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The solvability conditions for the second order nonlinear differential equation with unbounded coefficients in $L_2(\mathbb{R})$

The article deals with the existence of a generalized solution for the second order nonlinear differential equation in an unbounded domain. Intermediate and lower coefficients of the equation depends on the required function and considered smooth. The novelty of the work is that we prove the solvability of a nonlinear singular equation with the leading coefficient not separated from zero. In contrast to the works considered earlier, the leading coefficient of the equation can tend to zero, while the intermediate coefficient tends to infinity and does not depend on the growth of the lower coefficient. The result obtained formulated in terms of the coefficients of the equation themselves; there are no conditions on any derivatives of these coefficients.

Keywords: second order differential equation, nonlinear differential equation, differential equation in an unbounded domain, generalized solution, solvability.

Introduction

We investigate the following second-order singular differential equation

$$-\rho(x)\left(\rho(x)y'\right)' + r(x,y)y' + s(x,y)y = f(x), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, ρ is a twice continuously differentiable function, r is a continuously differentiable function, and s is a continuous function, $f \in L_2 \overset{def}{=} L_2(\mathbb{R})$, $\|\cdot\|_2$ is the norm in L_2. The singularity of the equation (1) means that it is given in a non-compact domain, and its coefficients can be unbounded.

The study of the equation (1) and its multidimensional generalizations is related to applications in quantum mechanics, stochastic analysis and stochastic differential equations [1–4]. In the above references the linear case is considered and results are obtained for $s(x,y) = s(x) > \delta > 0$, and the growth of $|r(x,y)| = |r(x)|$ at infinity is bounded by some positive power of $s(x)$. In the following researches [5–8] the linear case of equation (1) is also considered and it is assumed that the intermediate coefficient $r(x)$ can not grow faster than $|x|\ln|x|$ at infinity. In [5–8] issues on solvability of the equation (1) were considered only for the case $\rho(x) \geqslant \delta > 0$. The issue on solvability of the equation (1) stays unresolved for the case when the growth of $|r(x)|$ is faster than $|x|\ln|x|$ and is not dependent on s, and also when the coefficient $\rho(x)$ approaches zero as $x \to +\infty$ or as $x \to -\infty$.

For the case when $\rho \equiv 1$ and $|r|$ grows rapidly and does not depend on the coefficient s the equation (1) was analyzed in [9]. Here it was determined the solvability and the maximal regularity for the solution. The linear case for the equation (1) with a fast-growing growing intermediate coefficient was studied in [10] (when $f \in L_2$), [11] (when $f \in L_1(\mathbb{R})$) and [12] (when $f \in L_p(\mathbb{R})$, $1 < p < +\infty$). In [10–12] the function $\rho(x)$ is assumed to be separated from zero and bounded, or equal to 1. The study of the solvability of different classes of partial differential equations with unbounded coefficients is presented in [13–16].

Note that the rapid and independent growth of the absolute value of the intermediate coefficient r makes a big difference for solvability of the equation (1). Firstly, in this case the coefficient s can be unbounded from below. Moreover it can approach to $-\infty$ with certain rate [11, 12], where the rate of approaching s to $-\infty$ depends on the growth rate of $|r|$. Also let us note that in the study of the Sturm-Liouville equation (the case $\rho \equiv 1$, $r \equiv 0$, $s(x,y) = s(x)$) it is usually assumed that $s \geqslant -kx^2$ for some k [2]. Such condition in the case of equation (1) with unbounded r is not necessary.

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Secondly, due to the growth of the absolute value of r in the equation (1) it turns out we can assume approaching zero at infinity for the coefficient ρ in the leading term, thereby considering the so-called case of degeneration. The theorem 1 presented below shows that the rate of approaching ρ to zero also depends on the growth of $|r|$.

In the work [17] the results of the correct solvability and also a coercive estimate for the equation (1) was established in the case $r(x,y) = r(x)$, $s(x,y) = s(x)$ and $\rho(x) > 0$. In this paper we propose to extend some of the results obtained in [17] to the case of nonlinear generalization of equation (1).

