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## Upper Estimates of the angle best approximations of generalized Liouville-Weyl derivatives

In this article we consider continuous functions  $f$  with period  $2\pi$  and their approximation by trigonometric polynomials. This article is devoted to the study of estimates of the best angular approximations of generalized Liouville-Weyl derivatives by angular approximation of functions in the three-dimensional case. We consider generalized Liouville-Weyl derivatives instead of the classical mixed Weyl derivative. In choosing the issues to be considered, we followed the general approach that emerged after the work of the second author of this article. Our main goal is to prove analogs of the results of in the three-dimensional case. The concept of general monotonic sequences plays a key role in our study. Several well-known inequalities are indicated for the norms, best approximations of the  $r$ -th derivative with respect to the best approximations of the function  $f$ . The issues considered in this paper are related to the range of issues studied in the works of Bernstein. Later Stechkin and Konyushkov obtained an inequality for the best approximation  $f^{(r)}$ . Also, in the works of Potapov, using the angle approximation, some classes of functions are considered. In subsection 1 we give the necessary notation and useful lemmas. Estimates for the norms and best approximations of the generalized Liouville-Weyl derivative in the three-dimensional case are obtained.

*Keywords:* Lebesgue space, best approximation by three-dimensional angle, trigonometric polynomial, Liouville-Weyl derivative.

### Introduction

Let us mention several well-known inequalities for norms and best approximations of the  $r$ -th derivative in terms of best approximations of the function  $f$ .

The following result was proved by Bernstein for  $p = \infty$  (for  $1 \leq p < \infty$ , see [2]) if  $f \in L_p$ ,  $1 \leq p \leq \infty$ , and  $\sum_{k=0}^{\infty} (k+1)^{r-1} E_k(f)_p < \infty$ ,  $r \in \mathbb{N}$ , then  $\|f^{(r)}\|_p \leq C(r) \sum_{k=0}^{\infty} (k+1)^{r-1} E_k(f)_p$  [1].

Later on, Stechkin [3] for  $p = \infty$  and Konyushkov [4] for  $1 < p < \infty$  obtained the following inequality for the best approximations of  $f^{(r)}$ :

$$E_n(f^{(r)})_p \leq C(r, p) \left( n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_p \right) \quad r, n \in \mathbb{N}.$$

The last inequality was extended by the formula Timan [5] for the case of  $1 < p < \infty$  as follows:

$$E_n(f^{(r)})_p \leq C(r) \left( n^r E_n(f)_p + \left( \sum_{k=n+1}^{\infty} k^{\theta r - 1} E_k^\theta(f)_p \right)^{\frac{1}{\theta}} \right), \quad \theta = \min(2, p) \quad r, n \in \mathbb{N}.$$

Also, A. Jumabayeva and B. Simonov obtained estimates of norms and the angle best approximations of the generalized Liouville-Weyl derivatives by the angle approximation of functions in the two-dimensional case [6, 7].

Let  $L_p(T^3)$ ,  $1 < p < \infty$  be the space of measurable functions of three variables that are  $2\pi$  periodic in each variable and such that

$$\|f\|_p = \left( \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |f(x_1, x_2, x_3)|^p dx_1 dx_2 dx_3 \right)^{1/p} < \infty.$$

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$L_p^0$  is the set of functions  $f \in L_p$  such that  $\int_0^{2\pi} f(x_1, x_2, x_3) dx_1 = 0$  for almost everyone  $x_2, x_3$ ,  $\int_0^{2\pi} f(x_1, x_2, x_3) dx_2 = 0$  for almost everyone  $x_1, x_3$  and  $\int_0^{2\pi} f(x_1, x_2, x_3) dx_3 = 0$  for almost everyone  $x_1, x_2$ .

Let  $Y_{m_1, m_2, m_3}(f)_p$  be the best approximation by a three-dimensional angle of the function  $f \in L_p(\mathbb{T}^3)$ , i.e.

$$Y_{m_1, m_2, m_3}(f)_p = \inf_{T_{m_1, \infty, \infty}, T_{\infty, m_2, \infty}, T_{\infty, \infty, m_3}} \|f - T_{m_1, \infty, \infty} - T_{\infty, m_2, \infty} - T_{\infty, \infty, m_3}\|_p,$$

where the function  $T_{m_1, \infty, \infty}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$  is a trigonometric polynomial of order at most  $m_1$  in  $x_1$ , the function  $T_{\infty, m_2, \infty}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$  is a trigonometric polynomial of order at most  $m_2$  in  $x_2$  and the function  $T_{\infty, \infty, m_3}(x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$  is a trigonometric polynomial of order at most  $m_3$  in  $x_3$ . In the work of Potapov using the angle approximation, some classes of functions are considered [8, 9].

By  $\sigma(f)$  we denote the Fourier series of a function  $f \in L_p(\mathbb{T}^3)$ , that is

$$\sigma(f) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{k_3=-\infty}^{\infty} c_{k_1, k_2, k_3} e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} A_{n_1, n_2, n_3}(x_1, x_2, x_3), \quad (1)$$

where  $c_{k_1, k_2, k_3} = \frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(x_1, x_2, x_3) e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3)} dx_1 dx_2 dx_3$ .

The transformed Fourier series of  $\sigma(f)$  is given by

$$\begin{aligned} & \sigma(f, \lambda, \beta_1, \beta_2, \beta_3) = \\ & = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} \lambda_{n_1, n_2, n_3} [c_{n_1, n_2, n_3} e^{i(n_1 x_1 + \beta_1 \frac{x_1}{2})} e^{i(n_2 x_2 + \beta_2 \frac{x_2}{2})} e^{i(n_3 x_3 + \beta_3 \frac{x_3}{2})} |n_1|^{\beta_1} |n_2|^{\beta_2} |n_3|^{\beta_3}], \end{aligned}$$

where  $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$  and  $\lambda = \{\lambda_{n_1, n_2, n_3}\}_{n_1 n_2 n_3 \in \mathbb{N}}$  is a sequence of real numbers.

Let  $\varphi(x_1 x_2 x_3) \sim \sigma(f, \lambda, \beta_1, \beta_2, \beta_3)$  is the  $(\lambda, \beta_1, \beta_2, \beta_3)$  is the mixed derivative of the function  $f$  (or Liouville–Weyl derivative) and denote it by  $f^{(\lambda, \beta_1, \beta_2, \beta_3)}(x_1 x_2 x_3)$ . For example, if  $\lambda_{n_1, n_2, n_3} = n_1^{r_1} n_2^{r_2} n_3^{r_3}$ ,  $r_i > 0$ ,  $\beta_i = r_i$  ( $i = 1, 2, \dots$ )  $\Rightarrow f^{(\lambda, \beta_1, \beta_2, \beta_3)} = f^{(r_1, r_2, r_3)}$ , where  $f^{(r_1, r_2, r_3)}$ -mixed derivative of the function  $f$  in the sense of Weyl.

