DOI 10.31489/2022M2/4-17

UDC 519.633.2

Y.S. Aldanov[1], T.Zh. Toleuov[2,*], N. Tasbolatuly[1,3]

[1] *Astana International University, Nur-Sultan, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan;*
[3] *Al-Farabi Kazakh National University, Almaty, Kazakhstan*
(E-mail: aldanyersat@gmail.com, timur-toleuov@mail.ru, tasbolatuly@gmail.com)

Approximate solutions of the Riemann problem for a two-phase flow of immiscible liquids based on the Buckley–Leverett model

The article proposes an approximate method based on the "vanishing viscosity"method, which ensures the smoothness of the solution without taking into account the capillary pressure. We will consider the vanishing viscosity solution to the Riemann problem and to the boundary Riemann problem. It is not a weak solution, unless the system is conservative. One can prove that it is a viscosity solution actually meaning the extension of the semigroup of the vanishing viscosity solution to piecewise constant initial and boundary data. It is known that without taking into account the capillary pressure, the Buckley–Leverett model is the main one. Typically, from a computational point of view, approximate models are required for time slicing when creating computational algorithms. Analysis of the flow of a mixture of two immiscible liquids, the viscosity of which depends on pressure, leads to a further extension of the classical Buckley–Leverett model. Some two-phase flow models based on the expansion of Darcy's law include the effect of capillary pressure. This is motivated by the fact that some fluids, e.g., crude oil, have a pressure-dependent viscosity and are noticeably sensitive to pressure fluctuations. Results confirm the insignificant influence of cross-coupling terms compared to the classical Darcy approach.

Keywords: Darcy's law, two-phase flows, phases coupling, Buckley–Leverett theory, isothermal filtration, capillary pressure.

Introduction

S. Bianchini and A. Bressan [1] show that the solutions of the viscous approximations $u_t+A(u)u_x = \varepsilon u_{xx}$ are defined globally in time and satisfy uniform BV estimates, independent of ε. Letting $\varepsilon \to 0$, these viscous solutions converge to a unique limit. In the conservative case where $A = Df$ is the Jacobian of some flux function $f : R^n \to R^n$, the vanishing viscosity limits are the unique entropy-weak solutions to the system of conservation laws $u_t + f(u)u_x = 0$.

Buckley and Leverett proposed and calculated a model of fluid behavior in a porous medium in 1942. In the process of further development, many different calculation methods were proposed, in particular [2–5] and many others.

To read scientific studies on the relationships between phases in multiphase flow modeling, we refer to [6] for an analysis and links to these papers.

We will provide a clear description of the displacement of an incompressible fluid during the formation of a porous medium given by Dominique Guerillot et al. [6]. This is the mass conservation equation for two phases (oil and water):

$$\frac{\partial \rho_o S_o \phi}{\partial t} + \nabla \cdot (\rho_o \nu_o) = 0,$$
$$\frac{\partial \rho_\varpi S_\varpi \phi}{\partial t} + \nabla \cdot (\rho_\varpi \nu_\varpi) = 0$$

with the natural physical constraint $S_o + S_w = 1$, where ϕ — the effective porosity of the reservoir;

[*] Corresponding author.
E-mail: timur-toleuov@mail.ru

ρ_o, S_o and ρ_ϖ, S_ϖ — the density and saturations of oil and water, respectively; ν_o, p_o and ν_ϖ, p_ϖ — the superficial velocity and the pressure of the oil and the water phases, respectively.

Fluid flow through a porous medium is common in many areas of technology and science. At the same time, the problem of single-phase flow has been well studied both from an engineering and mathematical point of view [7]. The classical Darcy's law, widely used for practical purposes, can be obtained by modeling a sluggishly current incompressible flow. In practice, a porous medium is considered a periodic array of cells filled with a Newtonian fluid. The problem is formulated at the cell scale (microscale), and then scaled by homogenization in the entire area, providing the classical Darcy's law.

According to Darcy's equation, a porous solid has a resistance to the liquid in the pores, which is directly proportional to the speed of the liquid relative to the solid, usually called the drag coefficient.

Oil production in most cases occurs when it is displaced in the pore space of the productive reservoir by water or gas. This process is used in natural operating modes and in artificial methods of maintaining reservoir pressure by flooding or gas injection. The theory of isothermal filtration serves as the basis for calculating such processes [8–10].

Simulation can be without taking into account nonlinear effects [11], assuming that the flow of immiscible two fluids is separated by a smooth boundary layer [12]. In such a formulation of the problem, a solid matrix is considered an impenetrable rigid body, and the classical no-slip condition is imposed on its boundary. The result is a system of equations for saturation and pressure. Such a system is reduced to the classical Buckley–Leverett equation, when the viscosities of both fluids are independent of pressure. It is found that the relative permeabilities depend on the pressure of the liquid and when the solid matrix is considered rigid.

Some models consider the exchange of momentum between the phases of flows of two immiscible fluids in a porous medium. Sometimes creeping flow models are used that include an explicit relationship between two phases by adding cross-terms to the generalized Darcy's law [13]. These models show that cross-terms in macroscale models can significantly affect flow compared to results obtained using generalized Darcy's laws without cross-terms. Investigations with the availability of experimental data for analytical solutions suggest that the influence of this dependence on the dynamics of saturation fronts and stationary profiles is very sensitive to gravitational effects, the ratio of viscosity between two phases and permeability. These results indicate that the effects of momentum exchange on two-phase flow can increase with increasing porous medium permeability when the effect of liquid-liquid interfaces becomes similar to the effect of solid-liquid interfaces.

