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On the existence of the resolvent and separability of a class of the Korteweg-de Vries type linear singular operators

Partial differential equations of the third order are the basis of mathematical models of many phenomena and processes, such as the phenomenon of energy transfer of hydrolysis of adenosine triphosphate molecules along protein molecules in the form of solitary waves, i.e. solitons, the process of transferring soil moisture in the aeration zone, taking into account its movement against the moisture potential. In particular, this class includes the nonlinear Korteweg-de Vries equation, which is the main equation of modern mathematical physics. It is known that various problems have been studied for the Korteweg-de Vries equation and many fundamental results obtained. In this paper, issues about the existence of a resolvent and separability (maximum smoothness of solutions) of a class of linear singular operators of the Korteweg-de Vries type in the case of an unbounded domain with strongly increasing coefficients are investigated.

Keywords: resolvent, Korteweg-de Vries type singular operator, separability.

1 Introduction

The solvability of boundary value problems for odd order differential equations, in particular, to the Korteweg-de Vries equation is dealt with in a significant literature [1–9] and the papers cited there.

In contrast to these interesting papers, this article deals with the problem of the existence of a resolvent and the separability of a class of linear singular operators of the Korteweg-de Vries type in the case of an unbounded domain with strongly increasing coefficients.

Note that if the boundary regime operates sufficiently long, then due to the friction inherent in any real physical system, the influence of the initial data weakens with time. Thus, we arrive at a problem without initial conditions [10].

Therefore, taking this remark into account in the present work, we consider the differential operator

$$Lu + \lambda u = \frac{\partial u}{\partial y} + R_2(y) \frac{\partial^3 u}{\partial x^3} + R_1(y) \frac{\partial u}{\partial x} + R_0(y)u + \lambda u \quad (1)$$

initially defined on $C_{0,\pi}^\infty(\bar{\Omega})$, where $\bar{\Omega} = \{(x, y) : -\pi \leq x \leq \pi, -\infty < y < \infty\}$, $\lambda \geq 0$.

$C_{0,\pi}^\infty$ is a set of infinitely differentiable functions satisfying the conditions:

$$u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \quad i = 0, 1, 2 \quad (2)$$

and compactly supported with respect to the variable y .

Further assume that the coefficients $R_0(y), R_1(y), R_2(y)$ satisfy the following conditions:

- i) $R_0(y) \geq \delta_0 > 0, R_1(y) \geq \delta_1 > 0, -R_2(y) \geq \delta_2 > 0$ are continuous functions in $R(-\infty, +\infty)$;
- ii) $\mu_0 = \sup_{|y-t| \leq 1} \frac{R_0(y)}{R_0(t)} < \infty, \mu_1 = \sup_{|y-t| \leq 1} \frac{R_1(y)}{R_1(t)} < \infty, \mu_2 = \sup_{|y-t| \leq 1} \frac{R_2(y)}{R_2(t)} < \infty$.

The operator $L + \lambda I$ admits closure in $L_2(\Omega)$, which we also denote by $L + \lambda I$.

Theorem 1. Let the condition i) be fulfilled. Then the operator $L + \lambda I$ is continuously invertible in $L_2(\Omega)$ for $\lambda \geq 0$.

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Definition. We called the operator L is separable in $u \in D(L)$ if the estimate

$$\left\| \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} + \left\| R_2(y) \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(\Omega)} + \left\| R_1(y) \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)} + \|R_0(y)u\|_{L_2(\Omega)} \leq C(\|Lu\|_2 + \|u\|_{L_2(\Omega)}),$$

holds for $u(x, y)$, where C is independent of $u(x, y)$, $\|\cdot\|_2$ is the norm of $L_2(\Omega)$.

Theorem 2. Let conditions i) - ii) be fulfilled. Then the operator L is separable.

Example. Let $R_0(y) = |y| + 1$, $R_1(y) = e^{|y|}$, $R_2(y) = -10 \cdot e^{|y|}$, $-\infty < y < \infty$. It is easy to verify that all the conditions of Theorem 2 are satisfied. Consequently, the operator L is separable, i.e.

$$\left\| \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} + \left\| 10 \cdot e^{|y|} \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(\Omega)} + \left\| e^{|y|} \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)} + \|(|y| + 1)u\|_{L_2(\Omega)} \leq C(\|Lu\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}),$$

C is a constant.