Preliminaries

Let $C_0^{(k)}(\mathbb{R})(k = 1, 2, \ldots)$ be the set of k times continuously differentiable functions on $\mathbb{R}$ with compact support and $C_{loc}^{(j)}(\mathbb{R}) \overset{def}{=} \left\{ y : \psi y \in C_0^{(j)}(\mathbb{R}), \forall \psi \in C_0^{(j)}(\mathbb{R}) \right\}$ $(j = 1, 2)$. Consider the following linear equation

$$-\rho(x)\left(\rho(x)y'\right)' + r(x)y' + s(x)y = F(x). \tag{2}$$

Let g and $h \neq 0$ be given continuous functions. We denote

$$\alpha_{g,h}(t) \overset{def}{=} \|g\|_{L_2(0,t)} \left\|h^{-1}\right\|_{L_2(t,+\infty)} \quad (t > 0), \qquad \beta_{g,h}(\tau) \overset{def}{=} \|g\|_{L_2(\tau,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\tau)} \quad (\tau < 0),$$

$$\alpha_{g,h} \overset{def}{=} \sup_{t>0} \alpha_{g,h}(t), \ \beta_{g,h} \overset{def}{=} \sup_{\tau<0} \beta_{g,h}(\tau), \ \gamma_{g,h} \overset{def}{=} \max\left(\alpha_{g,h}, \beta_{g,h}\right).$$

The following statement is proved in [9].

Lemma 1. If g and h are continuous functions such that $\gamma_{g,h} < +\infty$. Then for $y \in C_0^{(1)}(\mathbb{R})$ the following inequality holds

$$\int_{-\infty}^{+\infty} |g(x)y(x)|^2 \, dx \leqslant C_1 \int_{-\infty}^{+\infty} \left|h(x)y'(x)\right|^2 dx.$$

Moreover we have $\left(\min\left(\alpha_{g,h}, \beta_{g,h}\right)\right)^2 \leqslant C_1 \leqslant 4\left(\gamma_{g,h}\right)^2$.

Let the operator $l_0 y = -\rho(x)(\rho(x)y')' + r(x)y' + s(x)y$ is defined on the set $C_0^{(2)}(\mathbb{R})$, we denote the closure of the operator l_0 by l in L_2. The function $y \in D(l)$ such that $ly = f$ is said to be a solution of the equation (2).

The following statement is proved in [17].

Lemma 2. If $0 < \rho(x) < +\infty$ is a twice continuously differentiable function, $r(x) \geqslant 1$ is a continuously differentiable function, and $s(x)$ is a continuous function, $r(x) \geqslant \rho^2(x)$, $\gamma_{1,\sqrt{r}} < +\infty$, $\gamma_{s,r} < +\infty$ and there exists $a \in \mathbb{R}$ such that

$$\sup_{x<a} \left\{ \rho(x) \exp\left(-\int_x^a \frac{r(t)}{\rho^2(t)} dt \right) \right\} < +\infty.$$

Next, let there be $C_2 > 1$ such that

$$C_2^{-1} \leqslant \frac{\rho(x)}{\rho(\nu)} \leqslant C_2, \quad C_2^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C_2, \quad \text{as} \ \ |x - \nu| \leqslant 1.$$

Then for any right-hand side $F \in L_2$ the linear equation (2) has a unique solution y and for y the following inequality holds

$$\left\| -\rho\left(\rho y'\right)' \right\|_2 + \|ry'\|_2 + \|sy\|_2 \leqslant C_3 \|F\|_2,$$

where C_3 depends only on C_2, $\gamma_{1,\sqrt{r}}$ and $\gamma_{s,r}$.

