DOI 10.31489/2021M3/25-35

UDC 517, 519.2

H.M. Hasan, D.F. Ahmed, M.F. Hama, K.H.F. Jwamer*

University of Sulaimani, Kurdistan Region, Sulaimani, Iraq
(E-mail: hataw.hasan@univsul.edu.iq, dilan.ahmed@univsul.edu.iq, mudhafar.hama@univsul.edu.iq, karwan.jwamer@univsul.edu.iq)

Central Limit Theorem in View of Subspace Convex-Cyclic Operators

In our work, we have defined an operator called subspace convex-cyclic operator. The property of this newly defined operator relates eigenvalues which have eigenvectors of modulus one with kernels of the operator. We have also illustrated the effect of the subspace convex-cyclic operator when we let it function in linear dynamics and joining it with functional analysis. The work is done on infinite dimensional spaces which may make linear operators have dense orbits. Its property of measure preserving puts together probability space with measurable dynamics and widens the subject to ergodic theory. We have also applied Birkhoff's Ergodic Theorem to give a modified version of subspace convex-cyclic operator. To work on a separable infinite Hilbert space, it is important to have Gaussian invariant measure from which we use eigenvectors of modulus one to get what we need to have. One of the important results that we have got from this paper is the study of Central Limit Theorem. We have shown that providing Gaussian measure, Central Limit Theorem holds under the certain conditions that are given to the defined operator. In general, our work is theoretically new and is combining three basic concepts dynamical system, operator theory, and ergodic theory under the measure and statistics theory.

Keywords: Central limit theory, Subspace convex-cyclic operator, Gaussian measures.

Introduction

Linear dynamics is a branch of functional analysis. It studies the dynamics of linear operators connecting functional analysis with dynamics. Linear dynamics is mostly dealing with the behaviour of iterates of linear transformations. Linear transformations designated by their Jordan canonical form makes linear dynamics easier to understand when on finite-dimensional spaces. However, when infinite-dimensional space taken into account, linear operators may have dense orbits. One of the focal branches of dynamical system is ergodic theory [1], which relates analysis with probability theory and deals with measurable dynamics. It exerts measure theory to the study of the behavior of dynamical systems. Measure-preserving transformations and measure spaces are the main study subjects in ergodic theory. In probability theory, one of the most substantial results is Central limit theorem, in which under specific conditions the sum of a large number of random variables approaches the normal distribution. This distribution is important since it is suitable for a lot of natural phenomena and social sciences.

In this paper each section demonstrates some of the concepts described above whilst the operator subspace convex-cyclic operator is working on them and illustrates the connections between those notions as follows. In section two, subspace convex-cyclic operator is defined with such a property that correlates eigenvalues having eigenvectors of modulus 1 with kernels of the operators. In section three we have shown that operators with eigenvectors of modulus 1 are subspace convex-cyclic operators. It is also shown that those operators having measure 1 Section four is to come up with a connection between linear dynamics and measurable dynamics with the help of subspace convex-cyclic operators. In this section we have spelled some basic definitions in order to be able to clarify the ergodicity of a transformation. The modified version of subspace convex-cyclic operator is given by applying Birkhoff's Ergodic Theorem on the given operator. In section five Gaussian measure is studied. Here eigenvectors of modulus 1 are used to get Gaussian invariant measure which is crucial for working on a separable infinite Hilbert space. We have given a result that connects the concepts described together. In the last section we have shown that Central Limit Theorem holds under the certain conditions that are given to the defined operator after providing Gaussian measure.

*Corresponding author.
E-mail: karwan.jwamer@univsul.edu.iq

New subspace convex-cyclic operators

We try to define a new result for showing operators are subspace convex-cyclic operators for subspace $\mathcal{M}$. In this section we try to find subspace convex-cyclic operators in a new point of view, that relates eigenvalues of modulus 1 and kernels of operators. They play an important role in the following sections.

Theorem 1. Let $T \in B(\mathcal{H})$. Suppose $\bigcup_{|\lambda|>1} ker(T-\lambda)$ and $\bigcup_{|\lambda|<1} ker(T-\lambda)$ both span a dense subspace $\mathcal{M}$ of $\mathcal{H}$. Then T is subspace convex-cyclic operator for subspace $\mathcal{M}$.