In the parabolic case, solvability has been sufficiently studied by S.N. Antontsev, V.N. Monakhov, O.B. Bocharov [14–16], and others. It should be noted that equations in the form (1) are the simplest mathematical models of many natural phenomena, sometimes reflecting the essence of these phenomena. In particular, the Leverett function is determined experimentally according to the materials of Kern. This approach does not give the desired results in problems of filtration theory.

The Cauchy problem for a system of conservation laws in one space dimension takes the form [7]:

$$u_t + f(u)_x = 0. \tag{1}$$

Here $u(0,x) = \tilde{u}(x)$ is initial conditions, $u = (u_o, u_\varpi)$ is the vector of conserved quantities (oil and water, respectively), while the components of $f = (f_o, f_\varpi)$ are the fluxes of oil and water, respectively. We assume that the flux function $f : \mathbf{R}^2 \to \mathbf{R}^2$ is smooth and that the system is strictly hyperbolic; i.e., at each point u the Jacobian matrix $A = Df(u)$ has u real, distinct eigenvalues

$$\lambda_1(u) < ... < \lambda_n(u).$$

One can then select bases of right t and left eigenvectors $r_i(u), l_i(u)$ normalized so that

$$|r_i| \equiv 1, l_i \cdot r_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Several fundamental laws of physics take the form of a conservation equation.

The lack of regularity is a major source of complexity since most of the standard differential calculus tools are not applicable. Special methods are needed, in particular, the main building block is the so-called Riemann problem [1, 17], in which the initial data are piecewise constant with one jump at the origin:

$$u\left(0,x\right)=\begin{cases} u^{-} & \text{if} \quad x<0, \\ u^{+} & \text{if} \quad x>0. \end{cases}$$

The viscosity solution of a Cauchy problem is unique and coincides with the limit of Glimm, and the front-tracking approximations for a strictly hyperbolic system of conservation laws satisfy the standard assumptions.

For each $i\in\{1,...,n\}$, the i-th characteristic field is either linearly degenerate, so that

$$D\lambda_i\left(u\right)\cdot r_i\left(u\right)=0$$

for all u, or else it is genuinely nonlinear, i.e.,

$$D\lambda_i\left(u\right)\cdot r_i\left(u\right)>0,$$

0 for all u.

The definition given in [7] was motivated by a natural conjecture. Namely, the viscosity solutions (characterized in terms of local integral estimates) should coincide precisely with the limits of vanishing viscosity approximations. In the present paper, we adopt a similar definition of viscosity solutions and prove that the above conjecture is indeed true. Our results apply to the more general case of (possibly nonconservative) quasilinear strictly hyperbolic systems. In particular, we obtain the uniqueness of the vanishing viscosity limit.

For a comprehensive account of the recent uniqueness and stability theory, we refer to [7, 8].

A long-standing conjecture is that the entropic solutions of the hyperbolic system (1) actually coincide with the limits of solutions to the parabolic system

$$u_t+f(u)_x=\varepsilon u_{xx}$$

when the viscosity coefficient $\varepsilon\to 0$. In view of the recent uniqueness results, it looks indeed plausible that the vanishing viscosity limit should single out the unique "good" solution of the Cauchy problem, satisfying the appropriate entropy conditions. In earlier literature, results in this direction were based on three main techniques [7]: Comparison principles for parabolic equations; Singular perturbations; Compensated compactness.

In our point of view, to develop a satisfactory theory of vanishing viscosity limits, the heart of the matter is to establish a priori BV bounds on solutions $u\left(t,\cdot\right)$ of (1.8) ε, uniformly valid for all $t\in[0,\infty)$ and $\varepsilon>0$. This is indeed what we will accomplish in the present paper. Our results apply, more generally, to strictly hyperbolic systems with viscosity, not necessarily in conservation form:

$$u_t+A\left(u\right)u_x=\varepsilon u_{xx}.$$

The modeling multiphase flows in porous media is of major importance in many fields of applications. In particularly in enhanced oil recovery applications of petroleum engineering. The classical mathematical models for multiphase flows are based on a straightforward generalization of Darcy's law for a single-phase flow [9]. A natural question arises: How important is the influence of one phase on the other phase? In some applications, it is shown that the coupling effects are small, and therefore negligible.

In [18–20], it was developed a mathematical model to apply Buckley–Leverett frontal advance theory to immiscible displacement in non-communicating stratified reservoirs. The influence of the coefficient of viscosity and coefficient of variation of the Dykstra-Parsons permeability (VDP) on productivity has been investigated. The introduction of pseudo-relative permeability functions is discussed. It was shown in [21] that mass conservation in several layers unconnected with each other can be used to derive the interlayer ratios of various emerging fronts. The ratio of flow areas in an immiscible two-phase flow in a porous medium was studied in [22, 23]. They used dynamic pore network modeling that uses an interface model used to simulate steady state two-phase flow.

Numerical modeling of the flow of immiscible fluids is of great importance in many areas for the proper management of underground resources, in particular water. Recently presented high-resolution numerical model that simulates a three-phase immiscible fluid flow in both unsaturated and saturated zones in a porous aquifer [22] is relevant.

In the theory of immiscible two-phase flow presented in [23], the conservation of mass is provided by general equations, which require some additions for a porous medium. The basic equation can be derived from the relative permeability data. It turns out that it has a surprisingly simple form when expressed in the correct variables [24]. The resulting system of equations can then be solved for a structured porous medium. However, the question remains what happens when the porous medium has a nontrivial structure along its entire length.

1 Mathematical modelling

Consider filtration of a two-phase liquid in a porous medium in water-pressure mode. The field is covered by a network of wells and their location schemes can be different. The oil-bearing formation is considered unlimited, of constant thickness, the porous medium is non-deformable, and the ratio of capillary pressure to the total hydrodynamic pressure drop is small, which allows to consider the problem obeying the classical Buckley–Leverett model.