The solvability conditions for the second order non-linear differential equation

For continuous functions of two variables $g(x,y)$ and $h(x,y) \neq 0$ we denote

$$\alpha_{g,h}(t,y) \overset{def}{=} \left(\int_0^t |g(x,y)|^2 dx \right)^{\frac{1}{2}} \left(\int_t^{+\infty} \frac{dx}{|h(x,y)|^2} \right)^{\frac{1}{2}} \quad (t > 0),$$

$$\beta_{g,h}(\tau,y) \stackrel{def}{=} \left(\int_{\tau}^{0} |g(x,y)|^2 dx\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\tau} \frac{dx}{|h(x,y)|^2}\right)^{\frac{1}{2}} \quad (\tau < 0),$$

$$\alpha_{g,h}(y) \stackrel{def}{=} \sup_{t>0} \alpha_{g,h}(t,y),\ \beta_{g,h}(y) \stackrel{def}{=} \sup_{\tau<0} \beta_{g,h}(\tau,y),$$

$$\gamma_{g,h}(y) \stackrel{def}{=} \max\left(\alpha_{g,h}(y), \beta_{g,h}(y)\right).$$

Definition 1. Let $y \in L_2$. y is said to be a solution of the equation (1), if there exist a sequence $\{y_n\} \subset C_{loc}^{(2)}(\mathbb{R})$ such that

$$\|\psi(y_n - y)\|_2 \to 0 \text{ and } \|\psi(Ly_n - f)\|_2 \to 0 \text{ as } n \to +\infty,\ \ \forall \psi \in C_0^{(\infty)}(\mathbb{R}).$$

Theorem 1. Let $\rho(x)$ be a twice continuous differentiable and bounded function, $r(x,t)$ be a continuous differentiable function, $s(x,t)$ be a continuous function and

$$\inf_{t\in\mathbb{R}} r(x,t) \geqslant \rho^2(x), \qquad \sup_{y\in\mathbb{R}} \gamma_{1,\sqrt{r(x,y)}} < +\infty, \qquad \sup_{t\in\mathbb{R}} \gamma_{s(\cdot,t),r(\cdot,t)} < +\infty,$$

there exists $a \in \mathbb{R}$ such that

$$\sup_{x<a}\left\{\rho(x)\exp\left(-\int_{x}^{a} \frac{\inf_{t\in\mathbb{R}} r(v,t)}{\rho^2(v)} dv\right)\right\} < +\infty.$$

Also for some $\delta > 0$ and $\forall A > 0$ the inequalities holds

$$r(x,y) \geqslant (1+x^2)^{\frac{3}{4}+\delta}, \tag{3}$$

$$\sup_{|x-\nu|\leqslant 1}\ \sup_{|C'-C''|\leqslant A} \frac{r(x,C')}{r(x,C'')} \leqslant T(A) < +\infty, \quad C_4^{-1} \leqslant \frac{\rho(x)}{\rho(\nu)} \leqslant C_4,\ \text{as } |x-\nu| < 1.$$

Then the equation (1) have a solution y, and for y the following inequality holds

$$\|-\rho(x)(\rho(x)y')'\|_2 + \|r(x,y)y'\|_2 + \|s(x,y)y\|_2 < +\infty.$$

Proof. Let $C(\mathbb{R})$ be a space of continuous and bounded functions with the norm $\|y\|_{C(\mathbb{R})} \stackrel{def}{=} \sup_{t\in\mathbb{R}} |y(t)|$, and ε and A are given positive numbers. We consider the following set

$$B_A \stackrel{def}{=} \left\{z \in C(\mathbb{R}) : \|z\|_{C(\mathbb{R})} \leqslant A\right\}.$$

Let $v \in B_A$, and $L_{v,\varepsilon}$ be a closure in L_2 of linear differential operator

$$l_{v,\varepsilon} = -\rho(x)(\rho(x)y')' + (r(x,v(x)) + \varepsilon(1+x^2))y' + s(x,v(x))y,$$

defined on the set $C_0^{(2)}(\mathbb{R})$. Now we consider the equation

$$L_{v,\varepsilon} y = f. \tag{4}$$

A function $y \in D(L_{v,\varepsilon})$ satisfying the equation (4) we call a solution of that equation. Since the conditions of lemma 2 hold for the functions $\rho(x)$, $r(x,v(x)) + \varepsilon(1+x^2)$, $s(x,v(x))$ then for any $f \in L_2$ the equation (4) has unique solution $y = y_\varepsilon(x)$, and for y the following estimate holds

$$\|-\rho(x)(\rho(x)y')'\|_2 + \left\|\left(r(x,v(x)) + \varepsilon(1+x^2)\right) y'\right\|_2 + \|s(x,v(x))y\|_2 \leqslant C_5\|f\|_2. \tag{5}$$

