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Interpolation of nonlinear integral Urysohn operators in net spaces

In this paper, we study the interpolation properties of the net spaces $N_{p,q}(M)$, in the case when M is a sufficiently general arbitrary system of measurable subsets from $\mathbb{R}^n$. The integral Urysohn operator is considered. This operator generalizes all linear, integral operators, and non-linear integral operators. The Urysohn operator is not a quasilinear or subadditive operator. Therefore, the classical interpolation theorems for these operators do not hold. A certain analogue of the Marcinkiewicz-type interpolation theorem for this class of operators is obtained. This theorem allows to obtain, in a sense, a strong estimate for Urysohn operators in net spaces from weak estimates for these operators in net spaces with local nets. For example, in order for the Urysohn integral operator in a net space, where the net is the set of all balls in $\mathbb{R}^n$, it is sufficient for it to be of weak type for net spaces, where the net is concentric balls.

Keywords: interpolation spaces, net spaces, Urysohn integral operators.

Introduction

Let (V,ν), (U,μ) are measurable spaces and $Z(U)$, $M(V)$ are normed spaces of ν-measurable and μ-measurable functions, respectively. Let $K : \mathbb{R}\times U\times V \to \mathbb{R}$, and the operator $T : Z(U) \to M(V)$ is defined by the following equality: For any $f \in Z(U)$

$$T(f,y) = \int_U K(f(x),x,y)d\mu \quad , \quad y\in V \tag{1}$$

and assume that this integral exists and is finite for almost all $y \in V$. This operator is called the Urysohn integral operator.

In the paper [1], new interpolation theorems were proved for these operators in Morrey-type spaces. Analogs of the interpolation theorems of Marcinkiewicz-Calderon, Stein-Weiss, Petre were obtained.

In this paper, we study the interpolation properties of the net spaces $N_{p,q}(M)$. Also, we prove a certain analogue of the Marcinkiewicz-type interpolation theorem for the Urysohn operator (1). We use the ideas developed in [1–3], where an interpolation theorem of Marcinkiewicz type for Morrey spaces was obtained.

Let in $\mathbb{R}^n$ is given n-dimensional Lebesgue measure μ, M is an arbitrary system of measurable subsets from $\mathbb{R}^n$. For a function $f(x)$, defined and integrable on each e from M, we define the function

$$\bar{f}(t,M) = \sup_{\substack{e\in M\\ |e|\geq t}} \frac{1}{|e|}\left|\int_e f(x)dx\right|, \quad t>0,$$

where the supremum is taken over all $e \in M$, whose measure is $|e| \overset{\text{def}}{=} \mu e \geq t$. In the case, when $\sup\{|e| : e\in M\} = \alpha < \infty$ and $t > \alpha$ assuming that $\bar{f}(t,M) = 0$.

Let p,q parameters satisfy the conditions $0 < p \leq \infty$, $0 < q \leq \infty$. We define the net spaces $N_{p,q}(M)$, as the set of all functions f, such that for $q < \infty$

$$\|f\|_{N_{p,q}(M)} = \left(\int_0^\infty \left(t^{\frac{1}{p}}\bar{f}(t,M)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,$$

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and for $q=\infty$

$$\|f\|_{N_{p,\infty}(M)}=\sup_{t>0} t^{\frac{1}{p}}\bar{f}(t,M)<\infty.$$

These spaces were introduced in the work [4]. Net spaces have found important applications in various problems of harmonic analysis, operator theory and the theory of stochastic processes [5–13].

Marcinkiewicz-type interpolation theorem for Urysohn operators

A family of measurable sets $G=\{G_t\}_{t>0}$ is called a local net if it satisfies the following conditions $G_t\subset G_s$ for $t\le s$ and $|G_t|=t$. An example of a local net is the set $\{B_t(x)\}_{t>0}$ of all balls centered at the point x.

Let $G=\{G_t\}_{t>0}$ be a local net. We define the net $F_{G,\Omega}=\bigcup_{x\in\Omega} G+x$, where $G+x=\{G_t+x\}_{t>o}$. The net $F_{G,\Omega}$ will be called the generated by local net G and the set Ω.