High precision modeling of immiscible two-phase flows in porous media is paramount. Nevertheless even with such high-precision numerical modeling, the lack of information or its fuzziness, for example, on the relative permeability and functions of capillary pressure in them does not allow a detailed comparison with experiments [25].

Without taking into account gravity, two-phase filtration for the case of straight-parallel displacement was considered by S. Buckley and M. Leverett in 1942, and later independently by A.M. Pirverdyan, who also studied the case of a more general filtration law for two-phase flow [10].

In the case of one-dimensional flow of incompressible immiscible liquids under conditions where capillary pressure and the influence of gravity can be ignored, the displacement process allows a simple mathematical description.

It should be noted that the equation of the form (1):

$$u_t + f(u)_x = 0$$

one-dimensional space variables have been considered by many researchers. A significant contribution to the non-local theory of the Cauchy problem for this equation was made by O.A. Oleinik, A.N. Tikhonov, A.A. Samarsky, and I.M. Gelfand. The Buckley–Leverett mathematical model belongs to equation (1).

A detailed specification allows to define a porous medium as either hydrophilic or hydrophobic.

It is known that if $\varepsilon > 0$ is the coefficient of viscosity, then the viscous friction force acting on each particle of the porous medium $x\,(t)$ and related to the unit of mass can be assumed to be equal to $\varepsilon \cdot u_{xx}$. Then returning to the mathematical model of Buckley–Leverett (then instead of $u\,(t, x)$ we will write $s\,(t, x)$ - water saturation)

$$s_t + s \cdot s_x = \varepsilon \cdot s_{xx} \tag{2}$$

where $F'\,(s) = \frac{1}{2}s$ is the Leverett function.

The equation of the form (2) was studied by O.B. Bocharov, B.N. Monakhov, I.L. Telegin [14, 15, 17].

If the viscous term tends to zero, the uniqueness of the vanishing viscosity limit is proved based on comparative estimates for the solutions of the corresponding Hamilton-Jacobi equation, as stated in [26, 27]. As an application, they obtained the existence and uniqueness of solutions for the class of triangular systems of 2×2 conservation laws with hyperbolic degeneracy. However, the forecast calculations did not give the desired results.

The assumed method at $\varepsilon \to 0$ is called the "vanishing viscosity"method. Given that

$$s_t = \left(\varepsilon \cdot s_x - \frac{s^2}{2}\right)_x$$

we introduce the potential $u\,(x,t)$ defined by the equality

$$du + \left(\varepsilon \cdot s_x - \frac{s^2}{2}\right)_x dt.$$

In this case

$$u_x = s,$$
$$u_t = \varepsilon \cdot s_x - \tfrac{s^2}{2} = \varepsilon \cdot u_{xx} - \tfrac{u^2 x}{2},$$

that is, the function $u\,(x,t)$ satisfies the equation

$$u_t + \frac{1}{2}u_x^2 = \varepsilon \cdot u_{xx}. \tag{3}$$

Make a replacement in (3)

$$u = -2\varepsilon \cdot \ln z.$$

Then

$$u_t = -2\varepsilon \cdot \tfrac{z_t}{z},$$
$$u_x = -2\varepsilon \cdot \tfrac{z_x}{z},$$
$$u_{xx} = -2\varepsilon \cdot \tfrac{z_{xx}}{z} + 2\varepsilon \cdot \tfrac{z_x^2}{z^2},$$

Equation (3) will take the form

$$-2\varepsilon \cdot \frac{z_t}{z} + 2\varepsilon^2 \cdot \frac{z_x^2}{z^2} = -2\varepsilon^2 \cdot \frac{z_{xx}}{z} + 2\varepsilon^2 \cdot \frac{z_x^2}{z^2},$$

in other words, the thermal conductivity equation is obtained regarding to $z\,(x,t)$:

$$z_t = \varepsilon \cdot z_{xx}. \tag{4}$$

This method is often called the Florin–Hopf–Cole transformation. From the made, substitutions it follows that the solution to equation (2) has the form:

$$s = u_x = -2\varepsilon \cdot \frac{z_x}{z}$$

where $z\,(x,t)$ is the solution (4).

Suppose that a wave of the form propagates through an injection well:

$$s\,(x,t) = s_- + \frac{s_+ - s_-}{2} \cdot (1 + sign(x - \omega t)) = \begin{cases} s_-, ats < \omega t \\ s_+, ats > \omega t \end{cases} \tag{5}$$

where $\omega = const$. Suppose that there is a generalized solution of the equation of the form (1) in the sense of fulfilling the integral identity. To do this, it is necessary and sufficient that the condition is met on the break line $\omega = const$

$$\omega = \frac{dx}{dt} = \frac{F(s_+) - F(s_-)}{s_+ - s_-}. \tag{6}$$

The idea of the "vanishing viscosity"method in this case is that this solution (discontinuous) of the form (5), (6) is acceptable. That is, for $x \neq \omega$ solutions of of $s^\varepsilon(x,t)$ the equation

$$s_t^\varepsilon = +(F(s^\varepsilon))_x = \varepsilon \cdot s_{xx}^\varepsilon \tag{7}$$

for $\varepsilon \to 0$, it is obtained as a pointwise limit.

Below, the proposed method by I.M. Gelfand has the desired result in applied problems.

Given the structure of the solution we will look for a solution $s(x,t)$ to equation (7) and (8) in the form:

$$s^\varepsilon(x,t) = u(\xi), \xi = \frac{x - \omega t}{\varepsilon}. \tag{8}$$

Substituting a solution of this type in (7), we get that the function $U(\xi)$ is the solution of the equation

$$-\omega \cdot v' + (F(v))' = v''. \tag{9}$$

At $x \neq \omega t$, the function $s^\varepsilon = v\left(\frac{x-\omega t}{\varepsilon}\right)$ pointwise approximates for $\varepsilon \to 0$ function $s(x,t)$ of the form (5) if and only if the function $v(\xi)$ satisfies the boundary conditions:

$$s(-n,t) = s_-, \quad s(n,t) = s_+ \tag{10}$$

where n is a sufficiently large distance from the well.

