Investigations in the Theory of Differential Equations. Minvuz Turkmen. SSR, Ashkhabad. — 1987. — P. 31–43.

14 Ashyralyev A.O. Coercive solvability of nonlocal boundary value problem for parabolic equations / A.O. Ashyralyev, A. Hanalyev, P.E. Sobolevskii // Abstract and Applied Analysis. — 2001. — 6. — No. 1. — P. 53–61.

15 Умаров С.Р. О некоторых краевых задачах для эллиптических уравнений с граничным оператором дробного порядка / С.Р. Умаров // Докл. РАН. — 1993. — 333. — № 6. — С. 708–710.

16 Ashurov R.R. On some boundary value problems for equations with boundary operators of fractional order / R.R. Ashurov, Yu.E. Fayziev // International Journal of Applied Mathematics. — 2021. — 34. — No. 2. — P. 283–295. DOI: 10.12732/ijam.v34i2.6

17 Yuldashev T.K. Inverse problem for a partial differential equation with Gerasimova-Caputo-type operator and degeneration / T.K. Yuldashev, B.J. Kadirkulov // Fractal Fract. — 2021. — 5. — No. 58. https://doi.org/10.3390/fractalfract5020058

18 Liu Y. Inverse problems of determining sources of the fractional partial differential equations / Liu Y., Li Z., Yamamoto M. // Handbook of Fractional Calculus with Applications. — J.A.T. Marchado Ed. DeGruyter. — 2019. — 2. — P. 411–430.

19 Ashyralyev A. Time-dependent source identification problem for the Schrodinger equation with nonlocal boundary conditions / A. Ashyralyev, M. Urun // AIP Conf. Proc. — 2019. — 2183. — Art. 070016.

20 Ashyralyev A. On the Crank-Nicolson difference scheme for the time-dependent source identification problem / A. Ashyralyev, M. Urun // Bulletin of the Karaganda University. Mathematics. — 2021. — 102(2). — P. 35–44.

21 Ashyralyev A. Time-dependent source identification Schrodinger type problem / A. Ashyralyev, M. Urun // International Journal of Applied Mathematics. — 2021. — 34(2). — P. 297–310.

22 Zhang Y. Inverse source problem for a fractional differential equations / Y. Zhang, X. Xu // Inverse Prob. — 2011. — 27. — No. 3. — P. 31–42.

23 Furati K.M. An inverse problem for a generalized fractional diffusion / K.M. Furati, O.S. Iyiola, M. Kirane // Applied Mathematics and Computation. — 2014. — 249. — P. 24–31.

24 Ismailov M.I. Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions / M.I. Ismailov, M. Cicek // Applied Mathematical Modelling. — 2016. — 40. — P. 4891–4899.

25 Kirane M. Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time / M. Kirane, A.S. Malik // Applied Mathematics and Computation. — 2011. — 218. — P. 163–170.

26 Kirane M. Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data / M. Kirane, B. Samet, B.T. Torebek // Electronic Journal of Differential Equations. — 2017. — 217. — P. 1–13.

27 Nguyen H.T. Regularized solution of an inverse source problem for a time fractional diffusion equation / H.T. Nguyen, D.L. Le, V.T. Nguyen // Applied Mathematical Modelling. — 2016. — 40. — P. 8244–8264.

28 Torebek B.T. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative / B.T. Torebek, R. Tapdigoglu // Mathematical Methods in Applied Sciences. — 2017. — 40. — P. 6468–6479.

29 Li Z. Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients / Z. Li, Y. Liu, M. Yamamoto // Applied Mathematica and Computation. — 2015. — 257. — P. 381–397.

30 Rundell W. Recovering an unknown source in a fractional diffusion problem / W. Rundell, Z. Zhang // Journal of Computational Physics. — 2018. — 368. — P. 299–314.

31 Asl N.A. Identifying an unknown time-dependent boundary source ib time-fractional diffusion equation with a non-local boundary condition / N.A. Asl, D. Rostamy // Journal of Computational and Applied Mathematics. — 2019. — 335. — P. 36–50.