Example. Let $\Omega=\mathbb{R}^n$, $G=\{Q_t\}_{t>0}$ be a set of cubes centered at 0, then $F_{G,\Omega}=\{Q_t+x\}_{t>0,\ x\in\mathbb{R}^n}$ is the set of all cubes in $\mathbb{R}^n$.

Lemma 1. Let T be the Urysohn operator of the form (1), then for an arbitrary function $f\in Z(U)$ from the domain and for any μ measurable set $w\subset U$ the following condition holds:

$$T(f,y)=T(f\chi_w,y)+T(f\chi_{U\setminus w},y)-T(0,y).$$

Proof. Due to the additivity of the integral with respect to measure

$$\begin{aligned}T(f,y)-T(f\chi_w,y)&=\int_U K(f(x),xy)d\mu-\int_U K(f(x)\chi_w(x),x,y)d\mu\\
&=\int_{U\setminus w} K(f(x)\chi_{U\setminus w}(x),x,y)d\mu+\int_w K(f(x)\chi_w(x),x,y)d\mu\\
&\quad-\int_{U\setminus w} K(0,x,y)d\mu-\int_w K(f(x)\chi_w(x),x,y)d\mu\\
&=\int_U K(f(x)\chi_{U\setminus w}(x),x,y)d\mu-\int_w K(0,x,y)d\mu-\int_{U\setminus w} K(0,x,y)d\mu\\
&=\int_U K(f(x)\chi_{U\setminus w}(x),x,y)d\mu-\int_U K(0,x,y)d\mu=T(f\chi_{U\setminus w},y)-T(0,y).\end{aligned}$$

Lemma 2. (Hardy's inequalities) Let $\mu>0,-\infty<\nu<\infty$ and $0<\sigma\le\tau\le\infty$, then the following inequalities hold

$$\left(\int_0^\infty\left(y^{-\mu}\left(\int_0^y\left(r^{-\nu}|g(r)|\right)^\sigma\frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\le(\mu\sigma)^{-\frac{1}{\sigma}}\left(\int_0^\infty\left(y^{-\mu-\nu}|g(y)|\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}$$

and

$$\left(\int_0^\infty\left(y^{\mu}\left(\int_0^y\left(r^{-\nu}|g(r)|\right)^\sigma\frac{dr}{r}\right)^{\frac{1}{\sigma}}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\le(\mu\sigma)^{-\frac{1}{\sigma}}\left(\int_0^\infty\left(y^{\mu-\nu}|g(y)|\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}.$$

Theorem 1. Let $\Omega\subset\mathbb{R}^n$, $G=\{G_t\}_{t>0}$ is the local net, $F=\bigcup_{x\in\Omega} G+x$. Let $0<p_0<p_1<\infty$ and $0<q_0,q_1\le\infty$, $q_0\ne q_1$, $0<\theta<1$, $1\le\tau\le\infty$,

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

If for the Urysohn operator T and some $M_0,M_1>0$ the following inequalities hold

$$\|T(f)-T(0)\|_{N_{q_i,\infty}(G+x)}\le M_i\|f\|_{N_{p_i,1}(G+x)},\ i=0,1,\ x\in\Omega, \tag{2}$$

then

$$\|T(f)-T(0)\|_{N_{q,\tau}(F)}\le cM_0^{1-\theta}M_1^\theta\|f\|_{N_{p,\tau}(F)}, \tag{3}$$

for all functions $f\in N_{p,\tau}(F)$, where $c>0$ depends only on the parameters $p_0,p_1,q_0,q_1,p,q,\tau,\theta$.

Proof.