It should be noted that $v(t)$ is not the only solution, i.e., there can be $\tilde{v} = v(\xi - \xi_0)$, for any $\xi_0 \in R$.

Integrating (9) and (10), we get

$$\begin{array}{l} v' = -\omega \cdot v + \Phi(v) + C = \tilde{\Phi}(v) + C, \\ C = const. \end{array} \tag{11}$$

If these conditions are met, the solutions of equation (9) that interest us are given by the formula.

Following the method of I.M. Gelfand, in order for an autonomous equation (11) with a smooth right part of $\tilde{\Phi}(v) + C$ to have a solution that tends to the constants s_- at $n \to -\infty$ and s_+ at $n \to +\infty$, it is necessary and sufficient to meet the following conditions:

a) s_- and s_+ -special points of the original equation, i.e., zero the right side of the equation (11):

$$\Phi(v) + C = \tilde{\Phi}(v) + C = 0,$$

that is, as a result, we have

$$\tilde{\Phi}(s_-) = \tilde{\Phi}(s_+) = -C;$$

b) another option between s_- and s_+ there are no other special points and the right part (11) on the specified interval:

1) positive at $s_- < s_+$ the solution increases, i.e.,

$$\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_-) > 0, \forall \upsilon \in (s_-, s_+) \tag{12}$$

2) negative at $s_- > s_+$, i.e., the solution decreases:

$$\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_+) < 0, \forall \upsilon \in (s_+, s_-). \tag{13}$$

If these conditions are met, the solutions of equation (9) that interest us are given by the formula

$$\int_{\upsilon_0}^{\upsilon} \frac{d\upsilon}{\tilde{\Phi}(\upsilon) - \tilde{\Phi}(s_-)} = \xi - \xi_0$$

where $\upsilon_0 = \frac{s_+ + s_-}{2}$ — location of wells.

The given conditions (12), (13) are an analytical record of the tolerance condition.

By varying s_-, s_+, and $F(s)$, various converging sequences of valid generalized solutions can be constructed. At the same time, any point-to-point limits of acceptable solutions are also considered acceptable.

2 Numerical Results

As a result, we get that the solution $s(x,t)$ can jump from s_- to s_+ (in the direction of increasing x). That is, in fact, this jump occurs during the transition from the water phase to the oil phase. In this case, the conditions for an acceptable gap are met (Fig. 1):

1) for $s_- < s_+$, the graph of the function $F(s)$ on the segment $[s_-, s_+]$ must be located below the chord with the ends $(s_-, F(s_-))$ and $(s_+, F(s_+))$;

2) in the case of $s_- > s_+$, the graph of the function $F(s)$ on the segment $[s_+, s_-]$ must be located no higher than the chord with the ends $(s_-, F(s_-))$ and $(s_+, F(s_+))$.

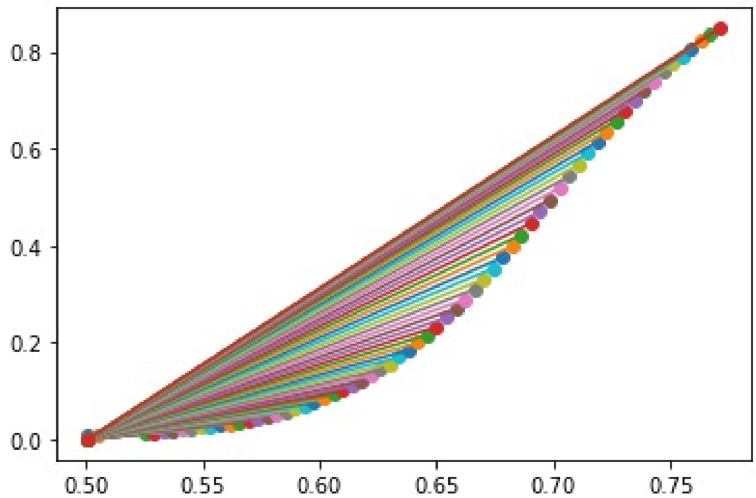

Figure 1. Construction of chord (s, F(s)) front saturation

The obtained conditions make it possible to regulate filtration processes in the bottomhole formation zone taking into account the initial information, in particular, some data from Table 1.

Table 1

Initial data used in modeling with the one-dimensional Buckley-Leverett problem

Parameter	Value
Porosity	0.28
Oil viscosity	1.e-4 kg/ms
Water viscosity	0.5e-4 kg/ms
Oil density	881 kg/m^3
Water density	1000 kg/m^3
Water relative perm calculation for a given water saturation	11.174
Oil relative perm calculation for a given water saturation	3.326
Cross-sectional area	0.4 m^2

The gap tolerance conditions obtained by the "vanishing viscosity"method are in perfect agreement with the forecast calculations. Indeed, the convexity property of the function $F(s)$ in the Buckley-Laverett mathematical model (up) down by definition means that any chord connecting points in a straight line shows the validity of the Buckley-Laverett mathematical model itself.

Figure 2 presents water saturation profile.