2 Auxiliary lemmas and inequalities

Lemma 2.1. Let the condition i) be fulfilled and $\lambda \geq 0$. Then the inequality

$$\|(L + \lambda I)u\|_{L_2(\Omega)} \geq (\delta_0 + \lambda) \|u\|_{L_2(\Omega)}, \tag{3}$$

holds for all $u \in D(L)$, where $\delta_0 > 0$.

Proof. Let $u \in C_{0,\pi}^\infty(\bar{\Omega})$ and consider the scalar product

$$\begin{aligned} \langle (L + \lambda I)u, u \rangle &= \int_{\Omega} \left(\frac{\partial u}{\partial y} + R_2(y) \frac{\partial^3 u}{\partial x^3} + R_1(y) \frac{\partial u}{\partial x} + (R_0(y) + \lambda)u \right) u dx dy = \\ &= \int_{\Omega} \frac{\partial u}{\partial y} u dx dy + \int_{\Omega} R_2(y) \frac{\partial^3 u}{\partial x^3} u dx dy + \int_{\Omega} R_1(y) \frac{\partial u}{\partial x} u dx dy + \int_{\Omega} (R_0(y) + \lambda)u^2 dx dy. \end{aligned} \tag{4}$$

Since $u \in C_{0,\pi}^\infty(\Omega)$ and (2) holds, then the following equalities:

$$\int_{\Omega} \frac{\partial u}{\partial y} u dx dy = 0; \quad \int_{\Omega} R_1(y) \frac{\partial u}{\partial x} u dx dy = 0; \quad \int_{\Omega} R_2(y) \frac{\partial^3 u}{\partial x^3} u dx dy = 0. \tag{5}$$

hold. Using the equality (5) from (4) we have

$$\langle (L + \lambda I)u, u \rangle = \int_{\Omega} (R_0(y) + \lambda)u^2 dx dy. \tag{6}$$

From (6), using the Cauchy-Bunyakovsky inequality and the condition i), we obtain

$$\|(L + \lambda I)u\|_{L_2(\Omega)} \geq (\delta_0 + \lambda) \|u\|_{L_2(\Omega)}, \tag{7}$$

for all $u \in C_{0,\pi}^\infty(\bar{\Omega})$. By virtue of the continuity of the norm, the last estimate holds for all $u \in D(L)$. Lemma 2.1 is proved.

Remark. Due to the realness of the coefficients of (1) the estimate (7) holds for complex-valued functions. Consider the operator

$$(l_{n,j} + \lambda I)z = z'(y) + (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y))z(y),$$

where $R_{2,j}(y)$, $R_{1,j}(y)$, $R_{0,j}(y)$ are bounded periodic functions of the same period $\Delta_j = (j - 1, j + 1)$, $j = 0, \pm 1, \pm 2$, $z(y) \in C_0^\infty(R)$, $z(y) = u(y) + i\vartheta(y)$.

Lemma 2.2. Let the condition i) be fulfilled. Then the operator $(l_{n,j} + \lambda I)$ considered on $C_0^\infty(R)$ is a closable operator in $L_2(R)$.

Proof. Let $u_n \in C_0^\infty(R)$ and $u_n \xrightarrow{L_2(R)} 0$, $(l_{n,j} + \lambda I)u_n \xrightarrow{L_2(R)} \vartheta$. If you use the operator $(l_{n,j} + \lambda I)^*$ formally associated with the operator $(l_{n,j} + \lambda I)$, then for arbitrary function $\omega \in C_0^\infty(R)$

$$\langle (l_{n,j} + \lambda I)u_n, \omega \rangle = \langle u_n, (l_{n,j} + \lambda I)^* \omega \rangle$$

where $(l_{n,j} + \lambda I)^* \omega = -\omega' + (in^3 R_{2,j}(y) - inR_{1,j}(y) + R_{0,j}(y))\vartheta = 0$.

Hence

$$\langle \vartheta, \omega \rangle = \langle 0, (l_{n,j} + \lambda I)^* \omega \rangle = 0$$

Consequently, $\langle \vartheta, \omega \rangle = 0$ for any $\omega \in C_0^\infty(R)$ and it means $\vartheta = 0$. Lemma 2.2 is proved.