32 Sun L. Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation / Sun L., Y. Zhang, T. Wei // Applied Numerical Mathematics. — 2019. — 135. — P. 228–245.

33 Malik S.A. An inverse source problem for a two-parameter anomalous diffusion equation with nonlocal boundary conditions / S.A. Malik, S. Aziz // Computers and Mathematics with applications. — 2017. — 3. — P. 7–19.

34 Ruzhansky M. Inverse source problems for positive operators / M. Ruzhansky, N. Tokmagambetov, B.T. Torebek // I: Hypoelliptic diffusion and subdiffusion equations. J. Inverse Ill-Possed Probl. — 2019. — 27. — P. 891–911.

35 Ashurov R. Inverse problem of determining the heat source density for the subdiffusion equation / R. Ashurov, O. Muhiddinova // Differential equations. — 2020. — 56. — No. 12. P. 1550–1563.

36 Ashurov R. Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator / R. Ashurov, O. Muhiddinova // Lobachevskii Journal of Mathematics. — 2021. — 42. — No. 3.

37 Shuang Zh. Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time / Zh. Shuang, R. Saima, R. Asia, K. Khadija, M.A. Abdullah // AIMS Mathematics. — 2021. — 6. — No. 11. — P. 12114–12132. DOI: 10.3934/math.2021703

38 Kirane M. An inverse source problem for a two-dimensional time fractional diffusion equation with nonlocal boundary conditions / Kirane M., Salman A.M., Mohammed A. Al-Gwaiz // Math. Meth. Appl. Sci. — 2012, doi: 10.1002/mma.2661

39 Muhammad A. An inverse problem for a family of time fractional diffusion equations /A. Muhammad, A.M. Salman, // Inverse Problems in Science and Engineering. — 2016. — 25(9). — P. 1299–1322. DOI: 10.1080/17415977.2016.1255738

40 Ashurov R.R. Determination of fractional order and source term in a fractional subdiffusion equation [Electronic resource] / R.R. Ashurov, Yu.E. Fayziev. — Access mode: https://www.researchgate.net/publication/354997348

41 Алимов Ш.А. Об одной задаче управления процессом теплообмена / Ш.А. Алимов // Докл. АН РАН. — 2008. — 421. — № 5. — С. 583–585.

42 Alimov Sh.O. On a time-optimal control problem associated with the heat exchange process / Sh.O. Alimov // Appl. Math. Optim. — 2008. — 57. — P. 58–68.

43 Fayziev Yu.E. On one control problem related to the process of heat transfer into rectangles / Yu.E. Fayziev, A. Kuchkarov, D. Nosirova // Acta NUUz. — 2017. — No. 2/2. — P. 239–244.

44 Fayziev Yu.E. On the control of the heat conduction / Yu.E. Fayziev // IIUM Engineering Journal. — 2018. — 19. — No. 1. — P. 168–177.

45 Джрбашян М.М. Интегральные преобразования и представления функций в комплексной области / М.М. Джрбашян. — М.: Наука, 1966.

46 Gorenflo R. Mittag-Leffler functions, related topics and applications / R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogozin. Springer, 2014.

47 Ашуров Р.Р. О построении решений линейного дифференциального уравнения с дробными производными и постоянными коэффициентами / Р.Р. Ашуров, Ю.Э. Файзиев // УзМЖ. — 2017. — № 3. — С. 23–41.