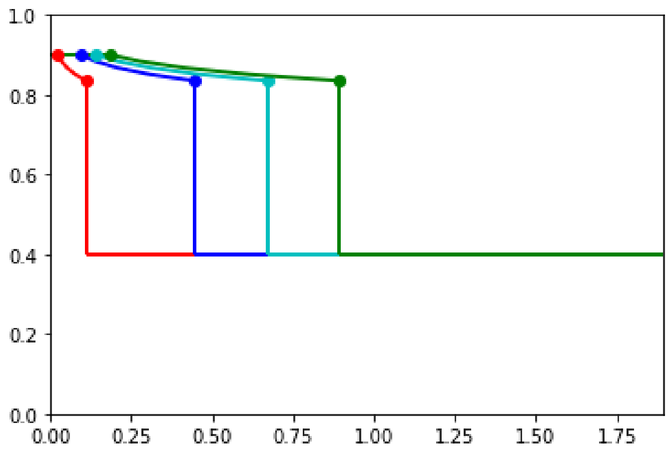

Figure 2. Water saturation profile as a function of time "t" and distance "x"

Figure 3 illustrates derivative of fractional flow curve.

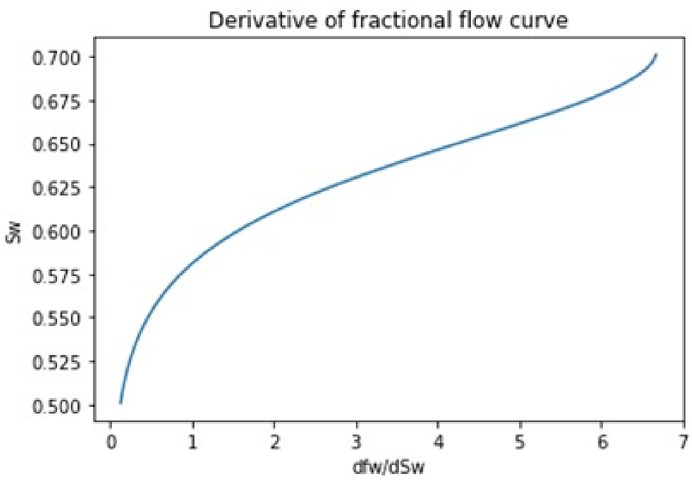

Figure 3. Derivative of fractional flow dF/ds

Table 2 shows the results of derivative of the fractional flow rate curve calculation.

Table 2

Results of numerical calculation of the derivative of the fractional flow rate curve

S_w	$\frac{dF_s}{ds_w}$
0.500	0.152
0.525	0.313
0.550	0.487
0.575	0.889
0.600	1.519
0.625	2.721
0.650	4.219
0.675	5.817
0.700	6.613

Presentation of the model of nonlinear wave propagation and how the use of the method allows one to cope with sharp fronts (or discontinuities) and develop them correctly, as well as to follow the formation of a jump and rarefaction (Fig. 4a, 4b). The formation of an abrupt jump (jump) is observed.

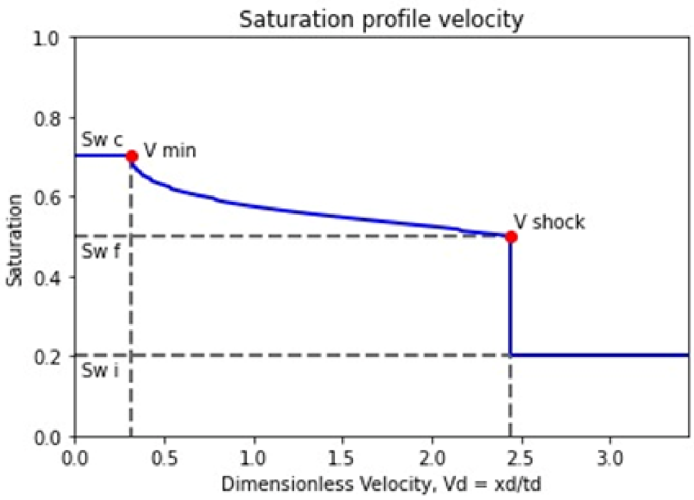

Figure 4. a) Shock and rarefaction formation

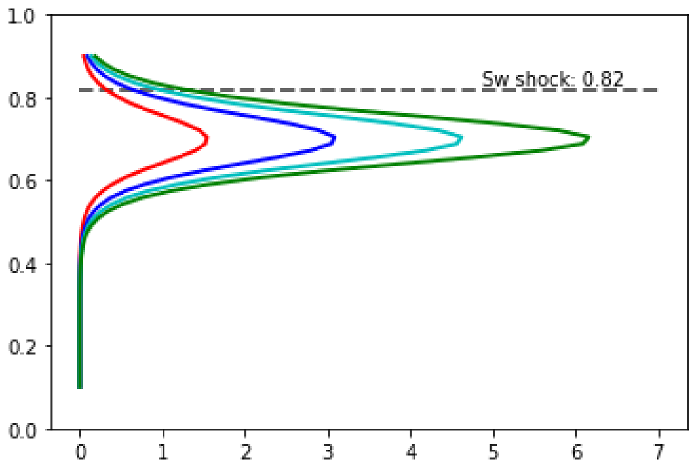

Figure 4. b) Shock and rarefaction formation

Conclusions

The article proposes one of the methods for solving the problem of filtration of a two-phase incompressible fluid. The problem of mathematical filtering is posed on the basis of the classical Buckley-Leverett model and an approximate solution is constructed. For the effective use of the described method, relevant data are needed, such as the coefficient of fluid viscosity, the density of formation fluids, etc., to plot the curves of the viscosity ratio. The considered method, based on the Buckley-Leverett theory, uses vanishing viscosity for frontal advance, but, in general, it can be applied to various systems that use different technological approaches and open the way for further research. In particular, stochastic analysis of two-phase flow in stratified porous media seems promising [28, 29]. Stochastic models, which include some assumptions about porous media, simplify and stabilize fuzzy information. In the future, we plan to use stochastic data and analyze them.