Proof. We show that T satisfies the subspace convex-cyclic criterion by letting

$$X := Span\left(\bigcup_{|\lambda|<1} ker(T-\lambda)\right) \quad and$$

$$Y := Span\left(\bigcup_{|\lambda|>1} ker(T-\lambda)\right).$$

The sequence of functions $x_k : Y \to H$ are defined as $x_k(y) = \frac{1}{\lambda^k} y$. Also, we define $P_k(T)y = \lambda^k y$. and we use technique Theorem 2.8 in [2] for extending x_k to Y by linear functional. This makes sense because the subspace Y is linearly independent. Thus for any $y \neq 0$ and $y \in Y$, it may uniquely be written as $y = y_1 + \cdots + y_k$, with $y_i \in ker(T-\lambda_i) \setminus \{0\}$ and $|\lambda_i| > 1$. These vectors x and y can be expressed as the following form

$$x := \sum_{i=1}^{k} \alpha_i x_i \qquad and \qquad y := \sum_{i=1}^{k} \beta_i y_i.$$

where $P_k(T)x_i = \lambda x_i$ and $P_k(T)y_i = \mu y_i$ and the scalars $\alpha_i, \beta_i, \lambda, \mu \in \mathbb{C}$ such that $|\lambda_i| < 1$ and $|\mu_i| > 1$ for $i = 1, 2, \cdots, k$. Since

$$P_k(T)(x) = \sum_{i=1}^{k} \alpha_i \lambda^m x_i \to 0 \qquad and$$

$$x_k(y) = \sum_{i=1}^{k} \beta_i \frac{1}{\mu^m} y_i \to 0 \quad as\ m \to \infty$$

$$and \qquad P_k(T)x_k(y) = y,$$

then the first and the second conditions of Theorem 3 in [3] are hold, for showing the third, the space $\mathcal{H}$ is an infinite dimensional (Real or Complex) separable Hilbert space. We observe that

$$x \in \bigcap_{j=1}^{\infty} \bigcup_{P_k \in \mathcal{P}} P_k(T)^{-1}(B_j)$$

if and only if for all $j \in \mathbb{N}$ there exist a convex polynomials P_k such that $x \in P_k(T)^{-1}(B_j)$ which implies $P_k(T)(x) \in B_j$. But since $\{B_j\}$ is a basis for the relatively topology of $\mathcal{M}$, this occurs if and only if $\widehat{Orb(T,x)} \cap \mathcal{M}$ is dense in $\mathcal{M}$, which means T is subspace convex-cyclic transitive and by definition of $\mathcal{M}$ convex-cyclic transitive, there exist $U\ and\ V$ relatively open subsets in $\mathcal{M}$ such that

$$W := P(T)^{-1}(U) \cap V \neq \phi \quad [3].$$

In particular, non-empty subset W relatively in $\mathcal{M}$, and $W \subset P(T)^{-1}(U)$. Then $P(T)(W) \subset U$ and $U \subset \mathcal{M}$, so we get that

$$P(T)(W) \subset \mathcal{M}.$$

Let $x \in \mathcal{M}$, we must show that $P(T)(\mathcal{M}) \subset \mathcal{M}$. Take $w_0 \in W$, since W is relatively open in $\mathcal{M}$ and $x \in \mathcal{M}$ so there exist $r > 0$ such that $w_0 + rx \in W$. But $P(T)(W) \subseteq \mathcal{M}$, that is,

$$P(T)(w_0 + rx) = P(T)(w_0) + rP(T)x \in \mathcal{M},$$

then $P(T)(w_0) \in \mathcal{M}$ and $\mathcal{M}$ is subspace. So,

$$r^{-1}\left(-P(T)(w_0) + P(T)(w_0) + rP(T)(x)\right) \in \mathcal{M},$$

that is, $P(T)(x) \in \mathcal{M}$. This is true for any $x \in \mathcal{M}$, hence for $P(T)(x) \in \mathcal{M}$, that is $P(T)(\mathcal{M}) \subseteq \mathcal{M}$. All conditions are satisfied. We get that T is subspace convex-cyclic for $\mathcal{M}$.

Remark 1. The inverse of the above theorem is not true, see Proposition 2.

Eigenvalue measure

We aim to show that operators with sufficiently many eigenvectors of modulus 1 are subspace convex-cyclic operators. Let us first construct the following definition from Theorems 3.1 and 3.9 in [2] and Theorem 1. Here we have not mentioned any density or properties related to density for details you can see [4], we just join it with collection of eigenvalues and associated eigenvectors of the subspace convex-cyclic orbits that we have named them orbit eigen-spaces. Such operator with unit measure property will help us for proofing the next outcomes.

Definition 1. Let $T \in B(\mathcal{H})$ be subspace convex-cyclic operator for a nontrivial subspace $\mathcal{M}$ of $\mathcal{H}$. Then for any scalar $\lambda \in \mathbb{T}$. We define $\mathfrak{A}$ as an orbit eigen-spaces if

$$\mathfrak{A} := Span\left[ker(\lambda I - T)\widehat{Orb(T, \mathcal{M})}\right].$$

Definition 2. Let $T \in B(\mathcal{H})$ be subspace convex-cyclic operator for a nontrivial subspace $\mathcal{M}$ of $\mathcal{H}$. Then for any scalar $\lambda \in \mathbb{T}$. We define unit measure μ if for every measurable subset $\Omega \subset \mathbb{T}$, $\mu(\Omega) = 1$. And $\mu\{ker(\lambda I - T)\} = 0$.