*Definition 1.1.* [10, 11] A sequence  $\lambda := \{\lambda_n\}_{n=1}^{\infty}$  is said to be general monotone, written  $\lambda \in GM^3$ , if the relations

$$\begin{aligned} \sum_{k_1=n_1}^{2n_1} |\lambda_{k_1, n_2, n_3} - \lambda_{k_1+1, n_2, n_3}| & \leq C |\lambda_{n_1, n_2, n_3}|, \quad \sum_{k_2=n_2}^{2n_2} |\lambda_{n_1, k_2, n_3} - \lambda_{n_1, k_2+1, n_3}| \leq C |\lambda_{n_1, n_2, n_3}|, \\ \sum_{k_3=n_3}^{2n_3} |\lambda_{n_1, n_2, k_3} - \lambda_{n_1, n_2, k_3+1}| & \leq C |\lambda_{n_1, n_2, n_3}|, \end{aligned}$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} |\lambda_{k_1, k_2, n_3} - \lambda_{k_1+1, k_2, n_3} - \lambda_{k_1, k_2+1, n_3} + \lambda_{k_1+1, k_2+1, n_3}| \leq C |\lambda_{n_1, n_2, n_3}|,$$

$$\sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} |\lambda_{n_1, k_2, k_3} - \lambda_{n_1, k_2+1, k_3} - \lambda_{n_1, k_2, k_3+1} + \lambda_{n_1, k_2+1, k_3+1}| \leq C |\lambda_{n_1, n_2, n_3}|,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_3=n_3}^{2n_3} |\lambda_{k_1, n_2, k_3} - \lambda_{k_1+1, n_2, k_3} - \lambda_{k_1, n_2, k_3+1} + \lambda_{k_1+1, n_2, k_3+1}| \leq C |\lambda_{n_1, n_2, n_3}|,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} |\lambda_{k_1, k_2, k_3} - \lambda_{k_1+1, k_2, k_3} - \lambda_{k_1, k_2+1, k_3} - \lambda_{k_1, k_2, k_3+1} +$$

$$+ \lambda_{k_1, k_2+1, k_3+1} + \lambda_{k_1+1, k_2, k_3+1} + \lambda_{k_1+1, k_2+1, k_3} - \lambda_{k_1+1, k_2+1, k_3+1}| \leq C |\lambda_{n_1, n_2, n_3}|$$

hold for all integers  $n_1, n_2$  and  $n_3$ , where the constant  $C$  is independent of  $n_1, n_2$  and  $n_3$ .

Auxiliary results

In order to prove the main result, we formulate auxiliary statements. We denote

$$\Delta_{m_1, m_2, m_3} := \sum_{n_1=2^{m_1-1}}^{2^{m_1}-1} \sum_{n_2=2^{m_2-1}}^{2^{m_2}-1} \sum_{n_3=2^{m_3-1}}^{2^{m_3}-1} A_{n_1, n_2, n_3}(x_1, x_2, x_3), \quad m_1, m_2, m_3 = 1, 2, \dots$$

Lemma 2.1. [11]  $\{\lambda_n\} \in GM$  if and only if there exists  $C > 0$ , such that

$$(i) \quad |\lambda_k| \leq C |\lambda_n| \quad \text{for } n \leq k \leq 2n; \quad (ii) \quad \sum_{k=n}^N |\Delta \lambda_k| \leq C(|\lambda_n| + \sum_{k=n+1}^N \frac{|\lambda_k|}{k}) \quad \text{for any } n < N.$$

By [11], it follows that if  $\{\lambda_{n_1 n_2 n_3}\} \in GM^3$ , then

$$|\lambda_{k_1, k_2, k_3}| \leq C |\lambda_{n_1, n_2, n_3}| \quad \text{for } n_1 \leq k_1 \leq 2n_1, n_2 \leq k_2 \leq 2n_2, n_3 \leq k_3 \leq 2n_3.$$

This implies that the condition

$$\begin{aligned} & \sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} |\lambda_{k_1, k_2, k_3} - \lambda_{k_1+1, k_2, k_3} - \lambda_{k_1, k_2+1, k_3} - \lambda_{k_1, k_2, k_3+1} + \lambda_{k_1, k_2+1, k_3+1} + \\ & + \lambda_{k_1+1, k_2, k_3+1} + \lambda_{k_1+1, k_2+1, k_3} - \lambda_{k_1+1, k_2+1, k_3+1}| \leq C(|\lambda_{n_1, n_2, n_3}| + |\lambda_{2n_1, 2n_2, 2n_3}|) \end{aligned}$$

is equivalent to the condition

$$\begin{aligned} & \sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} \sum_{k_3=n_3}^{2n_3} |\lambda_{k_1, k_2, k_3} - \lambda_{k_1+1, k_2, k_3} - \lambda_{k_1, k_2+1, k_3} - \lambda_{k_1, k_2, k_3+1} + \\ & + \lambda_{k_1, k_2+1, k_3+1} + \lambda_{k_1+1, k_2, k_3+1} + \lambda_{k_1+1, k_2+1, k_3} - \lambda_{k_1+1, k_2+1, k_3+1}| \leq C |\lambda_{n_1, n_2, n_3}|. \end{aligned}$$

Lemma 2.2. (Minkowskii inequality [12]) Let  $1 \leq p < \infty$  and  $a_{\nu k} \geq 0$ , then

$$(a) \quad \left( \sum_{k=1}^{\infty} \left( \sum_{\nu=1}^k a_{\nu k} \right)^p \right)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} \left( \sum_{k=\nu}^{\infty} a_{\nu k}^p \right)^{\frac{1}{p}}, \quad (b) \quad \left( \sum_{k=1}^{\infty} \left( \sum_{\nu=k}^{\infty} a_{\nu k} \right)^p \right)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} \left( \sum_{k=1}^{\nu} a_{\nu k}^p \right)^{\frac{1}{p}}.$$

Lemma 2.3. [12] For a function  $f(u, y)$  defined on measurable set  $E = E_1 \times E_2 \subset \mathbb{R}_n$ , where  $x = (u, y)$ ,  $u = (x_1, \dots, x_m)$ ,  $y = (x_{m+1}, \dots, x_n)$ , the following inequality holds

$$\left( \int_{E_1} \left| \int_{E_2} f(u, y) dy \right|^p du \right)^{\frac{1}{p}} \leq \int_{E_2} \left( \int_{E_1} |f(u, y)|^p du \right)^{\frac{1}{p}} dy.$$

Lemma 2.4. [8] Let  $f \in L_p(\mathbb{T}^2)$ ,  $1 < p < \infty$ ,  $m_i \in \mathbb{N} \cup 0$  ( $i = 1, 2$ ). Then

$$\|f - S_{m_1, \infty}(f) - S_{\infty, m_2}(f) + S_{m_1, m_2}(f)\|_p \asymp Y_{m_1, m_2}(f)_p,$$

where  $S_{m_1, m_2}$  are the partial sums of the Fourier series of a function  $f$ .