Let $k > 0$. Using the Hölder's inequality we get

$$\left|(1+x^2)^k y\right| = \int_{-\infty}^{x} ((1+t^2)^k y)' dt \leqslant \int_{-\infty}^{x} (1+t^2)^k |y'| dt + k \int_{-\infty}^{x} (1+t^2)^k |y| dt =$$

$$= \int_{-\infty}^{x} (1+t^2)^{-\alpha} (1+t^2)^{k+\alpha} |y'| dt + k \int_{-\infty}^{x} (1+t^2)^{-\beta} (1+t^2)^{k+\beta} |y| dt \leqslant$$

$$\leqslant \left(\int_{-\infty}^{x} (1+t^2)^{-2\alpha} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{x} (1+t^2)^{2(k+\alpha)} |y'|^2 dt \right)^{\frac{1}{2}} +$$

$$+ \left(\int_{-\infty}^{x} (1+t^2)^{-2\beta} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{x} (1+t^2)^{2(k+\beta)} |y|^2 dt \right)^{\frac{1}{2}}. \quad (6)$$

We choose the numbers α and β so that $\frac{3}{4} < \alpha$, $\frac{1}{4} < \beta \leqslant \alpha - \frac{1}{2}$. Then

$$\left(\int_{-\infty}^{x} (1+t^2)^{-2\alpha} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{x} (1+t^2)^{2(k+\alpha)} |y'|^2 dt \right)^{\frac{1}{2}} +$$

$$+ \left(\int_{-\infty}^{x} (1+t^2)^{-2\beta} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{x} (1+t^2)^{2(k+\beta)} |y|^2 dt \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant C_6 \left\|(1+x^2)^{k+\alpha} y'\right\|_2 + C_7 \left\|(1+x^2)^{k+\beta} y\right\|_2. \quad (7)$$

Using the lemma 1 and the condition $\alpha \geqslant \beta + \frac{1}{2}$ we obtain

$$C_6 \left\|(1+x^2)^{k+\alpha} y'\right\|_2 + C_7 \left\|(1+x^2)^{k+\beta} y\right\|_2 \leqslant C_8 \left\|(1+x^2)^{k+\alpha} y'\right\|_2.$$

Let $\alpha = \frac{3}{4} + \frac{\delta}{2}$ and $\beta = \frac{1}{4} + \frac{\delta}{2}$, where δ is the number from the condition (3). Then from (6) and (7) we receive

$$\sup_{x\in\mathbb{R}} \left|(1+x^2)^k y\right| \leqslant C_8 \left\|(1+x^2)^{k+\frac{3}{4}+\frac{\delta}{2}} y'\right\|_2.$$

Finally, by putting $k = \frac{\delta}{2}$ and taking into account the condition (3), we obtain the following estimate

$$\sup_{x\in\mathbb{R}} \left|(1+x^2)^{\frac{\delta}{2}} y\right| \leqslant C_8 \left\|r y'\right\|_2.$$

Therefore due to (5), lemma 1 and the condition (3) we have

$$\|y\|_W \stackrel{def}{=} \| -\rho(x)(\rho(x)y')'\|_2 + \left\|\left(r(x,v(x)) + (1+x^2)\right) y'\right\|_2 +$$

$$+ \left\|\left(s(x,v(x)) + (1+x^2)^{\frac{1}{4}}\right) y\right\|_2 + \sup_{x\in\mathbb{R}} \left|\left(1+x^2\right)^{\frac{\delta}{2}} y(x)\right| \leqslant C_9 \|f\|_2, \quad (8)$$

where the constant C_9 does not depend on y.