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Риман-Лиувилль туындысы бар субдиффузия теңдеулері үшін уақыт бойынша локальдыемес есептер туралы

Уақыт бойынша бөлшек ретті Риман-Лиувилль туындылары бар субдиффузия теңдеулері үшін уақыт бойынша локальдыемес шарты бар бастапқы-шеттік есептер қарастырылған. Теңдеудің эллипстік бөлігі $\partial\Omega$ жеткілікті тегіс шекарасы бар кез келген $N-$ өлшемді Ω облысында анықталған Лаплас операторын береді. Қарастырылып отырған есептердің шешімінің бар болуы мен жалғыздығы дәлелденді. Теңдеудің оң жағын және уақыт бойынша локальдыемес шартты функцияны анықтау үшін кері есептер зерттелді. Фурье әдісі зерттеудің негізгі құралы болып табылады, сондықтан алынған нәтижелер анағұрлым жалпы эллипстік операторы бар субдиффузия теңдеулеріне таралуы мүмкін.

Кілт сөздер: уақыт бойынша локальдыемес есептер, Риман-Лиувилль туындылары, субдиффузия теңдеуі, кері есептер.

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О нелокальных задачах по времени для уравнений субдиффузии с производными Римана–Лиувилля

Рассмотрены начально-краевые задачи с нелокальным по времени условием для уравнения субдиффузии с дробными по времени производными Римана–Лиувилля. Эллиптическая часть уравнения представляет собой оператор Лапласа, определенный в произвольной N-размерной области Ω с достаточно гладкой границей $\partial\Omega$. Доказаны существование и единственность решения рассматриваемых задач. Исследованы обратные задачи для определения правой части уравнения и функции в нелокальном во времени условии. Основным инструментом исследования является метод Фурье, поэтому полученные результаты могут быть распространены на уравнения субдиффузии с более общим эллиптическим оператором.

Ключевые слова: нелокальные по времени задачи, производные Римана-Лиувилля, уравнение субдиффузии, обратные задачи.

References

1 Pskhu, A.V. (2005). *Uravneniia v chastnykh proizvodnykh drobnogo poriadka [Fractional partial differential equations].* Moscow: Nauka [in Russian].

2 Il'in, V.A. (1967). O fundamentalnykh sistemakh funktsii operatora Laplasa v proizvolnoi oblasti i ob asimptoticheskoi otsenke fundamentalnykh chisel [Fundamental systems of functions of the Laplace operator in an arbitrary domain and asymptotic estimates for fundamental numbers]. *Doklady Akademii nauk SSSR – Proceedings of the USSR Academy of Sciences, 177*, 1, 25–28 [in Russian].

3 Il'in, V.A. (1960). O razreshimosti smeshannykh zadach dlia giperbolicheskogo i parabolicheskogo uravnenii [On the solvability of mixed problems for hyperbolic and parabolic equations]. *Uspekhi matematicheskikh nauk – Russian Math. Surveys, 15*, 2, 97–154 [in Russian].

4 Ladyzhenskaya, O.A. (1953). *Smeshannaia zadacha dlia giperbolicheskogo uravneniia [Mixed problem for a hyperbolic equation].* Moscow: Gosudarstvennoe izdatelstvo tekhniko-teoreticheskoi literatury [in Russian].

5 Alimov, Sh.A., & Ashurov, R.R. (2021). On the backward problems in time for time-fractional subdiffusion equations. *Fractional Differential Calculus, 11*, 2, 203–217. DOI: 10.7153/fdc-2021-11-14.

6 Liu, J., & Yamamoto, M. (2010). A backward problem for the time-fractional diffusion equation. *Appl. Anal., 89,* 1769–1788.

7 Sakamoto, K., & Yamamoto, M. (2011). Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl., 382,* 426–447.

8 Floridia, G., Li, Z., & Yamamoto, M. (2020). Well-posedness for the backward problems in time for general time-fractional diffusion equation. *Rend. Lincei Mat. Appl., 31,* 593–610.

9 Ashurov, R., & Fayziev, Yu. (2022). On the nonlocal problems in time for time-fractional subdiffusion equations. *Fractal Fract., 6,* 41. https://doi.org/10.3390/ fractalfract6010041

10 Krasnoselskii, M.A., Zabreiko, P.P., Pustilnik, E.I., & Sobolevskii, P.S. (1966). *Integralnye operatory v prostranstvakh summiruemykh funktsii [Integral operators in the spaces of integrable functions].* Moscow: Nauka [in Russian].