Let $1 \le \tau < \infty$, $f \in N_{p,\tau}(F)$, for arbitrary $x \in \Omega$, $s > 0$, we define the functions

$$f_{0,s} = f\chi_{G_s+x}, \quad f_{1,s} = f - f_{0,s},$$

where χ_{G_s+x} denotes the characteristic function of the set $G_s + x$. It is easily seen that $f_{0,s} \in N_{p_0,1}$ and $f_{1,s} \in N_{p_1,1}$. Then $f = f_{0,s} + f_{1,s}$ and

$$\sup_{\xi \ge t} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} (T(f, y) - T(0, y))dy \right|$$

$$= \sup_{\xi \ge t} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} (T(f\chi_{G_s+x}, y) + T(f\chi_{\mathbb{R}^n \setminus G_s+x}, y) - 2T(0, y))dy \right|$$

$$\le \sup_{\xi \ge t} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} (T(f_{0,s}, y) - T(0, y))dy \right|$$

$$+ \sup_{\xi \ge t} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} (T(f_{1,s}, y) - T(0, y))dy \right| = I_1 + I_2.$$

First, we estimate I_1, according to the inequality (2) we have

$$I_1 = \sup_{\xi \ge t} \frac{1}{|G_s|} \left| \int_{G_\xi + x} (T(f_{0,s}, y) - T(0, y))dy \right|$$

$$\le t^{-\frac{1}{q_0}} \sup_{r>0} r^{\frac{1}{q_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} (T(f_{0,s}, y) - T(0, y))dy \right|$$

$$= t^{-\frac{1}{q_0}} \|(T(f_{0,s}, y) - T(0, y))\|_{N_{q_0,\infty}(G+x)} \le M_0 t^{-\frac{1}{q_0}} \|f_{0,s}\|_{N_{p_0,1}(G+x)}$$

$$= M_0 t^{-\frac{1}{q_0}} \left(\int_0^s r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} f_{0,s}(y)dy \right| \frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \left| \int_{G_\xi + x} f_{0,s}(y)dy \right| \frac{dr}{r} \right).$$

Let $0 < r \le s$, if $\xi \le s$, $y \in G_\xi + x$, we have $f_{0,s}(y) = f(y)\chi_{G_s+x} = f(y)$, if $\xi > s$, then

$$\left| \int_{G_\xi + x} f_{0,s}(y)dy \right| = \left| \int_{G_s + x} f(y)dy \right|.$$

By the first integral, we have the following,

$$\int_0^s r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi} \left| \int_{G_\xi + x} f_{0,s}(y)dy \right| \frac{dr}{r} = \int_0^s r^{\frac{1}{p_0}} \sup_{s \ge \xi \ge r} \frac{1}{|G_\xi} \left| \int_{G_\xi + x} f(y)dy \right| \frac{dr}{r}$$

$$\le \int_0^s r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi} \left| \int_{G_\xi + x} f(y)dy \right| \frac{dr}{r} \le \int_0^s r^{\frac{1}{p_0}} \bar{f}(r, F) \frac{dr}{r}.$$

By the second integral, we have

$$\int_s^\infty r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi} \left| \int_{G_\xi + x} f_{0,s}(y)dy \right| \frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi} \left| \int_{G_s + x} f(y)dy \right| \frac{dr}{r}$$

$$= \left| \int_{G_s + x} f(y)dy \right| \int_s^\infty r^{\frac{1}{p_0}} \sup_{\xi \ge r} \frac{1}{|G_\xi|} \frac{dr}{r} = \left| \int_{G_s + x} f(y)dy \right| \int_s^\infty r^{\frac{1}{p_0} - 1} \frac{dr}{r}$$

$$= p_0' s^{\frac{1}{p_0}} \frac{1}{|G_s|} \left| \int_{G_s + x} f(y)dy \right| \le p_0' s^{\frac{1}{p_0}} \bar{f}(s, F).$$

Thus, we get

$$I_1 \lesssim M_0 t^{-\frac{1}{q_0}} \left(\int_0^s r^{\frac{1}{p_0}} \bar{f}(r, F) \frac{dr}{r} + s^{\frac{1}{p_0}} \bar{f}(s, F) \right).$$