For imagining the Definition 1 and Definition 2 see the following example. For skipped steps we refer the reader to review Example 4 in [3].

Example 1. Let $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, and consider $T := \lambda B$ where B is the backward shift on ℓ^2. Let $\mathcal{M}$ be the subspace of ℓ^2 consisting of all sequences with zeros on the even entries as $\mathcal{M} = \left\{\{a_n\}_{n=0}^{\infty} \in \ell^2 : a_{2k} = 0 \text{ for all } k\right\}$ [3].

Solution. For operators like T defined above, surely we have an eigenvalue γ under the condition $|\lambda| > |\gamma|$, so in this situation we have $ker(\gamma I - T) = ker(\gamma I - \lambda B)$ such vector like $x_\lambda \in \mathbb{R}$ will span them as

$$x_\gamma = \sum_{i=0}^{\infty} \left(\frac{\gamma}{\lambda}\right)^i e_i,$$

where (e_i) is a canonical basis for $i = 1, \cdots$ Let μ be the measure on the unit circle that normalized $\mathbb{T}$, and suppose that a measurable set $\Omega \subset \mathbb{T}$ let $h \in \mathcal{H}$ be an orthogonal vector such that $\langle h, x_\gamma \rangle = 0$ for every $\gamma \in \Omega$ by Hahn-Banach Theorem we have a well known linear functional defined as

$$\zeta(\gamma) = \sum_{i=0}^{\infty} \langle x, e_i \rangle \left(\frac{\gamma}{\lambda}\right)^i$$

$\zeta(\gamma) \to 0$, since ω is any subset of $\mathbb{T}$, so there are two choices, if it is countable then we get contradiction for been T as a subspace convex-cyclic operators, then it should be uncountable and in that case we have a limit points around the circle center and that leads to $\mu\{ker(\lambda I - T)\} = 0$ and its obvious that taking $\mu(\Omega) = max\{\rho(\Omega, \mathbb{T}), 1\} = 1$, where ρ is the metric that defined on the space depends on ℓ^2 space. Then the conditions in Definition 2 are satisfied.

Now, depending on Example 1 in [3] we can define $\mathcal{M} = \ell^2 \oplus \{0\}$. Consequently we get that

$$\left[\widehat{Orb}(T \oplus I, (x \oplus 0))\right] \cap \left[\ell^2 \oplus \{0\}\right] = \ell^2 \oplus \{0\} = \mathcal{M}$$

finally we can define

$$\mathfrak{A} := Span\left[ker(\lambda I - T)\mathcal{M}\right].$$

Subspace Conv-Cyc and Ergodic Theory

We previously obtained Subspace Convex-Cyclic Operators in Section 1 by another way. The aim of this section is to provide a bridge between linear dynamics and measurable dynamics. The most important concept to start with it is invariant measure, because it has the direct connection with Subspace Convex-Cyclic Transitive Operators (see Definition 2 and Theorem 1 in [3]), bounded $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{M}$ is subspace of $\mathcal{H}$. If for all non-empty open sets $U \subset \mathcal{M}$ and $V \subset \mathcal{M}$, there exist a convex P such that $U \cap P(T)(V) \neq \phi$ or $P(T)^{-1}(U) \cap V \neq \phi$ contains a relatively open non-empty subset of $\mathcal{M}$ if and only if $\mathcal{M}$ is an invariant subspace for $P_k(T)$ for all $k \geq 0$ [3].

We first start by recalling some basic definitions of Ergodic theory. For more details see [5] which is very useful related to that branch. In this section $\mathfrak{B}$ is Borel $\sigma - algebra$.

Definition 3. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space. We recall a measurable transformation $T : (\mathcal{H}, \mathfrak{B}, \mu) \to$ $\to (\mathcal{H}, \mathfrak{B}, \mu)$ is a measure-preserving transformation, or μ is $T - invariant$, if $\mu(f^{-1}(\mathfrak{U})) = \mu(\mathfrak{U})$ for all $\mathfrak{U} \in \mathfrak{B}$.

Definition 4. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and T is measure-preserving transformation. For any non-empty subset G of $\mathcal{H}$, We say μ is positive measure if $\mu(G) > 0$, as well For any non-empty open subset U of $\mathcal{H}$, We say μ is fully support if $\mu(U) > 0$.