References

1 Bianchini S. Vanishing viscosity solutions of nonlinear hyperbolic systems / S. Bianchini, A. Bressan // Annals of Mathematics. — 2005. — 161. — P. 223–342.

2 Welge H.J. A simplified method for computing oil recovery by gas or water drive / H.J. Welge // J. Pet. Technol. — 1952. — 4. — P. 91–98.

3 Sheldon J.W. One-dimensional incompressible noncapillary two-phase fluid flow in a porous medium / J.W. Sheldon, W.T. Cardwell // Pet. Trans. AIME. — 1959. — 216. — P. 290–296.

4 McWhorter D.B. Exact integral solutions for two-phase flow / D.B. McWhorter, D.K. Sunada // Water Resour. Res. — 1990. — 26. — P. 399–413.

5 Muskat M. The Flow of Homogeneous Fluids through Porous Media / M. Muskat. — New York: The Maple Press Company, 1946.

6 Guerillot D. Buckley–Leverett Theory for Two-Phase Immiscible Fluids Flow Model with Explicit Phase-Coupling Terms / D. Guerillot, M. Kadiri, S. Trabelsi // Water, 2020. — 12(11). — P. 3041.

7 Buckley S.E. Mechanism of fluid displacement in sands / S.E. Buckley, M.C. Leverett // Trans. AIME. — 1942. — 146. — P. 107–116.

8 Басниев К.С. Подземная гидравлика: учеб. для вузов / К.С. Басниев, А.М. Власов, И.Н. Кочина, В.М. Максимов. — М.: Недра, 1986. — 303 с.

9 Коновалов А.Н. Задачи фильтрации многофазной несжимаемой жидкости / А.Н. Коновалов. — Новосибирск: Наука, 1988. — 166 с. — ISBN 5-02-028569 -2.

10 Пирвердян А.М. Физика и гидравлика нефтяного пласта / А.М. Пирвердян. — М.: Недра, 1982. — 192 с.

11 Alimhan K. Further Results on Output Tracking for a Class of Uncertain High-Order Nonlinear Time-Delay Systems / K. Alimhan, M.N. Kalimoldayev, A.A. Adamov, O.Zh. Mamyrbayev, N. Tasbolatuly, A. Smolarz // Przeglad Elektrotechniczny. — 2019. — 95(5). — P. 88–91.

12 Fusi L. Buckley–Leverett Equation with viscosities and Relative Permeabilities Dependending on Pressure / L. Fusi, A. Farina, G. Saccomandi // SIAM Journal on Applied Mathematics. — 2015. — 75(5). — P. 1983–2000.

13 Pasquier S. Modeling two-phase flow of immiscible fluids in porous media: Buckley-Leverett theory with explicit coupling terms / S. Pasquier, M. Quintard, Y. Davit // Physical Review Fluids. — 2017. — 2. — P. 104101/1-104101/19.

14 Антонцев С.Н. Краевые задачи механики неоднородных жидкостей / С.Н. Антонцев, А.В. Кажихов, В.Н. Монахов. — Новосибирск: Наука, 1983. — 320 с.

15 Bocharov O.B. Numerical investigation of discontinuous solutions of conjugation of two non-isothermal filtration models / O.B. Bocharov, I.G. Telegin // Computational Technologies. — 2008. — 13(1). — P. 20–28.

16 Bocharov O.B. Boundary value problems of non-isothermal two-phase filtration in porous media / O.B. Bocharov, V.N. Monakhov // Dynamics of a continuous medium: Collection of scientific works, AS USSR Sibir Department. Institute of hydrodynamics. — 1988. — 86. — P. 47–69.

17 Daripa P. Polymer Floods: A Case Study of Nonlinear Wave Analysis And of Instability Control In Tertiary Oil Recovery / P. Daripa, J. Glimm, B. Lindquis, O. McBrayan // Society for Industrial and Applied Mathematics. — 1988. — 48(2). — P. 353–374.

18 El-Khatib N.A.F. The Application of Buckley-Leverett Displacement to Waterflooding in Non-Communicating Stratified Reservoirs / N.A.F. El-Khatib // In Proceedings of the SPE Middle East Oil Show. — 2001. — P. 91–102.

19 Prince A.O. Buckley-Leverett Displacement Theory for Waterflooding Performance in Stratified Reservoir / A.O. Prince // Petroleum and Coal. — 2014. — 56(3). — P. 267–281.

20 Zhao L. Analytical Model of Waterflood Sweep Efficiency in Vertical Heterogeneous Reservoirs under Constant Pressure / L. Zhao, L. Li, Zh. Wu, Ch. Zhang // Mathematical Problems in Engineering. — 2016. — Article ID 6273492. — P. 9.

21 Alisawi M. In Reservoir Engineering Method Buckley-Leverett method in water Flooding approach prepared by Mahdi Mohsen / M. Alisawi. — Research Proposal, 2020.

22 Feo A. High resolution shock capturing numerical simulations of three phase immiscible fluids from the unsaturated to the saturated zone / A. Feo, F. Celico // Scientific Reports. — 2011. — P. 1–11.

23 Roy S. Flow-Area Relations in Immiscible Two-Phase Flow in Porous Media / S. Roy, S. Sinha // Frontiers in Physics. — 2020. — 8.

24 Roy S. The Co-Moving Velocity in Immiscible Two-Phase Flow in Porous Media / S. Roy, H. Pedersen, S. Sinha, A. Hansen // arXiv:2108.10187v1 [physics.flu-dyn]. — 2021. — P. 1–27.

25 Riaz A. Numerical simulation of immiscible two-phase flow in porous media / A. Riaz, H.A. Tchelepi // Phys. Fluids. — 2006. — 18. — P. 1–27.