We estimate I_2 in a similar way applying the inequality (2), we obtain

$$I_2 = \sup_{s\geq t}\frac{1}{|G_s|}\Big|\int_{G_s+x}(T(f_{1,s},y)-T(0,y))dy\Big|$$
$$\leq t^{-\frac{1}{q_1}}\sup_{r>0} r^{\frac{1}{q_1}}\sup_{s\geq r}\frac{1}{|G_s|}\Big|\int_{G_s+x}(T(f_{1,s},y)-T(0,y))dy\Big|$$
$$= t^{-\frac{1}{q_1}}\|(T(f_{1,s},y)-T(0,y))\|_{N_{q_1,\infty}(G+x)} \leq M_1 t^{-\frac{1}{q_1}}\|f_{1,s}\|_{N_{p_1,1}(G+x)}$$
$$= M_1 t^{-\frac{1}{q_1}}\left(\int_0^\infty r^{\frac{1}{p_1}}\sup_{s\geq r}\frac{1}{|G_s|}\Big|\int_{G_s+x} f_{1,s}(y)dy\Big|\frac{dr}{r}\right)$$
$$= M_1 t^{-\frac{1}{q_1}}\left(\int_0^s r^{\frac{1}{p_1}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\Big|\int_{G_\xi+x} f_{1,s}(y)dy\Big|\frac{dr}{r} + \int_s^\infty r^{\frac{1}{p_1}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\Big|\int_{G_\xi+x} f_{1,s}(y)dy\Big|\frac{dr}{r}\right)$$
$$= M_1 t^{-\frac{1}{q_1}}\Big(J_1+J_2\Big).$$

To estimate J_1, J_2 note that

$$\Big|\int_{G_\xi+x} f_{1,s}(y)dy\Big| = \begin{cases} 0,\ \xi\leq s, \\ \Big|\int_{G_\xi+x\setminus G_s+x} f(y)dy\Big|,\ \xi>s \end{cases}$$

$$= \begin{cases} 0,\ \xi\leq s, \\ \Big|\int_{G_\xi+x} f(y)dy - \int_{G_s+x} f(y)dy\Big| \leq \Big|\int_{G_\xi+x} f(y)dy\Big| + \Big|\int_{G_s+x} f(y)dy\Big|,\ \xi>s. \end{cases}$$

Further,

$$J_1 \leq \int_0^s r^{\frac{1}{p_1}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\left(\Big|\int_{G_\xi+x} f(y)dy\Big| + \Big|\int_{G_s+x} f(y)dy\Big|\right)\frac{dr}{r}$$
$$\leq \int_0^s r^{\frac{1}{p_1}}\left(\bar{f}(s,F) + \Big|\int_{G_s+x} f(y)dy\Big|\sup_{\xi\geq r}\frac{1}{|G_\xi|}\right)\frac{dr}{r}$$
$$\leq 2\bar{f}(s,F)\int_0^s r^{\frac{1}{p_1}}\frac{dr}{r} = 2p_1 s^{\frac{1}{p_1}}\bar{f}(s,F),$$

and

$$J_2 \leq \int_s^\infty r^{\frac{1}{p_1}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\left(\Big|\int_{G_\xi+x} f(y)dy\Big| + \Big|\int_{G_s+x} f(y)dy\Big|\right)\frac{dr}{r}$$
$$\leq \int_s^\infty r^{\frac{1}{p_1}}\left(\bar{f}(s,F) + \Big|\int_{G_s+x} f(y)dy\Big|\sup_{\xi\geq r}\frac{1}{|G_\xi|}\right)\frac{dr}{r} \leq \int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}$$
$$+\Big|\int_{G_s+x} f(y)dy\Big|\int_s^\infty r^{\frac{1}{p_1}}\sup_{\xi\geq r}\frac{1}{|G_\xi|}\frac{dr}{r} = \int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}$$
$$+\Big|\int_{G_s+x} f(y)dy\Big|\frac{s^{\frac{1}{p_1}-1}}{p_1-1}p_1 \lesssim \int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_1}}\bar{f}(s,F).$$

Combining the estimates, we obtain the following estimate

$$I_2 \lesssim M_1 t^{\frac{1}{q_1}}\left(\int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_1}}\bar{f}(s,F)\right).$$