11 Pao, C.V. (1995). Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. *J. Math. Anal. Appl., 195,* 702–718.

12 Tuan, N.H., Triet, N.A., Luc, N.H., & Phuong, N.D. (2021). On a time fractional diffusion with nonlocal in time conditions. *Advances in Difference Equations, 204.* https://doi.org/10.1186/ s13662-021-03365-1

13 Ashyralyev, A.O., & Sobolevskii, P.E. (1987). Coercive stability of a multidimensional difference elliptic equation of 2m-th order with variable coefficients. *Investigations in the Theory of Differential Equations. Minvuz Turkmen. SSR,* 31–43.

14 Ashyralyev, A.O., Hanalyev, A., & Sobolevskii, P.E. (2001). Coercive solvability of nonlocal boundary value problem for parabolic equations. *Abstract and Applied Analysis, 6*, 1, 53–61.

15 Umarov, S.R. (1993). O nekotorykh kraevykh zadachakh dlia ellipticheskikh uravnenii s granichnym operatorom drobnogo poriadka [On some boundary value problems for elliptic equations with a boundary operator of fractional order]. *Doklady Rossiskoi akademi nauk – Proceeding of the Russian Academy of Sciences, 6,* 708–710 [in Russian].

16 Ashurov, R.R., & Fayziev, Yu.E. (2021). On some boundary value problems for equations with boundary operators of fractional order. *International Journal of Applied Mathematics, 34*, 2, 283–295. http://dx.doi.org/10.12732/ijam.v34i2.6b

17 Yuldashev, T.K., & Kadirkulov, B.J. (2021). Inverse problem for a partial differential equation with Gerasimova-Caputo-type operator and degeneration. *Fractal Fract., 5*, 58. https://doi.org/10.3390/fractalfract5020058

18 Liu, Y., Li Z., & Yamamoto, M. (2019). Inverse problems of determining sources of the fractional partial differential equations. *Handbook of Fractional Calculus with Applications. J.A.T. Marchado Ed. DeGruyter, 2,* 411–430.

19 Ashyralyev, A., & Urun, M. (2019). Time-dependent source identification problem for the Schrodinger equation with nonlocal boundary conditions. *AIP Conf. Proc., 2183,* art. 070016.

20 Ashyralyev, A., & Urun, M. (2021). On the Crank-Nicolson difference scheme for the time-dependent source identification problem. *Bulletin of the Karaganda university. Mathematics, 102*, 2, 35–44.

21 Ashyralyev, A., & Urun, M. (2021). Time-dependent source identification Schrodinger type problem. *International Journal of Applied Mathematics, 34*, 2, 297–310.

22 Zhang, Y., & Xu, X. (2011). Inverse source problem for a fractional differential equations. *Inverse Prob., 27*, 3, 31–42.

23 Furati, K.M., Iyiola, O.S., & Kirane, M. (2014). An inverse problem for a generalized fractional diffusion. *Applied Mathematics and Computation, 249,* 24–31.

24 Ismailov, M.I., & Cicek, M. (2016). Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. *Applied Mathematical Modelling, 40,* 4891–4899.

25 Kirane, M., & Malik, A.S. (2011). Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time. *Applied Mathematics and Computation, 218,* 163–170.

26 Kirane, M., Samet, B., & Torebek, B.T. (2017). Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data. *Electronic Journal of Differential Equations, 217,* 1–13.

27 Nguyen, H.T., Le, D.L., & Nguyen, V.T. (2016). Regularized solution of an inverse source problem for a time fractional diffusion equation. *Applied Mathematical Modelling, 40,* 8244–8264.

28 Torebek, B.T., & Tapdigoglu, R. (2017). Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. *Mathematical Methods in Applied Sciences, 40,* 6468–6479.

29 Li, Z., Liu, Y., & Yamamoto, M. (2015). Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients. *Applied Mathematica and Computation, 257,* 381–397.