Lemma 2.5. [8] a) Let  $1 < p < \infty$  and (1) be the Fourier series of  $f \in L_p(\mathbb{T}^3)$ , then

$$C_1(p) \|f\|_p \leq \left( \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \Delta_{m_1 m_2 m_3}^2 \right)^{\frac{p}{2}} dx_1 dx_2 dx_3 \right)^{\frac{1}{p}} \leq C_2(p) \|f\|_p.$$

b) Let  $1 < p < \infty$ . If (1) satisfies the following inequality

$$I_p = \left( \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \Delta_{m_1 m_2 m_3}^2 \right)^{\frac{p}{2}} dx_1 dx_2 dx_3 \right)^{\frac{1}{p}} < \infty.$$

Then (1) is the Fourier series of a function  $f = (x_1, x_2, x_3) \in L_p(\mathbb{T}^3)$  and  $\|f\|_p \leq C(p) I_p$ .

Main result

The aim of this paper is to prove the following theorem.

*Theorem 3.1.* Let  $1 < p < \infty$ ,  $0 < \theta \leq \min(p, 2)$ ,  $\lambda := \{\lambda_{n_1, n_2, n_3}\}_{n_1, n_2, n_3}$  be sequences of positive numbers such that  $\lambda \in GM^3$ ,  $\alpha_i \in \mathbb{R}_+$ ,  $r_i \in \mathbb{R}_+ \cup \{0\}$  and  $\beta_i \in \mathbb{R}$  ( $i = 1, 2$ ). If for  $f \in L_p^0(\mathbb{T}^3)$  the series

$$\begin{aligned} & \sum_{n_1=1}^{\infty} |\lambda_{n_1+1,1,1}^\theta - \lambda_{n_1,1,1}^\theta| Y_{n_1,0,0}^\theta(f)_p + \\ & + \sum_{n_2=1}^{\infty} |\lambda_{1,n_2+1,1}^\theta - \lambda_{1,n_2,1}^\theta| Y_{0,n_2,0}^\theta(f)_p + \sum_{n_3=1}^{\infty} |\lambda_{1,1,n_3+1}^\theta - \lambda_{1,1,n_3}^\theta| Y_{0,0,n_3}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} |\lambda_{n_1+1,n_2+1,1}^\theta - \lambda_{n_1+1,n_2,1}^\theta - \lambda_{n_1,n_2+1,1}^\theta + \lambda_{n_1,n_2,1}^\theta| Y_{n_1,n_2,0}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{n_1,1,n_3}^\theta - \lambda_{n_1+1,1,n_3}^\theta - \lambda_{n_1,1,n_3+1}^\theta + \lambda_{n_1+1,1,n_3+1}^\theta| Y_{n_1,0,n_3}^\theta(f)_p + \\ & + \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{1,n_2,n_3}^\theta - \lambda_{1,n_2+1,n_3}^\theta - \lambda_{1,n_2,n_3+1}^\theta + \lambda_{1,n_2+1,n_3+1}^\theta| Y_{0,n_2,n_3}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{n_1,n_2,n_3}^\theta - \lambda_{n_1+1,n_2,n_3}^\theta - \lambda_{n_1,n_2+1,n_3}^\theta - \lambda_{n_1,n_2,n_3+1}^\theta + \\ & + \lambda_{n_1,n_2+1,n_3+1}^\theta + \lambda_{n_1+1,n_2,n_3+1}^\theta + \lambda_{n_1+1,n_2+1,n_3}^\theta - \lambda_{n_1+1,n_2+1,n_3+1}^\theta| Y_{n_1,n_2,n_3}^\theta(f)_p \end{aligned} \tag{2}$$

converges, then there exists a function  $\varphi \in L_p^0(\mathbb{T}^3)$ , with the Fourier series  $\sigma(f, \lambda, \beta_1, \beta_2, \beta_3)$  and

$$\begin{aligned} \|\varphi\|_p & \leq \left( \lambda_{1,1,1}^\theta \|f\|_p^\theta + \sum_{n_1=1}^{\infty} |\lambda_{n_1+1,1,1}^\theta - \lambda_{n_1,1,1}^\theta| Y_{n_1,0,0}^\theta(f)_p + \right. \\ & + \sum_{n_2=1}^{\infty} |\lambda_{1,n_2+1,1}^\theta - \lambda_{1,n_2,1}^\theta| Y_{0,n_2,0}^\theta(f)_p + \sum_{n_3=1}^{\infty} |\lambda_{1,1,n_3+1}^\theta - \lambda_{1,1,n_3}^\theta| Y_{0,0,n_3}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} |\lambda_{n_1,n_2,1}^\theta - \lambda_{n_1+1,n_2,1}^\theta - \lambda_{n_1,n_2+1,1}^\theta + \lambda_{n_1+1,n_2+1,1}^\theta| Y_{n_1,n_2,0}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{n_1,1,n_3}^\theta - \lambda_{n_1+1,1,n_3}^\theta - \lambda_{n_1,1,n_3+1}^\theta + \lambda_{n_1+1,1,n_3+1}^\theta| Y_{n_1,0,n_3}^\theta(f)_p + \\ & + \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{1,n_2,n_3}^\theta - \lambda_{1,n_2+1,n_3}^\theta - \lambda_{1,n_2,n_3+1}^\theta + \lambda_{1,n_2+1,n_3+1}^\theta| Y_{0,n_2,n_3}^\theta(f)_p + \\ & + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\lambda_{n_1,n_2,n_3}^\theta - \lambda_{n_1+1,n_2,n_3}^\theta - \lambda_{n_1,n_2+1,n_3}^\theta - \lambda_{n_1,n_2,n_3+1}^\theta + \\ & + \lambda_{n_1,n_2+1,n_3+1}^\theta + \lambda_{n_1+1,n_2,n_3+1}^\theta + \lambda_{n_1+1,n_2+1,n_3}^\theta - \lambda_{n_1+1,n_2+1,n_3+1}^\theta| Y_{n_1,n_2,n_3}^\theta(f)_p \Big)^{\frac{1}{\theta}}, \\ Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)_p & \lesssim \left( \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}} Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(f)_p + \right. \\ & + \sum_{\nu_1=m_1}^{\infty} |\lambda_{2^{\nu_1}, 2^{m_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta| Y_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta(f)_p + \end{aligned} \tag{3}$$