Definition 5. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space. We recall a measurable transformation $T : (\mathcal{H}, \mathfrak{B}, \mu) \to$ $\to (\mathcal{H}, \mathfrak{B}, \mu)$ is Ergodic if it is a measure-preserving transformation and satisfies one of the following equivalent conditions [5]:

1 Given any measurable sets U and V with positive measures, one can find an integer $n \geq 0$ such that $T^n(U) \cap V \neq \phi$,

2 if $U \in \mathfrak{B}$ satisfies $T(U) \subset U$ then $\mu(U) = 0$ or $\mu(U) = 1$.

The following Theorem is known as Birkhoff's Ergodic Theorem.

Theorem 2. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. For any non-zero function $f \in L^1(\mathcal{H}, \mu)$,

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \to \int_{\mathcal{H}} f d\mu$$

as $N \to \infty$, *almost everywhere.*

In the following proposition, we demonstrate that Birkhoff's Ergodic Theorem can also be applied to our operator. So, this version will be our modified version with Subspace Convex-Cyclic Transitive Operators. The proof will depend on the Mahlar measure as worked in [6] of polynomial measure.

Proposition 1. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. Given any measurable sets U and V. Then

$$\frac{1}{N} \sum_{k=0}^{N-1} \mu(P(T)^{-1}(U) \cap V) = \lambda \mu(V)$$

as $N \to \infty$, *almost everywhere.*

Proof. Since $P(T)x$ as $P(T) = a_0 + a_1 T + a_2 T^2 + \cdots + a_n T^n, n \in \mathbb{N}$, so we write it as

$$P(T) = a_n \prod_{i=1}^{n} (T - a_i)$$

is defined by the formula

$$\mu(P) = a_n \prod_{i=1}^{n} max\{1, a_i\}$$

and was first considered by Mahler. If P and Q are non-zero polynomials, note that

$$\mu(P.Q) = \mu(P).\mu(Q).$$

The result directly comes after evaluating $P(T)^{-1}$ and letting $\lambda = a_n \prod_{i=1}^{n} max\{1, a_i\}$ for measure intersection from de Morgan's laws, a collection of subsets is $\sigma - algebra$ under the operations of taking complements and countable intersections.

Now one of our results can be stated. We add fully support measure because here we depend on open sets for taking any measure. The proof will depend the previous proposition.

Theorem 3. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space with fully support measure and $T : (\mathcal{H}, \mathfrak{B}, \mu) \to (\mathcal{H}, \mathfrak{B}, \mu)$ is measure-preserving and Ergodic transformation. Then T is Subspace Convex-Cyclic Operator.

Proof. Let (V_j) for $j \in \mathbb{N}$ be a countable basis of open sets for $\mathcal{H}$. Applying Proposition 1 to the collection of constant function that are convex, a_i where $\sum_{i=0}^{n} a_i = 1$ for any V_j, we get a sequence of sets $\{U_j\}$ for all j, such that

$$\frac{1}{N} \sum_{k=0}^{N-1} a_j (P(T)^{-1}(U_j) \cap V_j) = \lambda \mu(V_j)$$

as $N \to \infty$*, almost everywhere.*

Now, we have two cases depends on the measure of V_j and Ergodic definition. $\mu(V_j) = 1,\ j \in \mathbb{N}$, then we set $U := \bigcap_{j\in\mathbb{N}} U_j$, let U_1 and U_2 be open sets. Since by given

$$\frac{1}{N} \sum_{k=0}^{N-1} (P(T)^{-1}(U_1) \cap U_2) = \lambda$$

[3] which leads to $P(T)^{-1}(U) \cap V \neq \phi$ is Subspace Convex-Cyclic Transitive Operator and we can deduce that T is Subspace Convex-Cyclic Operator. The same thing is true for $\mu(V_j) = 0$.

Gaussian measure

We claimed that there is a special measure under an invariant bounded transformation preserving with its measure being ergodic T on $\mathcal{H}$. We are interested to add an additional tool that supports measure space $\mathcal{H}$. Gaussian measure is an atmosphere space that should be studied. As we know, working on an infinite Hilbert space without using Gaussian measure is not an easy way. To work with such a situation, we need Subspace convex-cyclic on Borel $\sigma - algebra$ that has sufficiently many eigenvectors of modulus 1.

For that purpose we need this section. Eigenvectors of modulus 1 are the fundamental tools we use to get Gaussian invariant measures. We need other definitions here. You can find more details in [7].

Definition 6. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space and $f : (\mathcal{H}, \mathfrak{B}, \mu) \to \mathbb{C}$ is a complex valued measurable function. Then f is said to have complex symmetric Gaussian distribution if the real and imaginary parts $\mathfrak{Re} f$ and $\mathfrak{Im} f$ of f have independent centered Gaussian distribution with the same variance.