30 Rundell, W., & Zhang, Z. (2018). Recovering an unknown source in a fractional diffusion problem. *Journal of Computational Physics, 368,* 299–314.

31 Asl, N.A., & Rostamy, D. (2019). Identifying an unknown time-dependent boundary source in time-fractional diffusion equation with a non-local boundary condition. *Journal of Computational and Applied Mathematics, 335,* 36–50.

32 Sun, L., Zhang, Y., & Wei, T. (2019). Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation. *Applied Numerical Mathematics, 135,* 228–245.

33 Malik, S.A., & Aziz, S. (2017). An inverse source problem for a two-parameter anomalous diffusion equation with nonlocal boundary conditions. *Computers and Mathematics with applications, 3,* 7–19.

34 Ruzhansky, M., Tokmagambetov, N., & Torebek B.T. (2019). Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. *J. Inverse Ill-Possed Probl., 27,* 891–911.

35 Ashurov, R., & Muhiddinova, O. (2020). Inverse problem of determining the heat source density for the subdiffusion equation. *Differential equations, 56,* 12, 1550–1563.

36 Ashurov, R,. & Muhiddinova, O. (2021). Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator. *Lobachevskii Journal of Mathematics, 42*, 3.

37 Shuang, Zh., Saima, R., Asia, R., Khadija, K., & Abdullah M.A. (2021). Initial boundary value problems for a multi-term time fractional diffusion equation with generalized fractional derivatives in time. *AIMS Mathematics, 6*, 11, 12114–12132. DOI: 10.3934/math.2021703

38 Kirane, M., Salman, A.M., & Mohammed, A. Al-Gwaiz (2012). An inverse source problem for a two-dimensional time fractional diffusion equation with nonlocal boundary conditions. *Math. Meth. Appl. Sci.* DOI: 10.1002/mma.2661

39 Muhammad, A., & Salman, A.M. (2016). An inverse problem for a family of time fractional diffusion equations. *Inverse Problems in Science and Engineering, 25*, 9, 1299–1322. DOI: 10.1080/17415977.2016.1255738

40 Ashurov, R.R., & Fayziev, Yu.E. Determination of fractional order and source term in a fractional subdiffusion equation. https://www.researchgate.net/publication/354997348

41 Alimov, Sh.O. (2008). Ob odnoi zadache upravleniia protsessom teploobmena [On a control problem associated with the heat exchange process]. *Doklady Rossiiskoi akademii nauk – Proceeding of the Russian Academy of Sciences, 421*, 5, 583–585 [in Russian].

42 Alimov, Sh.O., & Albeverio, S. (2008). On a time-optimal control problem associated with the heat exchange process. *Appl. Math. Optim., 57,* 58–68.

43 Fayziev, Yu.E., Kuchkarov, A., & Nosirova, D. (2017). On one control problem related to the process of heat transfer into rectangles. *Acta NUUz, 2,* 239–244.

44 Fayziev, Yu.E. (2018). On the control of the heat conduction. *IIUM Engineering Journal, 19*, 1, 168–177.

45 Djerbashian, M.M. (1966). *Integralnye preobrazovaniia i predstavleniia funktsii v kompleksnoi oblasti [Integral transforms and representation of functions in the complex domain].* Moscow: Nauka [in Russian].

46 Gorenflo, R., Kilbas, A.A., Mainardi, F., & Rogozin, S.V. (2014). Mittag-Leffler functions, related topics and applications. Springer.

47 Ashurov, R.R., & Fayziev, Yu.E. (2017). O postroenii reshenii lineinogo differentsialnogo uravneniia s drobnymi proizvodnymi i postoiannymi koeffitsientami [On construction of solutions of linear fractional differentional equations with constant coefficients and the fractional derivatives]. *Uzbekskii matematicheskii zhurnal – Uzbek Mathematical Journal, 3,* 3–21 [in Russian].