$$\begin{aligned}
 & + \sum_{\nu_2=m_2}^{\infty} \left| \lambda_{2^{m_1-1}, 2^{\nu_2}, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^{\theta} \right| Y_{2^{m_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^{\theta}(f)_p + \\
 & + \sum_{\nu_3=m_3}^{\infty} \left| \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3}}^{\theta} - \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta} \right| Y_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p + \\
 & \quad + \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_2=m_2}^{\infty} \left| \lambda_{2^{\nu_1}, 2^{\nu_2}, 2^{m_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 2^{m_3-1}}^{\theta} - \right. \\
 & \quad - \lambda_{2^{\nu_1}, 2^{\nu_2-1}, 2^{m_3-1}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^{\theta} \left. \right| Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^{\theta}(f)_p + \\
 & \quad + \sum_{\nu_2=m_2}^{\infty} \sum_{\nu_3=m_3}^{\infty} \left| \lambda_{2^{m_1-1}, 2^{\nu_2}, 2^{\nu_3}}^{\theta} - \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3}}^{\theta} - \right. \\
 & \quad - \lambda_{2^{m_1-1}, 2^{\nu_2}, 2^{\nu_3-1}}^{\theta} + \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} \left. \right| Y_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p + \\
 & \quad + \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_3=m_3}^{\infty} \left| \lambda_{2^{\nu_1}, 2^{m_2-1}, 2^{\nu_3}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3}}^{\theta} - \right. \\
 & \quad - \lambda_{2^{\nu_1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta} \left. \right| Y_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p + \\
 & + \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_2=m_2}^{k_2} \sum_{\nu_3=m_3}^{\infty} \left| \lambda_{2^{\nu_1}, 2^{\nu_2}, 2^{\nu_3}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 2^{\nu_3}}^{\theta} - \lambda_{2^{\nu_1}, 2^{\nu_2-1}, 2^{\nu_3}}^{\theta} - \lambda_{2^{\nu_1}, 2^{\nu_2}, 2^{\nu_3-1}}^{\theta} + \right. \\
 & \left. + \lambda_{2^{\nu_1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 2^{\nu_3-1}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} \right| Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p \Big)^{\frac{1}{\theta}}.
 \end{aligned}$$

*Proof.* Let the series (2) be convergent and  $f \in L_p^0(T^3)$ . We use the following inequality

$$\begin{aligned}
 & \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}}^{\theta} \leq \lambda_{111}^{\theta} + \sum_{m_3=2}^{n_3} \left| \lambda_{1,1,2^{m_3-1}}^{\theta} - \lambda_{1,1,2^{m_3-2}}^{\theta} \right| + \\
 & + \sum_{m_2=2}^{n_2} \left| \lambda_{1,2^{m_2-1},1}^{\theta} - \lambda_{1,2^{m_2-2},1}^{\theta} \right| + \sum_{m_1=2}^{n_1} \left| \lambda_{2^{m_1-1},1,1}^{\theta} - \lambda_{2^{m_1-2},1,1}^{\theta} \right| + \\
 & + \sum_{m_1=2}^{n_1} \sum_{m_2=2}^{n_2} \left| \lambda_{2^{m_1-1}, 2^{m_2-1}, 1}^{\theta} - \lambda_{2^{m_1-2}, 2^{m_2-1}, 1}^{\theta} - \lambda_{2^{m_1-1}, 2^{m_2-2}, 1}^{\theta} + \lambda_{2^{m_1-2}, 2^{m_2-2}, 1}^{\theta} \right| + \\
 & + \sum_{m_2=2}^{n_2} \sum_{m_3=2}^{n_3} \left| \lambda_{1, 2^{m_2-1}, 2^{m_3-1}}^{\theta} - \lambda_{1, 2^{m_2-2}, 2^{m_3-1}}^{\theta} - \lambda_{1, 2^{m_2-1}, 2^{m_3-2}}^{\theta} + \lambda_{1, 2^{m_2-2}, 2^{m_3-2}}^{\theta} \right| + \\
 & + \sum_{m_1=2}^{n_1} \sum_{m_3=2}^{n_3} \left| \lambda_{2^{m_1-1}, 1, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-2}, 1, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 1, 2^{m_3-2}}^{\theta} + \lambda_{2^{m_1-2}, 1, 2^{m_3-2}}^{\theta} \right| + \\
 & + \sum_{m_1=2}^{n_1} \sum_{m_2=2}^{n_2} \sum_{m_3=2}^{n_3} \left| \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{m_2-2}, 2^{m_3-2}}^{\theta} - \lambda_{2^{m_1-2}, 2^{m_2-1}, 2^{m_3-2}}^{\theta} - \right. \\
 & \left. - \lambda_{2^{m_1-2}, 2^{m_2-2}, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-2}}^{\theta} + \lambda_{2^{m_1-2}, 2^{m_2-1}, 2^{m_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{m_2-2}, 2^{m_3-1}}^{\theta} + \right. \\
 & \quad \left. + \lambda_{2^{m_1-2}, 2^{m_2-2}, 2^{m_3-2}}^{\theta} \right|.
 \end{aligned}$$

Let us denote  $\Delta_{n_1, n_2, n_3} = \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \sum_{\nu_3=2^{n_3-1}}^{2^{n_3}-1} A_{\nu_1, \nu_2, \nu_3}(f, x_1, x_2, x_3)(n_1, n_2, n_3 = 1, 2, \dots)$ . Using (5) and property of GM (Lemma 2.1), we get

$$I_1 = \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}}^2 \Delta_{n_1, n_2, n_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} =$$

$$\begin{aligned}
 &= \left\| \left[ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}}^2 \Delta_{n_1, n_2, n_3}^2 \right]^{\frac{1}{2}} \right\|_p = \\
 &= \left\| \left[ \lambda_{1,1,1}^2 \Delta_{1,1,1}^2 + \sum_{n_1=2}^{\infty} \lambda_{2^{n_1-1}, 1, 1}^2 \Delta_{n_1, 1, 1}^2 + \sum_{n_2=2}^{\infty} \lambda_{1, 2^{n_2-1}, 1}^2 \Delta_{1, n_2, 1}^2 + \sum_{n_3=2}^{\infty} \lambda_{1, 1, 2^{n_3-1}}^2 \Delta_{1, 1, n_3}^2 + \right. \right. \\
 &\quad \left. \left. + \sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} \lambda_{2^{n_1-1}, 2^{n_2-1}, 1}^2 \Delta_{n_1, n_2, 1}^2 + \sum_{n_1=2}^{\infty} \sum_{n_3=2}^{\infty} \lambda_{2^{n_1-1}, 1, 2^{n_3-1}}^2 \Delta_{n_1, 1, n_3}^2 + \right. \right. \\
 &\quad \left. \left. + \sum_{n_2=2}^{\infty} \sum_{n_3=2}^{\infty} \lambda_{1, 2^{n_2-1}, 2^{n_3-1}}^2 \Delta_{1, n_2, n_3}^2 + \sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_3=2}^{\infty} \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}}^2 \Delta_{n_1, n_2, n_3}^2 \right]^{\frac{1}{2}} \right\|_p \lesssim \\
 &\lesssim \lambda_{1,1,1} \left\| \left[ \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \Delta_{n_1, n_2, n_3}^2 \right]^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_{n_1=2}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_1=2}^{n_1} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \right]^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \\
 &\quad + \left\| \left( \sum_{n_1=1}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_3=1}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_2=2}^{n_2} |\lambda_{1, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{1, 2^{\nu_2-2}, 1}^{\theta}| \right]^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \\
 &\quad + \left\| \left( \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_3=2}^{n_3} |\lambda_{1, 1, 2^{\nu_3-1}}^{\theta} - \lambda_{1, 1, 2^{\nu_3-2}}^{\theta}| \right]^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \\
 &\quad + \left\| \left( \sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_3=1}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left( \sum_{\nu_1=2}^{n_1} \sum_{\nu_2=2}^{n_2} |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^{\theta} + \right. \right. \right. \\
 &\quad \left. \left. \left. + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^{\theta} \right)^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_{n_1=2}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left( \sum_{\nu_1=2}^{n_1} \sum_{\nu_3=2}^{n_3} |\lambda_{2^{\nu_1-1}, 1, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 2^{\nu_3-1}}^{\theta} - \right. \right. \right. \\
 &\quad \left. \left. \left. - \lambda_{2^{\nu_1-1}, 1, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-2}, 1, 2^{\nu_3-2}}^{\theta} \right)^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \left\| \left( \sum_{n_1=1}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_3=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left( \sum_{\nu_2=2}^{n_2} \sum_{\nu_3=2}^{n_3} |\lambda_{1, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \right. \right. \right. \\
 &\quad \left. \left. \left. - \lambda_{1, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} - \lambda_{1, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} + \lambda_{1, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} \right)^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p + \\
 &\quad + \left\| \left( \sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_3=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_1=2}^{n_1} \sum_{\nu_2=2}^{n_2} \sum_{\nu_3=2}^{n_3} |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} - \right. \right. \right. \\
 &\quad \left. \left. \left. - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} + \right. \right. \right. \\
 &\quad \left. \left. \left. + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} \right]^{\frac{2}{\theta}} \right)^{\frac{1}{2}} \right\|_p =: H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8.
 \end{aligned}$$