So, we got the following estimate

$$\sup_{s\geq t}\frac{1}{|G_s|}\Big|\int_{G_s+x}(T(f,y)-T(0,y))dy\Big| \lesssim M_0 t^{-\frac{1}{q_0}}\left(\int_0^s r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r} + s^{\frac{1}{p_0}}\bar{f}(s,F)\right)$$

$$+M_1 t^{-\frac{1}{q_1}}\left(\int_s^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}+s^{\frac{1}{p_1}}\bar{f}(s,F)\right).$$

Assuming that $s=ct^\gamma$, where $\gamma=\left(\frac{1}{q_1}-\frac{1}{q_0}\right)\Big/\left(\frac{1}{p_1}-\frac{1}{p_0}\right)$, then, taking into account the above estimates, we obtain

$$\|T(f)-T(0)\|_{N_{q,\tau}(F)}=\left(\int_0^\infty\left(t^{\frac{1}{q}}\sup_{\substack{s\geq t\\ x\in\mathbb{R}^n}}\frac{1}{|G_s|}\left|\int_{G_s+x}f(x)dx\right|\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}$$

$$\lesssim M_0A_1+M_0A_2+M_1A_3+M_1A_4,$$

where

$$A_1=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_0}}\int_0^{ct^\gamma}r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_2=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_0}}(ct^\gamma)^{\frac{1}{p_0}}\bar{f}(ct^\gamma,F)\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_3=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_1}}\int_{ct^\gamma}^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}},$$

$$A_4=\left(\int_0^\infty\left(t^{\frac{1}{q}-\frac{1}{q_1}}(ct^\gamma)^{\frac{1}{p_1}}\bar{f}(ct^\gamma,F)\right)^\tau\frac{dt}{t}\right)^{\frac{1}{\tau}}.$$

Using the change of variable $ct^\gamma=y$, we get

$$A_1=\gamma^{-\frac{1}{\tau}}c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_1,\quad A_2=\gamma^{-\frac{1}{\tau}}c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_2,$$

$$A_3=\gamma^{-\frac{1}{\tau}}c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_2,\quad A_4=\gamma^{-\frac{1}{\tau}}c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}B_3,$$

where

$$B_1=\left(\int_0^\infty\left(y^{\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\int_0^y r^{\frac{1}{p_0}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}},$$

$$B_2=\left(\int_0^\infty\left(y^{\frac{1}{p}}\bar{f}(y,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}=\|f\|_{N_{p,\tau}(F)},$$

$$B_3=\left(\int_0^\infty\left(y^{-(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\int_y^\infty r^{\frac{1}{p_1}}\bar{f}(r,F)\frac{dr}{r}\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}},$$

$$B_4=\left(\int_0^\infty\left(y^{\frac{1}{p}}\bar{f}(y,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}=\|f\|_{N_{p,\tau}(F)}.$$

To estimate B_1,B_3 we apply Hardy's inequalities from the lemma 2 and we obtain

$$B_1\lesssim\left(\int_0^\infty\left(y^{\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)+\frac{1}{p_0}}\bar{f}(r,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\lesssim\|f\|_{N_{p,\tau}(F)},$$

$$B_3\lesssim\left(\int_0^\infty\left(y^{(\theta-1)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)+\frac{1}{p_1}}\bar{f}(r,F)\right)^\tau\frac{dy}{y}\right)^{\frac{1}{\tau}}\lesssim\|f\|_{N_{p,\tau}(F)}.$$

Thus, from the above estimates, the following estimate was obtained

$$\|T(f)-T(0)\|_{N_{q,\tau}(F)}\lesssim\left(M_0c^{-\theta\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}+M_1c^{(1-\theta)\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}\right)\|f\|_{N_{p,\tau}(F)},$$

where the corresponding constants depend only on $p_0, p_1, q_0, q_1, p, q, \tau$ and θ.