Lemma 2.3. Let the condition i) be fulfilled. Then the estimate

$$\|(l_{n,j} + \lambda I)z\|_2 \geq (\delta_0 + \lambda) \|z\|_2 \tag{8}$$

holds for all $z(y) \in D(l_{n,j} + \lambda I)$, $\|\cdot\|_2$ is the norm of $L_2(R)$.

Proof. Let $z(y) \in C_0^\infty(R)$ and $z(y) = u(y) + i\vartheta(y)$. Consider the scalar product

$$\begin{aligned} \langle (l_{n,j} + \lambda I)z, z \rangle &= \int_{-\infty}^{\infty} (z' + (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \lambda)z) \bar{z} dy = \\ &= \int_{-\infty}^{\infty} z'(y)z(y) dy + \int_{-\infty}^{\infty} (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \lambda)|z|^2 dy = \\ &= i \left(\int_{-\infty}^{\infty} u\vartheta'(y) dy + \int_{-\infty}^{\infty} (-in^3 R_{2,j}(y) + inR_{1,j}(y))|z|^2 dy \right) + \int_{-\infty}^{\infty} (R_{0,j}(y) + \lambda)|z|^2 dy \end{aligned} \tag{9}$$

Hence, using the properties of complex numbers, we have

$$|\langle (l_{n,j} + \lambda I)z, z \rangle| \geq \left| \int_{-\infty}^{\infty} (R_{0,j}(y) + \lambda)|z|^2 dy \right|.$$

Since $R_{0,j}(y)$ does not change the sign, then

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| \geq \int_{-\infty}^{\infty} |(R_{0,j}(y) + \lambda)||z|^2 dy.$$

By virtue of condition i) and using the Cauchy-Bunyakovsky inequality from the latter inequality, we have

$$\|(l_{n,j} + \lambda I)z\|_2 \geq (\delta_0 + \lambda) \|z\|_2.$$

Lemma 2.3 is proved.

Lemma 2.4. Let the condition i) be fulfilled. Then the operator $(l_{n,j} + \lambda I)$ has a continuous inverse operator $(l_{n,j} + \lambda I)^{-1}$ defined on the whole $L_2(R)$.

Proof. By the estimate (8) it suffices to show that the range is dense in $L_2(R)$. Let's assume that the range is not dense in $L_2(R)$. Then there is an element $\vartheta \in L_2(R)$ such that $\langle (l_{n,j} + \lambda I)u, \vartheta \rangle = 0$ for all $u \in D(l_{n,j})$. This means that

$$(l_{n,j} + \lambda I)^* \vartheta = -\vartheta' + (in^3 R_{2,j}(y) - inR_{1,j}(y) + R_{0,j}(y))\vartheta = 0. \tag{10}$$

in terms of distribution theory. By the periodicity of functions $R_0(y)$, $R_1(y)$, $R_2(y)$, we have that $(in^3 R_{2,j}(y) - inR_{1,j}(y) + R_{0,j}(y))\vartheta \in L_2(R)$. From this and from (10) it follows that $\vartheta \in W_2^1(R)$, where $W_2^1(R)$ is the Sobolev space. The general theory of the embedding theorems implies that

$$\lim_{|y| \rightarrow \infty} \vartheta(y) = 0. \tag{11}$$

Now, using the equality (11) and the arguments used in the proof of the estimate (9), we obtain that

$$\|(l_{n,j} + \lambda I)^* \vartheta\|_2 \geq \delta_0 \|\vartheta\|_2. \tag{12}$$

From estimates (12) and (10) it follows that $\vartheta = 0$. Lemma 2.4 is proved.

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \in C_0^\infty(R)$ is a set of such functions that $\varphi_j(y) \geq 0$, $\text{supp } \varphi_j \subseteq \Delta_j (j \in Z)$, $\sum_{j=-\infty}^{\infty} \varphi_j^2(y) = 1$.

Here we note immediately that any point $y \in R$ can belong to no more than three segments from the system of segments $\{\text{supp } \varphi_j\}$ [11,12].