Let us estimate  $H_1$ . Applying Lemma 2.5, we have  $H_1 \leq C\lambda_{1,1,1}\|f\|_p < \infty$ . Now we estimate  $H_2$  :

$$H_2 = \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{n_1=2}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_1=2}^{n_1} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \right]^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}}.$$

Using Minkowski's inequality and Lemma 2.2 (a) for  $\frac{2}{\theta} \geq 1$ , we derive

$$\begin{aligned}
 &\sum_{n_3=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_1=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_1=2}^{n_1} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \right]^{\frac{2}{\theta}} = \\
 &= \sum_{n_1=2}^{\infty} \left( \left( \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \left[ \sum_{\nu_1=2}^{n_1} |\Delta_{n_1, n_2, n_3}^{\theta} \lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \right]^{\frac{\theta}{2}} \right)^{\frac{2}{\theta}} \right)^{\frac{2}{\theta}} \leq
 \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n_1=2}^{\infty} \left( \sum_{\nu_1=2}^{n_1} \left[ \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}|^{\frac{\theta}{2}} \right]^{\frac{\theta}{2}} \right)^{\frac{2}{\theta}} = \\ &= \left( \left( \sum_{n_1=2}^{\infty} \left\{ \sum_{\nu_1=2}^{n_1} \left[ \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}|^{\frac{\theta}{2}} \right]^{\frac{\theta}{2}} \right\} \right)^{\frac{2}{\theta}} \right)^{\frac{\theta}{2}} \leq \\ &\leq \left( \sum_{\nu_1=2}^{\infty} \left\{ \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}|^{\frac{\theta}{2}} \right\} \right)^{\frac{2}{\theta}} = \\ &= \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} \right)^{\frac{2}{\theta}}. \end{aligned}$$

Applying this inequality, we obtain

$$\begin{aligned} H_2 &\leq \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \left\{ \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} \right\}^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} = \\ &= \left( \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} \right)^{\frac{p}{\theta}} dx_1, dx_2, dx_3 \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}}. \end{aligned}$$

Further, using Minkowski's inequality for  $\frac{p}{\theta} \geq 1$ , Lemmas 2.4 and 2.5, we have

$$\begin{aligned} H_2 &\leq \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} = \\ &= \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| \left\| \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{1}{2}} \right\|_{\theta}^{\theta} \right)^{\frac{1}{\theta}} \lesssim \\ &\lesssim \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| Y_{2^{\nu_1-1}, 0, 0}^{\theta}(f)_p \right)^{\frac{1}{\theta}}. \end{aligned}$$

Thus, we obtain  $H_2 \lesssim \left( \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 1, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 1, 1}^{\theta}| Y_{2^{\nu_1-1}, 0, 0}^{\theta}(f)_p \right)^{\frac{1}{\theta}}$ . From (2) it follows that  $H_2 < \infty$ ,  $H_3, H_4$  can be estimated similarly to  $H_2$  and we have

$$H_3 \lesssim \left( \sum_{\nu_2=1}^{\infty} |\lambda_{1, 2^{\nu_2}, 1}^{\theta} - \lambda_{1, 2^{\nu_2-1}, 1}^{\theta}| Y_{0, 2^{\nu_2}-1, 0}^{\theta}(f)_p \right)^{\frac{1}{\theta}}, H_4 \lesssim \left( \sum_{\nu_3=1}^{\infty} |\lambda_{1, 1, 2^{\nu_3}}^{\theta} - \lambda_{1, 1, 2^{\nu_3-1}}^{\theta}| Y_{0, 0, 2^{\nu_3}-1}^{\theta}(f)_p \right)^{\frac{1}{\theta}}.$$

To estimate  $H_5$ , we apply the method of estimate for  $H_4$  as in article [9]. First, we obtain the upper estimate of the following sum. Applying Lemmas 2.2 and 2.3 twice for  $\frac{2}{\theta} \geq 1$ , we get

$$\begin{aligned} &\sum_{n_3=1}^{\infty} \sum_{n_2=2}^{\infty} \sum_{n_1=2}^{\infty} \Delta_{n_1, n_2, n_3}^2 \left[ \sum_{\nu_1=2}^{n_1} \sum_{\nu_2=2}^{n_2} |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^{\theta}| \right]^{2/\theta} \leq \\ &\leq \left( \sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^{\theta}| \left( \sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} \sum_{n_3=1}^{\infty} |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} \right)^{\frac{2}{\theta}}. \end{aligned}$$

Hence, Lemma 2.3 with  $\frac{p}{\theta} \geq 1$  implies that

$$H_5 \leq \left( \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 1}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^{\theta}| \right]^{\frac{p}{\theta}} dx_1, dx_2, dx_3 \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}}.$$

$$\begin{aligned}
 & -\lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^\theta \left| \left( \sum_{n_1=\nu_1}^\infty \sum_{n_2=\nu_2}^\infty \sum_{n_3=1}^\infty |\Delta_{n_1, n_2, n_3}|^2 \right)^{\frac{\theta}{2}} dx_1, dx_2, dx_3 \right|^{\frac{\theta}{p}} \Bigg|^{\frac{1}{\theta}} \leq \\
 & \leq \left( \sum_{\nu_2=2}^\infty \sum_{\nu_1=2}^\infty |\lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 1}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 1}^\theta + \right. \\
 & \left. + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 1}^\theta \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{n_1=\nu_1}^\infty \sum_{n_2=\nu_2}^\infty \sum_{n_3=1}^\infty |\Delta_{n_1, n_2, n_3}|^2 \right]^{\frac{\theta}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}}.
 \end{aligned}$$