26 Spinola L.V. Vanishing Viscosity Solutions of a 2x2 Triangular Hyperbolic System with Dirichlet Conditions on Two Boundaries / L.V. Spinola // Indiana University Mathematics Journals. — 2007. — 56(1). — P. 279–364.

27 Bressan A. The unique limit of the Glimm scheme / A. Bressan // Arch. Rational Mech. Anal. — 1995. — 130. — P. 205–230.

28 Bressan A. Vanishing Viscosity Solutions for Conservation Laws with Regulated Flux / A. Bressan, G. Guerra, W. Shen // Journal of Differential Equations. — 2019. — 266(1). — P. 312–351

29 Dias M.M. Network models for two-phase flow in porous media Part 1. Immiscible microdisplacement of non-wetting fluids / M.M. Dias, C.A. Payatakes // Journal of Fluid Mechanics. — 1986. — 164. — P. 305–336.

Е.С. Алданов[1], Т.Ж. Төлеуов[2], Н. Тасболатұлы[1,3]

[1] *«Астана» халықаралық университеті, Нұр-Сұлтан, Қазақстан;*
[2] *Қ. Жұбанов атындағы Ақтөбе өңірлік университеті, Ақтөбе, Қазақстан;*
[3] *Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы, Қазақстан*

Бакли–Леверетт моделінің негізінде екіфазалы араласпайтын сұйықтар ағыны үшін Риман есебінің жуық шешімдері

Мақалада капиллярлық қысымды есепке алмастан, шешімнің тұтастығын қамтамасыз ететін «жойылатын тұтқырлық» әдісіне негізделген жуықтау әдісі ұсынылған. Риман есебі мен шекаралық есебінің тұтқырлығы жойылған шешімі қарастырылған. Жүйе консервативті болмаса, бұл әлсіз шешім емес екенін ескеру керек, бірақ оның тұтқырлық шешім екенін дәлелдеуге болады, бұл шын мәнінде жойылып кететін тұтқырлық шешімнің жарты группаларды бөлшек-тұрақты бастапқы және шекаралық шарттарға дейін кеңейтуді білдіреді. Капиллярлық қысымды есепке алмағанда, Бакли-Леверетт моделі негізгі болып табылатыны белгілі. Нақты болжамдық есептеулерге сүйене отырып, модель көптеген салаларда өзін дәлелдеді. Әдетте, есептеу тұрғысынан алғанда, есептеу алгоритмдерін құру кезінде уақытты кванттау үшін жуықтау әдістері қажет. Тұтқырлығы қысымға тәуелді екі араласпайтын сұйықтықтардың ағынын талдау Бакли-Леверетт классикалық үлгісін одан әрі кеңейтуге әкеледі. Дарси заңының кеңеюіне негізделген кейбір екіфазалы ағын модельдері капиллярлық қысымның әсерін қосады. Бұл кейбір сұйықтықтардың, мысалы, шикі мұнайдың, қысымға тәуелді тұтқырлығына және қысымның ауытқуына айтарлықтай сезімталдығына негізделген. Нәтижелер классикалық Дарси әдісімен салыстырғанда кросс-байланыс жағдайларының елеусіз әсерін растайды.

Кілт сөздер: Дарси заңы, екіфазалы ағындар, фазалық байланыс, Бакли–Леверетт теориясы, изотермиялық сүзу, капиллярлық қысым.

Е.С. Алданов[1], Т.Ж. Толеуов[2], Н. Тасболатулы[1,3]

[1] *Международный университет «Астана», Нур-Султан, Казахстан;*
[2] *Актюбинский региональный государственный университет имени К. Жубанова, Актобе, Казахстан;*
[3] *Казахский национальный университет имени аль-Фараби, Алматы, Казахстан*

Приближенные решения задачи Римана для двухфазного потока несмешивающихся жидкостей на основе модели Бакли–Леверетта

В статье предложен приближенный метод, основанный на «исчезающей вязкости», которая обеспечивает гладкость раствора без учета капиллярного давления. Мы будем рассматривать решение задачи Римана и краевой задачи Римана с исчезающей вязкостью. Обратите внимание на то, что это не слабое решение, если система не является консервативной, то можно доказать, что это вязкостное решение, фактически означающее расширение полугруппы решения исчезающей вязкости до кусочно-постоянных начальных и граничных данных. Известно, что, без учета капиллярного давления, модель Бакли–Леверетта является основной. Основанная на реальных прогнозных расчетах модель положительно зарекомендовала себя во многих сферах. Обычно, с вычислительной точки зрения, приближенные модели требуются для квантования времени при создании вычислительных алгоритмов. Анализ потока из двух несмешивающихся жидкостей, вязкость которых зависит от давления, приводит к дальнейшему расширению классической модели Бакли–Леверетта. Некоторые модели двухфазного потока, основанные на расширении закона Дарси, включают эффект капиллярного давления. Это мотивировано тем фактом, что некоторые жидкости, например, сырая нефть, имеют вязкость, зависящую от давления, и заметно чувствительны к колебаниям давления. Результаты подтвердили незначительное влияние условий кросс-связывания по сравнению с классическим подходом Дарси.

Ключевые слова: закон Дарси, двухфазные потоки, связь фаз, теория Бакли–Леверетта, изотермическая фильтрация, капиллярное давление.

References

1 Bianchini, S., & Bressan, A. (2005). Vanishing viscosity solutions of nonlinear hyperbolic systems. *Annals of Mathematics*, 161, 223–342.

2 Welge, H.J. (1952). A simplified method for computing oil recovery by gas or water drive. *J. Pet. Technol.*, 4, 91–98.

3 Sheldon, J.W., & Cardwell, W.T. (1959). One-dimensional incompressible noncapillary two-phase fluid flow in a porous medium *Pet. Trans. AIME.*, 216, 290–296.