This is equivalent to saying that $\mathfrak{Re} f$ and $\mathfrak{Im} f$ are jointly normal and that f and λf have the same distribution for any λ of modulus 1 [8].

Definition 7. Let $(\mathcal{H}, \mathfrak{B}, \mu)$ be a probability space a Gaussian measure on $\mathcal{H}$ is a probability measure μ on $\mathcal{H}$ such that for every $x \in \mathcal{H}$, the function $f_x : y \to \langle x, y\rangle$ has symmetric complex Gaussian distribution.

In particular with this terminology, such a measure is centered:

$$\int_{\mathcal{H}} \langle x, y\rangle d\mu(y) = \int_{\mathcal{H}} f_x(y) d\mu(y) = \int_{\mathcal{H}} y d(f_x(\mu))(y) \quad [9].$$

Remark 2. A Gaussian measure is determined exactly by the operator S defined on $\mathcal{H}$ by the relation

$$\langle Sx, y\rangle = \int_{\mathcal{H}} \langle x, z\rangle \overline{\langle y, z\rangle} d\mu(z)$$

The operator S in probability books are called as covariance.

Proposition 2. Let T be a subspace convex-cyclic operator on a separable Hilbert space, and $(\mathcal{M}, \mathfrak{B}, \mu)$ be a probability space with Gaussian measure on $\mathcal{M}$ for any $\mathcal{M} \subset \mathcal{H}$. Then $T(\mu)$ also is a Gaussian measure on $\mathcal{M}$. Such that $ker(T - \lambda I)$ span $(f_k(\lambda)),\ k \geq 1$ for all $\lambda \in \mathbb{T}$.

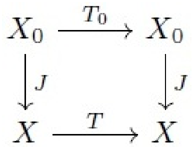
Proof. Since T is a subspace convex-cyclic operator, then each iteration of T will be a new element in $\mathcal{M}$, so directly by definition of subspace convex-cyclic operator we get dense set that itself is Gaussian measure. Now define a sequence of Borel $\sigma - algebra$ that sufficiently many eigenvectors points with certain eigenvectors of modulus 1 and polynomial operators as $f_k : \mathbb{T} \to \mathfrak{B}$ by $f_k(\lambda) = \tau_\lambda(x_k)$ such that for every $\lambda \in \mathbb{T}$, where $\tau_\lambda(x_k)$ be orthogonal onto $ker(T - \lambda I)$. Then each element of the sequence f_k is a Borel measure, so

$$\tau_\lambda(x_k) = x \Leftrightarrow T(x) = \lambda(x),$$

which means τ roll as a projection invariant and will span vectors $(f_k(\lambda))$ $k \geq 1$ is dense in $ker(T - \lambda I)$.

Quasi-factor is one of the most important concepts that has great application in dynamical system and operator theory you can find details and application in both [9] and [10].

Definition 8. Let $T_0 : X_0 \to X_0$ and $T : X \to X$ be two continuous maps acting on topological spaces X_0 and X. The map T is said to be a quasi-factor of T_0 if there exists a continuous map with dense range $J : X_0 \to X$ such that the diagram commutes, such that $TJ = JT_0$. When this can be achieved with a homeomorphism $J : X_0 \to X$, so that $T = JT_0J^{-1}$ we say that T_0 and T are topologically conjugate. Finally, when T_0 and T are linear operators and the factoring map (resp. the homeomorphism) J can be taken as linear, we say that T is a linear quasi-factor of T_0 (resp. that T_0 and T are linearly conjugate) [10].



We have similar results for the following theorem but with more conditions because they are defined with other operators. Our operator makes this easier. Now, it remains to state the theorem that connect all concepts to gather. Subspace convex-cyclic on Borel $\sigma - algebra$ that sufficiently many eigenvectors points with certain eigenvectors of modulus 1, and invariant Gaussian measure.

Theorem 4. If T is a subspace convex-cyclic operator on a separable Hilbert space X, then T admits a symmetry Gaussian invariant measure which is quasi-factor.

Proof. Let $(f_k(\lambda))$, $k \geq 1$ with the property that we deal with it in Proposition 2 for all $\lambda \in \mathbb{T}$ as defined in Definition 2, Let φ be defined on $\ell^2(\mathbb{T}, \sigma)$ of sequences (g_k) $k \geq 1$ of functions converges of $\ell^2(\mathbb{T}, \sigma)$

$$\varphi \sum_{k=1}^{\infty} g_k(\lambda) = \sum_{k=1}^{\infty} \lambda g_k(\lambda).$$

You can note that φ behaves as an operator of multiplication by λ on each component, now it is time to define κ as $\kappa : \sum\limits_{k=1}^{\infty} \ell^2(\mathbb{T}, \sigma) \to \mathcal{H}$ by Definition 7 we have,

$$\kappa \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \frac{1}{2^k} \int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda).$$