By Lemmas 2.4 and 2.5, we obtain

$$H_5 \lesssim \left( \sum_{\nu_2=1}^\infty \sum_{\nu_1=1}^\infty |\lambda_{2^{\nu_1}, 2^{\nu_2}, 1}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 1}^\theta - \lambda_{2^{\nu_1}, 2^{\nu_2-1}, 1}^\theta + \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1}^\theta | Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 0}^\theta(f)_p \right)^{\frac{1}{\theta}}.$$

From (2), it follows that  $H_5 < \infty$ .  $H_6, H_7, H_8$  can be estimated similarly to  $H_5$  and we have

$$H_6 \lesssim \left( \sum_{\nu_3=1}^\infty \sum_{\nu_1=1}^\infty |\lambda_{2^{\nu_1}, 1, 2^{\nu_3}}^\theta - \lambda_{2^{\nu_1-1}, 1, 2^{\nu_3}}^\theta - \lambda_{2^{\nu_1}, 1, 2^{\nu_3-1}}^\theta + \lambda_{2^{\nu_1-1}, 1, 2^{\nu_3-1}}^\theta | Y_{2^{\nu_1-1}, 0, 2^{\nu_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}},$$

$$H_7 \lesssim \left( \sum_{\nu_3=1}^\infty \sum_{\nu_2=1}^\infty |\lambda_{1, 2^{\nu_2}, 2^{\nu_3}}^\theta - \lambda_{1, 2^{\nu_2-1}, 2^{\nu_3}}^\theta - \lambda_{1, 2^{\nu_2}, 2^{\nu_3-1}}^\theta + \lambda_{1, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta | Y_{0, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}},$$

$$H_8 \lesssim \left( \sum_{\nu_3=1}^\infty \sum_{\nu_2=1}^\infty \sum_{\nu_1=1}^\infty |\lambda_{2^{\nu_1}, 2^{\nu_2}, 2^{\nu_3}}^\theta - \lambda_{2^{\nu_1}, 2^{\nu_2}, 2^{\nu_3-1}}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 2^{\nu_3}}^\theta - \lambda_{2^{\nu_1}, 2^{\nu_2-1}, 2^{\nu_3}}^\theta + \right.$$

$$\left. + \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta + \lambda_{2^{\nu_1-1}, 2^{\nu_2}, 2^{\nu_3-1}}^\theta + \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3}}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta | Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}}.$$

Collecting estimates of  $H_1 - H_8$  we get  $I_1 < \infty$ . Hence, by Lemma 2.5 (b), there exists a function  $g(x_1, x_2, x_3) \in L_p^0$ , with the Fourier series

$$\sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \sum_{n_3=1}^\infty \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}} \Delta_{n_1, n_2, n_3} \quad (6)$$

and

$$\|g\|_p \leq C(p)I_1. \quad (7)$$

We rewrite series (6) in the form of  $\sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \sum_{n_3=1}^\infty \gamma_{n_1, n_2, n_3} A_{n_1, n_2, n_3}(x_1, x_2, x_3)$ , where

$$\gamma_{1,1,1} = \lambda_{1,1,1}, \gamma_{1, \nu_2, \nu_3} = \lambda_{1, 2^{\nu_2-1}, 2^{\nu_3-1}} \text{ for } 2^{\nu_2-1} \leq \nu_2 \leq 2^{\nu_2} - 1, 2^{\nu_3-1} \leq \nu_3 \leq 2^{\nu_3} - 1 \quad (n_2 = 2, 3, \dots),$$

$$\gamma_{\nu_1, 1, 1} = \lambda_{2^{\nu_1-1}, 1, 1} \text{ for } 2^{\nu_1-1} \leq \nu_1 \leq 2^{\nu_1} - 1 \quad (n_2 = 2, 3, \dots),$$

$$\gamma_{\nu_1, 1, \nu_3} = \lambda_{2^{\nu_1-1}, 1, 2^{\nu_3-1}} \text{ for } 2^{\nu_1-1} \leq \nu_1 \leq 2^{\nu_1} - 1, 2^{\nu_3-1} \leq \nu_3 \leq 2^{\nu_3} - 1, \quad (n_1, n_3 = 2, 3, \dots),$$

$$\gamma_{\nu_1, \nu_2, 1} = \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 1} \text{ for } 2^{\nu_1-1} \leq \nu_1 \leq 2^{\nu_1} - 1, 2^{\nu_2-1} \leq \nu_2 \leq 2^{\nu_2} - 1, \quad (n_1, n_2 = 2, 3, \dots),$$

$$\gamma_{1, 1, \nu_3} = \lambda_{1, 1, 2^{\nu_3-1}} \text{ for } 2^{\nu_3-1} \leq \nu_3 \leq 2^{\nu_3} - 1, \quad (n_3 = 2, 3, \dots),$$

$$\gamma_{1, \nu_2, 1} = \lambda_{1, 2^{\nu_2-1}, 1} \text{ for } 2^{\nu_2-1} \leq \nu_2 \leq 2^{\nu_2} - 1, \quad (n_2 = 2, 3, \dots),$$

$$\gamma_{\nu_1, \nu_2, \nu_3} = \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}} \text{ for } 2^{\nu_1-1} \leq \nu_1 \leq 2^{\nu_1} - 1,$$

$$2^{\nu_2-1} \leq \nu_2 \leq 2^{\nu_2} - 1, 2^{\nu_3-1} \leq \nu_3 \leq 2^{\nu_3} - 1 \quad (n_1, n_2, n_3 = 2, 3, \dots).$$

Now we consider the following series

$$\sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \sum_{n_3=1}^\infty \lambda_{n_1, n_2, n_3} A_{n_1, n_2, n_3}(x_1, x_2, x_3) = \sum_{n_1=1}^\infty \sum_{n_2=1}^\infty \sum_{n_3=1}^\infty \gamma_{n_1, n_2, n_3} \Lambda_{n_1, n_2, n_3} A_{n_1, n_2, n_3}(x_1, x_2, x_3) \quad (8),$$