Let $A = C_9\|f\|_2$, and $L_{v,\varepsilon}^{-1}$ be an operator inverse to $L_{v,\varepsilon}$. We denote $P_\varepsilon(v) \stackrel{def}{=} L_{v,\varepsilon}^{-1} f$. It follows from (8) that $P_\varepsilon(v)$ maps the ball B_A into itself. Moreover B_A is mapped to the set

$$Q_A \stackrel{def}{=} \{y : \|y\|_W \leqslant C_9 \|f\|_2\}.$$

1. Since $Q_A \subset B_A$ then the set of the functions Q_A is uniformly bounded

2. According to Morrey's inequality [18, p. 282] with $p = 2$ for the functions $y \in W_2^1(\mathbb{R})$ the following inequality holds

$$\|y\|_{C^{0,\frac{1}{2}}(\mathbb{R})} \leqslant C_{10} \|y\|_{W_2^1(\mathbb{R})},$$

where $C^{0,\frac{1}{2}}(\mathbb{R})$ is a Hölder space with a norm

$$\|y\|_{C^{0,\frac{1}{2}}(\mathbb{R})} = \sup_{\substack{a,b\in\mathbb{R},\\ a\neq b}} \frac{y(a)-y(b)}{\sqrt{|a-b|}}.$$

Therefore, for any $y \in Q_A$

$$|y(t+h)-y(t)| \leqslant C_{11}\sqrt{|h|},$$

and hence functions from Q_A are equicontinuous.

3. It follows from the estimate (8) that

$$\sup_{x\in\mathbb{R}} \left|\left(1+x^2\right)^{\frac{\delta}{2}} y(x)\right| \leqslant A,$$

therefore, for any $y \in Q_A$ we have

$$\sup_{x\in\mathbb{R}} |y(x)| \leqslant \frac{A}{(1+x^2)^{\frac{\delta}{2}}} \to 0 \text{ as } |x| \to +\infty.$$

Hence, the set Q_A is compact in $C(\mathbb{R})$.

We consider a sequence of functions $\{v_n\}_{n=1}^{+\infty} \subset B_A$ such that $\|v-v_n\|_{C(\mathbb{R})} \to 0$ as $n \to +\infty$, and denote $P_\varepsilon(v_n) = y_n$. Then $L_{v_n,\varepsilon} y_n = f$ and by virtue of linearity of $L_{v,\varepsilon}$ we receive

$$L_{v,\varepsilon}(y_n - y) = \big(r(x,v(x)) - r(x,v_n(x))\big)y_n' + \big(s(x,v(x)) - s(x,v_n(x))\big)y_n.$$

Therefore, for any $N > 0$, taking into account that the functions $r(x,v(x)) - r(x,v_n(x))$ and $s(x,v(x)) - s(x,v_n(x))$ are continuous in $\mathbb{R}$, we get

$$\|y_n - y\|_{L_2(-N,N)} \leqslant C_{12} \max\left(\sup_{|x|\leqslant N}\big|r(x,v(x)) - r(x,v_n(x))\big|, \sup_{|x|\leqslant N}\big|s(x,v(x)) - s(x,v_n(x))\big|\right) \times$$
$$\times \left(\|y_n'\|_{L_2(-N,N)} + \|y_n\|_{L_2(-N,N)}\right) \to 0,$$

as $n \to +\infty$.

As $v_n \in B_A$ then $y_n \in Q_A$. Since Q_A is a compact in L_2, and the operator $L_{v,\varepsilon}$ is closed then the Cauchy sequence $\{y_n\}_{n=1}^{+\infty}$ converges to the element $y \in Q_A$ (due to the uniqueness of the limit). Therefore P_ε is a continuous operator.