Then κ is a well defined operator on Hilbert space; each $\kappa_k : \ell^2(\mathbb{T}, \sigma) \to \mathcal{H}$, which maps g_k onto $\int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda)$ is a kernel operator. So, for any element in this sequence to be 0 without one which is arbitrary, this implies that for every $x \in \mathcal{H}$

$$\langle x, \int\limits_{\mathcal{H}} g_k(\lambda) f_k(\lambda) d\sigma(\lambda) \rangle = 0.$$

This implies that $\langle x, f_k(\lambda) \rangle = 0$ in the sense $\sigma - algebra$ which means that x is orthogonal to $ker(T - \lambda I)$. This implies that T has a $\sigma - algebra$ set of eigenvectors that spanned. Which leads to κ having dense range.

Now, if we want to show that T is quasi-factor, we make a choice of the operators pair as κ, φ, use the fact that $f_k(\lambda) \in ker(T - \lambda I)$, we get that for every $g_k(\lambda) \in \sum\limits_{i=1}^{\infty} \ell^2(\mathbb{T}, \sigma)$,

$$
\begin{aligned}
T\kappa \sum_{i=1}^{\infty} g_k(\lambda) &= \sum_{i=1}^{\infty} \frac{1}{2^i} \int\limits_{\mathbb{T}} g_k(\lambda) T f_k(\lambda) d\sigma(\lambda) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \int\limits_{\mathbb{T}} g_k(\lambda) \lambda f_k(\lambda) d\sigma(\lambda) \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i} \int\limits_{\mathbb{T}} \varphi \sum_{i=1}^{\infty} g_k(\lambda) f_k(\lambda) d\sigma(\lambda) \\
&= \kappa\varphi \sum_{i=1}^{\infty} g_k(\lambda)
\end{aligned}
$$

Central Limit Theorem

There are many important applications of central limit theorem, which are related to many branch of mathematics such as probability, dynamical system, operator system and many others. Our focus is on dynamical system that deals with Gaussian measure. How far a dynamical system is form an independed under conditions that added to such operator T. We prove that the central limit theorem also holds after providing Gaussian measure. The concepts of central limit theorem can be defined as follows without losing generality and modifying the definition in [11].

Definition 9. Let A be a $\sigma - algebra$ with a Subspace Convex-Cyclic operator T and some operator f where $f \in L^1(\mathcal{H}, \mu)$. Then by assumption of Theorem 2 we say that

$$
\frac{f + f + \cdots + f \circ T^{n-1}}{\sqrt{n}} \to \int\limits_{\mathcal{H}} f d\mu
$$

converges in distribution to a $\sigma - algebra$ random variable.

Theorem 5. Let $T \in B(\mathcal{H})$. Suppose that T satisfies the following assertion:

1 Theorem 3 in [3], and then T be Subspace Convex-Cyclic operator.

2 Definition 2

3 Suppose that there exists $\mathcal{P}$ collection of polynomials, also $\alpha \in (1, \infty)$ such that for any $f, g \in \mathcal{P}$

$$
\langle f \circ T^n, g \rangle < C \frac{1}{n^{\alpha}} \ .
$$

Then T and the sequence of function $\frac{1}{\sqrt{n}}(f + f + \cdots + f \circ T^{n-1})$ converges in distribution to a Gaussian measure spaces.

Proof. Let ω be any non-decreasing function. Let $(x_k)_{k \in \mathbb{N}}$ be a dense sequence in $\mathcal{D}$ and $T^k x_k \leq \omega(k)$. By the first assumption we have that any $k \in \mathbb{N}$,

$$
T^k x_k \leq \frac{\omega(k)}{(1 + |k|)^{\alpha}}.
$$

We construct the measure exactly as we did in Proposition 2, with additional properties that the sequence (p_l) also satisfies:

$$
\begin{gathered}
\forall k \geq 1, \quad \forall l \geq q, \quad \omega(l)^q \leq p_l^{-1/2} \\
\forall k \geq 1, \quad \sum_{l \geq 1} (N_{l+1} - N_l) p_l^{\frac{1}{2q}},
\end{gathered}
$$

where the nature of l can be review in Lemma 2 [3] for choosing N_l and N_{l+1}, to prove that $P \subset \ell^2(\mathcal{H}, \mathfrak{B}, \mu)$, it enough to show that, for any $k \geq 1$, $\langle x, y \rangle^k \in \ell^2(\mathcal{H}, \mathfrak{B}, \mu)$

$$
\int\limits_{\mathcal{H}} \langle x, y \rangle^q = \int\limits_{\mathcal{M}} \left\langle \sum_{l \geq 1} \sum_{|k| = N_l}^{N_l + 1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k))
$$

$$\leq \sum_{l_1,\ldots,l_q \geq 1} \int_{\mathcal{M}} \left\langle \sum_{|k|=N_{l_i}}^{N_l+1} T^k x_{n_k}, y \right\rangle d\mu((n_k)).$$