where

$$\begin{aligned} \Lambda_{1,1,1} &= 1, \Lambda_{1,\nu_2,\nu_3} = \frac{\lambda_{1,\nu_2,\nu_3}}{\gamma_{1,\nu_2,\nu_3}} = \frac{\lambda_{1,\nu_2,\nu_3}}{\lambda_{1,2^{n_2-1},2^{n_3-1}}} \text{ for } 2^{n_2-1} \leq \nu_2 \leq 2^{n_2} - 1, \\ & \quad 2^{n_3-1} \leq \nu_3 \leq 2^{n_3} - 1 \quad (n_2, n_3 = 2, 3\dots), \\ \Lambda_{\nu_1,1,1} &= \frac{\lambda_{\nu_1,1,1}}{\gamma_{\nu_1,1,1}} = \frac{\lambda_{\nu_1,1,1}}{\lambda_{2^{n_1-1},1,1}} \text{ for } 2^{n_1-1} \leq \nu_1 \leq 2^{n_1} - 1, \quad (n_2 = 2, 3\dots), \\ \Lambda_{\nu_1,\nu_2,1} &= \frac{\lambda_{\nu_1,\nu_2,1}}{\gamma_{\nu_1,\nu_2,1}} = \frac{\lambda_{\nu_1,\nu_2,1}}{\lambda_{2^{n_1-1},2^{n_2-1},1}} \text{ for } 2^{n_1-1} \leq \nu_1 \leq 2^{n_1} - 1, 2^{n_2-1} \leq \nu_2 \leq 2^{n_2} - 1, \quad (n_1, n_2 = 2, 3\dots), \\ \Lambda_{\nu_1,1,\nu_3} &= \frac{\lambda_{\nu_1,1,\nu_3}}{\gamma_{\nu_1,1,\nu_3}} = \frac{\lambda_{\nu_1,1,\nu_3}}{\lambda_{2^{n_1-1},1,2^{n_3-1}}} \text{ for } 2^{n_1-1} \leq \nu_1 \leq 2^{n_1} - 1, 2^{n_3-1} \leq \nu_3 \leq 2^{n_3} - 1, \quad (n_1, n_2 = 2, 3\dots), \\ \Lambda_{1,1,\nu_3} &= \frac{\lambda_{1,1,\nu_3}}{\gamma_{1,1,\nu_3}} = \frac{\lambda_{1,1,\nu_3}}{\lambda_{1,1,2^{n_3-1}}} \text{ for } 2^{n_3-1} \leq \nu_3 \leq 2^{n_3} - 1, \\ \Lambda_{1,\nu_2,1} &= \frac{\lambda_{1,\nu_2,1}}{\gamma_{1,\nu_2,1}} = \frac{\lambda_{1,\nu_2,1}}{\lambda_{1,2^{n_2-1},1}} \text{ for } 2^{n_2-1} \leq \nu_2 \leq 2^{n_2} - 1, \\ \Lambda_{\nu_1,\nu_2,\nu_3} &= \frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\gamma_{\nu_1,\nu_2,\nu_3}} = \frac{\lambda_{\nu_1,\nu_2,\nu_3}}{\lambda_{2^{n_1-1},2^{n_2-1},2^{n_3-1}}} \text{ for } 2^{n_1-1} \leq \nu_1 \leq 2^{n_1} - 1, 2^{n_2-1} \leq \nu_2 \leq 2^{n_2} - 1, \\ & \quad 2^{n_3-1} \leq \nu_3 \leq 2^{n_3} - 1 \quad (n_2, n_3 = 2, 3\dots). \end{aligned}$$

As shown in [6], the sequence  $\{\Lambda_{n_1=1, n_2=1, n_3=1}\}_{n_1=1, n_2=1, n_3=1}^{\infty, \infty, \infty}$  satisfies the conditions of the Marcinkiewicz multiplier theorem [12], then the series (8) is the Fourier series of a function  $\varphi(x_1, x_2, x_3) \in L_p$  and  $\|\varphi\|_p \leq C(\rho, \lambda)\|g\|_p$ .

Taking into account (7) and the estimates of  $H_1 - H_8$  we get (3).

Let us estimate  $Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)_p$ . Using Lemma 2.4, we get

$$\begin{aligned} Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)_p &\leq C\|\varphi - S_{2^{m_1-1}, \infty, \infty}(\varphi) - S_{\infty, 2^{m_2-1}, \infty}(\varphi) - \\ & \quad - S_{\infty, \infty, 2^{m_3-1}}(\varphi) + 2S_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)\|_p. \end{aligned}$$

We consider the series (see (8))

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda_{n_1, n_2, n_3} A_{n_1, n_2, n_3}^*(x_1, x_2, x_3) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \gamma_{n_1, n_2, n_3} \Lambda_{n_1, n_2, n_3} A_{n_1, n_2, n_3}^*(x_1, x_2, x_3),$$

where  $A_{n_1, n_2, n_3}^*(x_1, x_2, x_3) = 0$ , if  $n_1 \leq 2^{m_1} - 1$  and  $n_2 \leq 2^{m_2} - 1, n_3 \leq 2^{m_3} - 1$  also  $A_{n_1, n_2, n_3}^*(x_1, x_2, x_3) = A_{n_1, n_2, n_3}(x_1, x_2, x_3)$  otherwise. Since the sequence  $\{\Lambda_{n_1=1, n_2=1, n_3=1}\}$  satisfies the conditions of the Marcinkiewicz multiplier theorem, then

$$\left\| \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda_{n_1, n_2, n_3} A_{n_1, n_2, n_3}^*(x_1, x_2, x_3) \right\|_p \leq C \left\| \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \lambda_{2^{n_1-1}, 2^{n_2-1}, 2^{n_3-1}} \Delta_{n_1, n_2, n_3}^* \right\|_p,$$

where  $\Delta_{n_1, n_2, n_3}^* = 0$ , if  $n_1 \leq m_1$  and  $n_2 \leq m_2, n_3 \leq m_3$   $\Delta_{n_1, n_2, n_3}^* = \Delta_{n_1 n_2, n_3}$  otherwise.

By Lemma 2.5, we have

$$\begin{aligned} Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)_p &\lesssim \\ &\lesssim \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \sum_{k_3=m_3+1}^{\infty} \lambda_{2^{k_1-1}, 2^{k_2-1}, 2^{k_3-1}}^2 \Delta_{k_1, k_2, k_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}}. \end{aligned} \tag{9}$$

For the sequence  $\lambda_{2^{k_1-1}, 2^{k_2-1}, 2^{k_3-1}}^\theta$ , we use inequality (5) where the index of the first element starts with  $2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}$ , and we take the sum from  $2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}$  to  $2^{k_1-1}, 2^{k_2-1}, 2^{k_3-1}$  respectively. The resulting inequality is substituted into inequality (9).

$$Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}(\varphi)_p \lesssim$$

$$\begin{aligned}
&\lesssim \left( \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_1=m_1+1}^{k_1} \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{m_3-1}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_2=m_2+1}^{k_2} \left| \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_3=m_3+1}^{k_3} \left| \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^\theta - \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-2}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_2=m_2+1}^{k_2} \sum_{\nu_3=m_3+1}^{k_3} \left| \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta - \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{\nu_3-1}}^\theta - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. - \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^\theta + \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{\nu_3-2}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_3=m_3+1}^{k_3} \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{\nu_3-1}}^\theta - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. - \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-2}}^\theta + \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{\nu_3-2}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right. \\
&\quad \times \left. \left. \left. \left( \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} \sum_{\nu_3=m_3+1}^{k_3} \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-1}}^\theta - \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-2}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} + \right. \\
&\quad \left. + \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & -\lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-1}}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^\theta + \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-2}}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-2}}^\theta + \\
 & + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-2}}^\theta \Big| \Big)^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1, dx_2, dx_3 \Big\}^{\frac{1}{p}} =: L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8.
 \end{aligned}$$

We estimate  $L_1$  as  $H_1$ , to get

$$\begin{aligned}
 L_1 & \leq \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{1}{p}} \lesssim \\
 & \lesssim \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta Y_{2^{m_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta(f)_p.
 \end{aligned}$$