4 McWhorter, D.B., & Sunada, D.K. (1990). Exact integral solutions for two-phase flow. *Water Resour. Res.*, 26, 399–413.

5 Muskat, M. (1946). *The Flow of Homogeneous Fluids through Porous Media.* New York: The Maple Press Company.

6 Guerillot, D., Kadiri, M., & Trabelsi, S. (2020). Buckley–Leverett Theory for Two-Phase Immiscible Fluids Flow Model with Explicit Phase-Coupling Terms. *Water*, 12(11), 3041.

7 Buckley, S.E., & Leverett, M.C. (1942). Mechanism of fluid displacement in sands. *Trans. AIME*, 146, 107–116.

8 Basniev, K.S., Vlasov, A.M., Kochina, I.N., & Maksimov, V.M. (1986). *Podzemnaia gidravlika [Underground Hydraulics].* Moscow: Nedra [in Russian].

9 Konovalov, A.N. (1988). *Zadachi filtratsii mnogofaznoi neszhimaemoi zhidkosti [Problems of filtering a multiphase incompressible fluid].* Novosibirsk: Nauka [in Russian]. ISBN 5-02-028569-2

10 Pirverdian, A.M. (1982). *Fizika i gidravlika neftianogo plasta [Physics and hydraulics of the oil reservoir].* Moscow: Nedra [in Russian].

11 Alimhan, K., Kalimoldayev, M.N., Adamov, A.A., Mamyrbayev, O.Zh., Tasbolatuly, N., & Smolarz, A. (2019). Further Results on Output Tracking for a Class of Uncertain High-Order Nonlinear Time-Delay Systems. *Przeglad Elektrotechniczny*, 95(5), 88–91.

12 Fusi, L., Farina, A., & Saccomandi, G. (2015). Buckley–Leverett Equation with viscosities and Relative Permeabilities Dependending on Pressure. *SIAM Journal on Applied Mathematics,* 75(5), 1983–2000.

13 Pasquier, S., Quintard, M., & Davit, Y. (2017). Modeling two-phase flow of immiscible fluids in porous media: Buckley-Leverett theory with explicit coupling terms. *Physical Review Fluids*, 2, 104101/1-104101/19.

14 Antontsev, S.N., Kazhikhov, A.V., & Monakhov, V.N. (1983). *Kraevye zadachi mekhaniki neodnorodnykh zhidkostei [Boundary value problems in the mechanics of inhomogeneous fluids].* Novosibirsk: Nauka [in Russian].

15 Bocharov, O.B., & Telegin, I.G. (2008). Numerical investigation of discontinuous solutions of conjugation of two non-isothermal filtration models. *Computational Technologies*, 13(1), 20–28.

16 Bocharov, O.B., & Monakhov, V.N. (1988). Boundary value problems of non-isothermal two-phase filtration in porous media. *Dynamics of a continuous medium: Collection of scientific works, AS USSR Sibir Department. Institute of hydrodynamics*, 86, 47–69.

17 Daripa, P., Glimm, J., Lindquis, B., & McBrayan, O. (1988). Polymer Floods: A Case Study of Nonlinear Wave Analysis And of Instability Control In Tertiary Oil Recovery. *Society for Industrial and Applied Mathematics*, 48(2), 353–374.

18 El-Khatib, N.A.F. (2001). The Application of Buckley-Leverett Displacement to Waterflooding in Non-Communicating Stratified Reservoirs. *In Proceedings of the SPE Middle East Oil Show*, 91–102.

19 Prince, A.O. (2014). Buckley-Leverett Displacement Theory for Waterflooding Performance in Stratified Reservoir. *Petroleum and Coal*, 56(3), 267–281.

20 Zhao, L., Li, L., Wu, Zh., & Zhang, Ch. (2016). Analytical Model of Waterflood Sweep Efficiency in Vertical Heterogeneous Reservoirs under Constant Pressure. *Mathematical Problems in Engineering*, Article ID 6273492, 9.

21 Alisawi, M. (2020). In Reservoir Engineering Method Buckley-Leverett method in water Flooding approach prepared by Mahdi Mohsen. *Research Proposal*, DOI: 10.13140/RG.2.2.33588.48007

22 Feo, A., & Celico, F. (2011). High resolution shock capturing numerical simulations of three phase immiscible fluids from the unsaturated to the saturated zone. *Scientific Reports*, 1–11.

23 Roy, S., & Sinha, S. (2020). Flow-Area Relations in Immiscible Two-Phase Flow in Porous Media. *Frontiers in Physics*, 8.

24 Roy, S., Pedersen, H., Sinha, S., & Hansen, A. (2021). The Co-Moving Velocity in Immiscible Two-Phase Flow in Porous Media. *arXiv:2108.10187v1 [physics.flu-dyn]*, 1–27.

25 Riaz, A., & Tchelepi, H.A. (2006). Numerical simulation of immiscible two-phase flow in porous media. *Phys. Fluids.*, 18, 1–27.

26 Spinola, L.V. (2007). Vanishing Viscosity Solutions of a 2x2 Triangular Hyperbolic System with Dirichlet Conditions on Two Boundaries. *Indiana University Mathematics Journals*, 56(1), 279–364.

27 Bressan, A. (1995). The unique limit of the Glimm scheme. *Arch. Rational Mech. Anal.*, 130, 205–230.

28 Bressan, A., Guerra, G., & Shen, W. (2019). Vanishing Viscosity Solutions for Conservation Laws with Regulated Flux. *Journal of Differential Equations*, 266(1), 312–351.

29 Dias, M.M., & Payatakes, C.A. (1986). Network models for two-phase flow in porous media Part 1. Immiscible microdisplacement of non-wetting fluids. *Journal of Fluid Mechanics*, 164, 305–336.