Let $c = \left(\frac{M_1}{M_0}\right)^{\frac{p_0 p_1}{p_1 - p_0}}$, then

$$\|T(f) - T(0)\|_{N_{q,\tau}(F)} \lesssim M_0^{1-\theta} M_1^{\theta} \|f\|_{N_{p,\tau}(F)}.$$

Consequently, we obtain the required estimate (3). The theorem is proved.

Corollary. Let F be the set of all balls in $\mathbb{R}^n$, F_x – the set of all balls centered at the point $x \in \mathbb{R}^n$. Let $0 < p_0 < p_1 < \infty$ and $0 < q_0, q_1 \leq \infty$, $q_0 \neq q_1$, $0 < \theta < 1$, $1 \leq \tau \leq \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If the following inequalities hold for the Urysohn operator T and some $M_0, M_1 > 0$

$$\|T(f) - T(0)\|_{N_{q_i,\infty}(G+x)} \leq M_i \|f\|_{N_{p_i,1}(G+x)}, \ i = 0, 1, \ x \in \mathbb{R}^n,$$

then for all functions $f \in N_{p,\tau}(F)$, holds

$$\|T(f) - T(0)\|_{N_{q,\tau}(F)} \leq c M_0^{1-\theta} M_1^{\theta} \|f\|_{N_{p,\tau}(F)},$$

where $c > 0$ depends only on parameters $p_0, p_1, q_0, q_1, p, q, \tau, \theta$.

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Сызықтық емес интегралды Урысон операторларының торлы кеңістіктеріндегі интерполяциясы

Мақалада $N_{p,q}(M)$ торлы кеңістіктерінің интерполяциялық қасиеттері зерттелген, мұндағы M – $\mathbb{R}^n$ жиынының өлшенетін ішкі жиындардың жеткілікті жалпы ерікті жүйесі. Интегралды Урысон операторы қарастырылған. Бұл оператор барлық сызықтық, интегралды операторларды, сондай-ақ сызықты емес интегралды операторларды жалпылайды. Урысон операторы, әдетте, квазисызықты немесе субаддитивті оператор емес, сондықтан бұл операторлар үшін классикалық интерполяциялық теоремалар орындалмайды. Осы операторлар класы үшін Марцинкевич типіндегі интерполяциялық теоремасының белгілі бір аналогы алынды. Бұл теорема белгілі бір мағынада локальды торы бар торлы кеңістіктердегі Урысон операторлары үшін әлсіз бағалаулардан торлы кеңістіктердегі осы операторлар үшін күшті бағалау алуға мүмкіндік береді. Мысалы, тор $\mathbb{R}^n$-дегі барлық шарлар жиынтығы болатын торлы кеңістігінде Урысон интегралды операторы болу үшін, оның тор концентрлі шарлар болатын торлы кеңістіктер үшін әлсіз типті болуы жеткілікті.

Кілт сөздер: интерполяция кеңістіктері, торлы кеңістіктер, Урысон операторлары.

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Интерполяция нелинейных интегральных операторов Урысона в сетевых пространствах

В статье изучены интерполяционные свойства сетевых пространств $N_{p,q}(M)$, в случае когда M есть достаточно общая произвольная система измеримых подмножеств из $\mathbb{R}^n$. Рассмотрен интегральный оператор Урысона. Данный оператор обобщает все линейные, интегральные, а также нелинейные интегральные операторы. Оператор Урысона, вообще говоря, не является квазилинейным либо субаддитивным, поэтому классические интерполяционные теоремы для этих операторов не имеют места. Получен некий аналог интерполяционной теоремы типа Марцинкевича для этого класса операторов. Настоящая теорема позволяет получать в некотором смысле сильную оценку для операторов Урысона в сетевых пространствах из слабых оценок для них. Так, например, для того, чтобы был интегральный оператор Урысона в сетевом пространстве, где сеть есть множество всех шаров в $\mathbb{R}^n$, достаточно, чтобы он был слабого типа для сетевых пространств, где сеть есть концентрические шары.

Ключевые слова: интерполяционные пространства, сетевое пространство, операторы Урысона, интегральный оператор.

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