We also have

$$\begin{aligned}
 L_2 & = \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \right. \right. \\
 & \times \left. \left( \sum_{\nu_1=m_1+1}^{k_1} \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{m_3-1}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \Big\}^{\frac{1}{p}} \lesssim \\
 & \lesssim \left( \sum_{\nu_1=m_1+1}^\infty \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{m_3-1}}^\theta \right| \times \right. \\
 & \times \left. \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=\nu_1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} \lesssim \\
 & \lesssim \left( \sum_{\nu_1=m_1+1}^\infty \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{m_3-1}}^\theta \right| Y_{2^{\nu_1-1}, 2^{m_2-1}, 2^{m_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}}.
 \end{aligned}$$

Similarly, we obtain the estimates for  $L_3, L_4$

$$\begin{aligned}
 L_3 & \lesssim \left( \sum_{\nu_2=m_2+1}^\infty \left| \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| Y_{2^{m_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}}, \\
 L_4 & \lesssim \left( \sum_{\nu_3=m_3+1}^\infty \left| \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^\theta - \lambda_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-2}}^\theta \right| Y_{2^{m_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}}.
 \end{aligned}$$

We estimate  $L_5$  as follows:

$$\begin{aligned}
 L_5 & = \left\{ \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=m_1+1}^\infty \sum_{k_2=m_2+1}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \times \left( \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \right. \right. \right. \\
 & \left. \left. \left. - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| \right)^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \Big\}^{\frac{1}{p}} \lesssim \\
 & \lesssim \left( \sum_{\nu_1=m_1+1}^\infty \sum_{\nu_2=m_2+1}^\infty \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta + \right. \right. \\
 & \left. \left. + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| \times \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ \sum_{k_1=\nu_1}^\infty \sum_{k_2=\nu_2}^\infty \sum_{k_3=m_3+1}^\infty \Delta_{k_1, k_2, k_3}^2 \right]^{\frac{p}{2}} dx_1, dx_2, dx_3 \Big\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}} \lesssim \\
 & \lesssim \left( \sum_{\nu_1=m_1+1}^\infty \sum_{\nu_2=m_2+1}^\infty \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta - \right. \right. \\
 & \left. \left. - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{m_3-1}}^\theta \right| Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{m_3-1}}^\theta(f)_p \right)^{\frac{1}{\theta}}.
 \end{aligned}$$

Similarly, we obtain the estimates for  $L_6, L_7$  :

$$L_6 \lesssim \left( \sum_{\nu_2=m_2+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} \left| \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{m_1-1}, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} \right| Y_{2^{m_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p \right)^{\frac{1}{\theta}}.$$

$$L_7 \lesssim \left( \sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_3=m_3+1}^{\infty} \left| \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{m_2-1}, 2^{\nu_3-2}}^{\theta} \right| Y_{2^{\nu_1-1}, 2^{m_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p \right)^{\frac{1}{\theta}}.$$

Finally, we estimate  $L_8$  as follows:

$$L_8 \lesssim \left( \sum_{\nu_1=m_1+1}^{\infty} \sum_{\nu_2=m_2+1}^{k_2} \sum_{\nu_3=m_3+1}^{\infty} \left| \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-1}, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-1}, 2^{\nu_3-2}}^{\theta} + \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-1}}^{\theta} - \lambda_{2^{\nu_1-2}, 2^{\nu_2-2}, 2^{\nu_3-2}}^{\theta} \right| Y_{2^{\nu_1-1}, 2^{\nu_2-1}, 2^{\nu_3-1}}^{\theta}(f)_p \right)^{\frac{1}{\theta}}.$$

Taking into account the estimates for  $L_1 - L_8$ , we obtain (4). The theorem is proved.

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## Жалпыланған Лиувилл-Вейл туындыларының бұрыштық ең жақын жуықтауларының жоғарғы бағалаулары

Мақалада  $2\pi$  периодты  $f$  үзіліссіз функциялар және оларды тригонометриялық көпмүшеліктермен жуықтауға жалпыланған Лиувиль-Вейл туындылары арқылы берілген үш өлшемді функциялардың бұрыштық ең жақын жуықтауын бастапқы берілген функциялардың бұрыштық ең жақын жуықтауы арқылы бағалауы қарастырылған. Авторлар классикалық Вейл аралас туындыларының орнына жалпыланған Лиувиль-Вейл туындыларын зерттеген. Қарастырылатын мәселелерді таңдағанда осы мақаладағы екінші автордың жұмысынан кейін қалыптасқан жалпы тәсілді ұстанған. Басты мақсат жұмыстың нәтижелерін үш өлшемді жағдайда дәлелдеу. Жалпы монотонды тізбектер туралы түсінік осы зерттеуде басты рөл атқарады. Функцияның ең жақын жуықтауларына қатысты  $r$ -туындысының ең жақын жуықтаулары, норма үшін бірнеше белгілі теңсіздіктер көрсетілген. Мақалада қарастырылған мәселелер Бернштейннің зерттелген еңбектерінің мәселелеріне жатады. Кейінірек Стечкин және Конюшков  $f^{(r)}$  ең жақын жуықтау үшін теңсіздік алынды. Сонымен қатар Потаповтың еңбектерінде бұрыштарды жақындату арқылы функциялардың кейбір кластары қарастырылған. Бірінші бөлімде қажетті түсініктермен және пайдалы леммалар берілген. Үш өлшемді жағдайда жалпыланған Лиувиль-Вейл туындысы арқылы берілген функциялардың нормасының және бұрыштық ең жақын жуықтауының бағалауы алынды.

*Кілт сөздер:* Лебег кеңістігі, үш өлшемді бұрышпен ең жақын жуықтау, тригонометриялық көпмүшелік, Лиувиль-Вейл туындысы.

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## Верхние оценки угловых наилучших приближений обобщенных производных Лиувилля-Вейля

В статье рассмотрены непрерывные функции  $f$  с периодом  $2\pi$  и их приближения тригонометрическими полиномами. Изучены оценки наилучших угловых приближений обобщены производных Лиувилля-Вейля угловым приближением функций в трехмерном случае. Авторами обобщенные производные Лиувилля-Вейля вместо классического смешанного производного Вейля. При выборе рассматриваемых вопросов они следовали общему подходу, сформировавшемуся после работы второго автора настоящей статьи. Главная цель — доказать аналоги результатов работы в трехмерном случае. Понятие общих монотонных последовательностей играет ключевую роль в исследовании. Указаны несколько известных неравенств для норм, наилучших приближений  $r$ -го производного по наилучшим приближениям функции  $f$ . Вопросы, рассмотренные в настоящей работе, относятся к кругу проблем, изученных в работах Бернштейном. Позднее Стечкин и Конюшков получили неравенство для наилучшего приближения  $f^{(r)}$ . Также в работах Потапова при помощи приближения углом изучены некоторые классы функций. В подразделе 1 авторами даны необходимые обозначения и полезные леммы. Получены оценки норм и наилучшие приближения обобщенного производного Лиувилля-Вейля в трехмерном случае.

*Ключевые слова:* пространство Лебега, наилучшее приближение трехмерным углом, тригонометрический полином, производная Лиувилля-Вейля.

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