We then apply Hölder's inequality to get

$$\int_{\mathcal{H}} \langle x, y\rangle^q \leq \sum_{l_1,\ldots,l_q \geq 1} \left(\int_{\mathcal{M}} \left\langle \sum_{|k|=N_{l_i}}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)) \right)^{\frac{1}{q}}.$$

We fix $l \geq 1$ and we want to calculate

$$\int_{\mathcal{M}} \left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)).$$

Let $(n_k) \subset \mathcal{M}$ and let us write

$$\left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q \leq 2^q \left(\left\langle \sum_{|k|=N_l}^{N_l+1} ((n_k \leq l)) T^k x_{n_k}, y \right\rangle^q + \left\langle \sum_{|k|=N_l}^{N_l+1} ((n_k \geq l)) T^k x_{n_k}, y \right\rangle^q \right)$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q-1}(N_{l+1} - N_l)^{q-1} \sum_{|k|=N_l}^{N_l+1} \left\langle T^k x_{n_k}, y \right\rangle^q.$$

We take integral to this inequality over $\mathcal{M}$ for getting

$$\int_{\mathcal{M}} \left\langle \sum_{|k|=N_l}^{N_l+1} T^k x_{n_k}, y \right\rangle^q d\mu((n_k)) \leq \frac{2^q}{2^{ld}} +$$
$$2^{2q-1}(N_{l+1} - N_l)^{q-1} \sum_{|k|=N_l}^{N_l+1} \sum_{m>l} p_m \left\langle T^k x_m, y \right\rangle^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q \sum_{m>l} p_m \omega(m)^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q p_l \max(\omega(l), \omega(q))^q$$
$$\leq \frac{2^q}{2^{ld}} + 2^{2q}(N_{l+1} - N_l)^q p_l^{1/2} \omega(q)^q.$$

Since we assumed $\omega(l)^q p_l \leq p_l^{1/2}$, we take the exponent $1/q$ and we collect the inequalities to get

$$\int_X \langle X, y\rangle^q \, d\mu(x) \leq$$
$$C \left(\sum_{l \geq 1} \left(\frac{2}{2^{1/2}} + 4(N_{l+1} - N_l) p_l^{\frac{1}{2q}} \right) \right)^q \omega(q)^q$$
$$\leq C_n.$$

Thus, $\langle x,y\rangle^k \in \ell^2(\mathcal{H},\mathfrak{B},\mu)$ by constant C_d which can be much bigger.

Because our operator is subspace convex-cyclic operator so such a polynomial, we choose a derivative function, as $f\in\omega(k)$ and we observe that

$$\left\langle D^k f(x_k), y_k\right\rangle \leq \left\langle D^k, f\right\rangle \cdot \langle x_k, y_k\rangle^k$$

so that, from the proof of the first point, we deduce

$$\left\langle D^k, f\right\rangle \leq C^k \omega(k)^k D^k(f).$$

Since $D^k f$ is convergent, then the series $\sum_k \frac{D^k f(x_k)}{k!}$ is convergent in $\ell^2(\mathcal{H},\mathfrak{B},\mu)$ converges in distribution to a Gaussian measure spaces.

The following example will help us to understand the above theorem more. We apply Theorem 5, because B_w satisfied all conditions that we had in the statement. It remains to show that how the inequality in the $3rd$ condition will happen.

Example 2. Let B_w be a bounded backward weighted shift on $\ell^2(\mathbb{N})$. Suppose moreover that $\sum\limits_{n=1}^{\infty}\frac{1}{(w_1\ldots w_n)^2}$ converge. Then B_w converges in distribution to a Gaussian measure spaces.

Proof. Let ω be any non-decreasing function. Let $(\alpha_n)_n\in\mathbb{N}$ be a dense sequence in $\mathbb{N}$ with $|\alpha_n^p|\leq\omega(n)$. We set $\mathcal{D}:=(x_n)\ n\geq 1$, with $x_n=\alpha_n e_l$, where $(e_l)\ l\geq 1$ is the standard basis of $\ell^2(\mathbb{N})$. We define S_n on $\mathcal{D}$ by $S_n(e_l)=\frac{1}{(w_1\ldots w_n)^2}e_n$.

Since $(\alpha_n)_n$ is dense in $\mathbb{N}$ then $span(S_n x_n\ n\geq 1)$ is dense in $\mathcal{H}$.