Thus, the continuous operator $P_\varepsilon : B_A \to B_A$ maps the ball B_A into itself, hence according to the Schauder theorem it has a fixed point, i. e. $\exists y \in B_A : P_\varepsilon(y) = y$. In other words y satisfies the equation

$$-\rho(x)(\rho(x)y')' + \big(r(x,y) + \varepsilon(1+x^2)\big)y' + s(x,y)y = f(x),$$

by virtue of (8) the following estimate holds

$$\|-\rho(x)(\rho(x)y')'\|_2 + \left\|\big(r(x,y) + \varepsilon(1+x^2)\big)y'\right\|_2 + \|s(x,y)y\|_2 \leqslant C_9\|f\|_2.$$

We consider a sequence of positive numbers $\{\varepsilon_k\}_{k=1}^{+\infty}$ tending to 0. If $y_k \in B_A$ is a fixed point of the operator P_{ε_k} then

$$-\rho(x)(\rho(x)y_k')' + \big(r(x,y_k) + \varepsilon_k(1+x^2)\big)y_k' + s(x,y_k)y_k = f(x),$$

and

$$\|-\rho(x)(\rho(x)y_k')'\|_2 + \left\|\big(r(x,y_k) + \varepsilon_k(1+x^2)\big)y_k'\right\|_2 + \|s(x,y_k)y_k\|_2 \leqslant C_{13}(\varepsilon_k)\|f\|_2. \qquad (9)$$

Let $[a,b] \subset \mathbb{R}$ be a finite segment. Since the space $W_2^2(a,b)$ is compactly embedded to $L_2(a,b)$ then there is a subsequence $\{y_{k_i}\}_{i=1}^{+\infty}$, converging to y by the norm of $L_2(a,b)$, that is

$$\lim_{i\to+\infty} \|y_{k_i} - y\|_{L_2(a,b)} = 0.$$

Then according to the definition 1, y is the solution of the equation (1), and by virtue of (9) the following estimate holds

$$\|-\rho(x)(\rho(x)y')'\|_2 + \|r(x,y)y'\|_2 + \|s(x,y)y\|_2 < +\infty.$$

Conclusion

In this work we considered the conditions of the correct solvability as well as established a coercive estimate for the second-order differential equation (1) in a non-compact domain, and with coefficients that can be unbounded. In the case of a Hilbert space, this work generalizes the results of [17] to the nonlinear differential equation.

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References

1 Naimark, M.A. (1968). *Linear Differential operators.* London: George G.Harrap & Co.

2 Reed, M., & Simon, B. (1975). *Methods of Modern Mathematical Physics: Fourier Analysis, Self-Adjointness.* Academic Press.

3 Fedoruk, M.V. (1993). *Asymptotic Analysis. Linear Ordinary Differential Equations.* Berlin: Springer.

4 Bogachev, V.I., Krylov, N.V., Röckner, M., & Shaposhnikov S.V. (2015). *Fokker-Planck-Kolmogorov equations: Mathematical surveys and monographs.* American Mathematical Society, Providence.

5 Fornaro, S., & Lorenzi, L. (2007). Generation results for elliptic operators with unbounded diffusion coefficients in L^p-and C_b-spaces. *Discrete and Continuous Dynamical Systems, 18*(4), 747-772.

6 Hieber, M., & Sawada, O. (2005). The Navier-Stokes Equations in $\mathbb{R}^n$ with Linearly Growing Initial Data. *Archive for Rational Mechanics and Analysis, 175,* 269-285.

7 Metafune, G., Pallara. D., & Vespri, V. (2005). L^p-estimates for a class of elliptic operators with unbounded coefficients in $\mathbb{R}^n$. *Houston Journal of Mathematics, 31,* 605-620.

8 Hieber, M., Lorenzi, L., Prüss, J., Rhandi, A., & Schnaubelt, R. (2009). Global properties of generalized Ornstein-Uhlenbeck operators on $L_p(\mathbb{R}^N, \mathbb{R}^N)$ with more than linearly growing coefficients. *J. Math. Anal. Appl., 350,* 100-121.

9 Ospanov, K.N,. & Akhmetkaliyeva R.D. (2012). Separation and the existence theorem for second order nonlinear differential equation. *Elec. J. Qual. Th. Dif. Eq., 66,* 1-12.