As we get $\sum\limits_{k\geq 1}^{\infty} B_w^k x_k=\alpha_n e_l$, also $\sum\limits_{k<1}^{\infty} S_k x_k=\alpha_n e_l$, now

$$\begin{aligned}\left\langle \sum_{k<1}^{\infty} S_k x_k, e_k\right\rangle &= \left\langle \sum_{k<1}^{\infty} \alpha_n e_k, e_k\right\rangle\\ &= \left(\sum_{k<1}^{\infty}\frac{|\alpha_n|}{(w_1\ldots w_n)^2}\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{k<1}^{\infty}\frac{\omega(n)}{(w_1\ldots w_n)^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\frac{\omega(n-1)}{w_1}\geq\frac{1}{w_1}$, this yields to

$$\left\langle \sum_{k<1}^{\infty} S_k x_k, e_k\right\rangle \leq C_w leq \left(\sum_{k<1}^{\infty}\frac{\omega(n)}{(w_1\ldots w_n)^2}\right).$$

So, B_w satisfied all conditions. We get the result.

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Х.М. Хасан, Д.Ф. Ахмед, М.Ф. Хама, К.Х.Ф. Джвамер

Сулеймания университеті, Күрдістан аймағы, Сулеймания, Ирак

Ішкі кеңістіктікті дөңес-циклдік операторларды ескере отырып, орталық шекті теорема

Мақалада дөңес-циклдік ішкі кеңістікті оператор деп аталатын оператор анықталған. Бұл жаңадан анықталған оператордың қасиеті өзінің векторлары бар меншікті мәндерді, бір модульді оператордың ядроларымен байланыстырады. Авторлар сызықтық динамикада жұмыс жасаған, оны функционалды талдаумен біріктірген кезде дөңес-циклдік ішкі кеңістік операторының әсерін суреттеген. Жұмыс шексіз өлшемді кеңістіктерде орындалды, бұл сызықтық операторлардың тығыз орбиталарға ие болуына әкелуі мүмкін. Оның өлшемді сақтау қасиеті ықтималдық кеңістігін өлшенетін динамикамен біріктіреді және эргодтық теорияның тақырыбын кеңейтеді. Сондай-ақ, дөңес-циклдік ішкі кеңістік операторының модификацияланған нұсқасын беру үшін Биркгоф эргодтық теоремасы қолданылған. Сепарабелді шексіз Гильберт кеңістігімен жұмыс істеу үшін Гаусстың инвариантты өлшемі болуы керек, оның көмегімен қажет нәрсені алу үшін модульдің жеке векторлары пайдаланылған. Осы мақалада алынған маңызды нәтижелердің бірі - орталық шекті теореманы зерттеу. Гаусс өлшемін қамтамасыз ете отырып, белгілі бір операторға берілген белгілі бір жағдайларда орталық шекті теорема дұрыс екені көрсетілген. Жалпы, жұмыс теориялық тұрғыдан жаңа және үш негізгі ұғымды біріктіреді: динамикалық жүйе, операторлар теориясы және өлшеу теориясы мен статистика шеңберіндегі эргодтық теория.

Кілт сөздер: орталық шекті теорема, ішкі кеңістікті дөңес-циклдік оператор, Гаусс өлшемдері.

Х.М. Хасан, Д.Ф. Ахмед, М.Ф. Хама, К.Х.Ф. Джвамер

Университет Сулеймани, Курдистан, Сулеймания, Ирак

Центральная предельная теорема с учетом подпространственных выпукло-циклических операторов

В статье определен оператор, называемый выпукло-циклическим оператором подпространства. Свойство этого вновь определенного оператора связывает собственные значения, имеющие собственные векторы, модуля один с ядрами оператора. Авторами проиллюстрирован эффект выпукло-циклического оператора подпространства, в случае когда показаны функции в линейной динамике и объединены с функциональным анализом. Работа выполнена в бесконечномерных пространствах, которые могут привести к тому, что линейные операторы будут иметь плотные орбиты. Его свойство сохранения меры объединяет вероятностное пространство с измеримой динамикой и расширяет предмет эргодической теории. Авторами статьи использована эргодическая теорема Биркгофа, дающая модифицированную версию выпукло-циклического оператора подпространства. Чтобы работать с сепарабельным бесконечным гильбертовым пространством, важно иметь гауссову инвариантную меру, из которой применяются собственные векторы модуля один, чтобы получить то, что необходимо. Одним из важных результатов, полученных в этой статье, является изучение центральной предельной теоремы. Показано, что, обеспечивая гауссову меру, центральная предельная теорема верна при определенных условиях, которые задаются определенному оператору. В целом, данная работа является теоретически новой и объединяет три основных понятия: динамическую систему, теорию операторов и эргодическую теорию в рамках теории меры и статистики.

Ключевые слова: центральная предельная теорема, выпукло-циклический оператор подпространства, гауссовы меры.