10 Ospanov, K.N., & Akhmetkaliyeva R.D. (2015). Some inequalities for secong order differential operators with unbounded drift. *Eurasian Mathematical Journal, 6*(2), 63-71.

11 Ospanov, K.N. (2015). L_1-maximal regularity for quasilinear second order differential equation with damped term. *Elec. J. Qual. Th. Dif. Eq., 39,* 1-9.

12 Ospanov, K.N. (2019). Maximal L_p-regularity for a second-order differential equation with unbounded intermediate coefficient. *Elec. J. Qual. Th. Dif. Eq., 65,* 1-13.

13 Muratbekov, M.B., Muratbekov, M.M., & Abylayeva, A.M. (2013). On existence of the resolvent and discreteness of the spectrum of a class of differential operators of hyperbolic type. *Elec. J. Qual. Th. Dif. Eq., 64,* 1-10.

14 Muratbekov, M.B., & Muratbekov, M.M. (2007). Estimates of the spectrum for a class of mixed type operators. *Differential equations, 43*(1), 143-146.

15 Zhapsarbayeva, L., & Ospanov, K. (2020). Solvability of nonlinear problem for some second-order nonstrongly elliptic system. *Complex Variables and Elliptic Equations, in appear.*

16 Ospanov, K.N., Yeskabylova, Zh.B., & Bekjan, T.N. (2019). The solvability results for the third-order singular non-linear differential equation. *Eurasian mathematical journal, 10*(4), 84-90.

17 Ospanov, K., & Yesbayev, A. (2020). Solvability and maximal regularity results for a differential equation with diffusion coefficient. *Turkish journal of mathematics, 44*(4), 1304-1316.

18 Evans, L.C. (2010). *Partial differential equations: Mathematical surveys and monographs.* American Mathematical Society, Providence.

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$L_2(\mathbb{R})$-де шенелмеген коэффициенттері бар екінші ретті сызықтықемес дифференциалдық теңдеудің шешілу шарттары

Мақалада шенелмеген облыста сызықтыемес екінші ретті дифференциалдық теңдеудің жалпы шешімінің бар болу мәселесі қарастырылған. Теңдеудің аралық және ең кіші коэффициенттері ізделген функцияға тәуелді және тегіс болып саналады. Жұмыстың жаңашылдығы — үлкен коэффициенті нөлден өзге болатын сызықтыемес сингулярлық теңдеудің шешілетіндігін дәлелдейтіндігімізде. Бұрын қарастырылғандардан айырмашылығы, теңдеудің үлкен коэффициенті нөлге ұмтылуы мүмкін, ал аралық коэффициент шексіздікке ұмтылады және ең кіші коэффициенттің өсуіне бағынбайды. Алынған нәтиже теңдеудің коэффициенттері бойынша тұжырымдалған; бұл коэффициенттердің кез-келген туындыларына шарттар қойылмайды.

Кілт сөздер: екінші ретті дифференциалдық теңдеу, сызықтыемес дифференциалдық теңдеу, шенелмеген облыстағы дифференциалдық теңдеу, жалпы шешім, шешімділік.

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Условия разрешимости нелинейного дифференциального уравнения второго порядка с неограниченными коэффициентами в $L_2(\mathbb{R})$

В статье рассмотрен вопрос существования обобщённого решения нелинейного дифференциального уравнения второго порядка в неограниченной области. Промежуточный и младший коэффициенты уравнения зависят от искомой функции и считаются гладкими. Новизна работы состоит в том, что мы доказываем разрешимость нелинейного сингулярного уравнения с неотделённым от нуля старшим коэффициентом. В отличие от работ, рассмотренных ранее, старший коэффициент уравнения может стремиться к нулю, а промежуточный — к бесконечности и не подчиняться росту младшего коэффициента. Полученный результат сформулирован в терминах самих коэффициентов уравнения, в нём не ставятся условия на какие-либо производные этих коэффициентов.

Ключевые слова: дифференциальное уравнение второго порядка, нелинейное дифференциальное уравнение, дифференциальное уравнение в неограниченной области, обобщённое решение, разрешимость.