Assume that

$$K_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j(y)(l_{n,j} + \lambda I)^{-1} \varphi_j f,$$

$$B_\lambda f = \sum_{j=-\infty}^{\infty} \varphi'_j(y)(l_{n,j} + \lambda I)^{-1} \varphi_j f, \quad f \in C_0^\infty(R), \quad \lambda \geq 0.$$

It is easy to verify that

$$(l_n + \lambda I)K_\lambda f = f + \sum_j \varphi'_j(y)(l_{n,j} + \lambda I)^{-1} \varphi_j f, \tag{13}$$

where

$$(l_n + \lambda I)z = -z'(y) + (-in^3 R_2(y) + inR_1(y) + R_0(y))z, \quad z \in D(l_n).$$

Lemma 2.5. Let the condition i) be fulfilled. Then there is a number $\lambda_0 > 0$ such that $\|B_\lambda\|_{2 \rightarrow 2} < 1$ for all $\lambda \geq \lambda_0$.

Proof. Let $f \in C_0^\infty(R)$. Only functions $\varphi_{j-1}, \varphi_j, \varphi_{j+1}$ are nonzero in the interval $\overline{\Delta_j} (j \in Z)$, consequently:

$$\|B_\lambda f\|_{L_2(R)}^2 = \int_{-\infty}^{\infty} \left| \sum_{j=-\infty}^{\infty} \varphi'_j(y)(l_{n,j} + \lambda I)^{-1} \varphi_j f \right|^2 dy \leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} \left| \sum_{k=j-1}^{j+1} [\varphi'_k(y)(l_{n,k} + \lambda I)^{-1} \varphi_k f] \right|^2 dy.$$

Hence, using the obvious inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and estimate (8), we have:

$$\begin{aligned} \|B_\lambda f\|_{L_2(R)}^2 &\leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} \left| \sum_{k=j-1}^{j+1} [\varphi'_k(l_{n,k} + \lambda I)^{-1} \varphi_k f] \right|^2 dy \leq 9 \sum_{j=-\infty}^{\infty} \|\varphi'_j(l_{n,j} + \lambda I)^{-1} \varphi_j f\|_{L_2(\Delta_j)}^2 \leq \\ &\leq 9 \sum_{j=-\infty}^{\infty} \|\varphi'_j(l_{n,j} + \lambda I)^{-1} \varphi_j f\|_{L_2(R)}^2 \leq 9 \cdot c \sum_{j=-\infty}^{\infty} \|(l_{n,j} + \lambda I)^{-1}\|_{L_2(R) \rightarrow L_2(R)}^2 \cdot \|\varphi_j f\|_{L_2(R)}^2 \leq \\ &\leq \frac{9 \cdot c}{(\delta_0 + \lambda)^2} \cdot \int_{-\infty}^{\infty} \left(\sum_j \varphi_j^2 \right) |f|^2 dy = \frac{9 \cdot c}{(\delta_0 + \lambda)^2} \cdot \|f\|_{L_2(R)}^2. \end{aligned}$$

This implies that

$$\|B_\lambda\|_{L_2(R) \rightarrow L_2(R)} \leq \frac{9 \cdot c}{(\delta_0 + \lambda)^2}. \tag{14}$$

From (14) it follows that it is easy to find a number $\lambda_0 > 0$, such that $\lambda \geq \lambda_0, \|B_\lambda\|_{L_2(R) \rightarrow L_2(R)} < 1$. Lemma 2.5 is proved.

Now consider the operator

$$(l_n + \lambda I)z = z'(y) + (-in^3 R_2(y) + inR_1(y) + R_0(y))z(y),$$

where $z(y) = u(y) + i\vartheta(y), z(y) \in C_0^\infty(R), (R = (-\infty, \infty))$.

Lemma 2.6. Let the condition i) be fulfilled. Then the estimate

$$\|(l_n + \lambda I)z\|_2 \geq (\delta_0 + \lambda) \|z\|_2. \tag{15}$$

holds for all $z \in D(l_n)$.

Proof. We obtain the proof of Lemma 2.3 by reproducing the computations and argument used in Lemma 2.6.

Lemma 2.7. Let the condition i) be fulfilled. Then the operator $l_n + \lambda I$ for $\lambda \geq \lambda_0$ is boundedly invertible and the equality

$$(l_n + \lambda I)^{-1} = K_\lambda(I - B_\lambda)^{-1}. \tag{16}$$

holds for the inverse operator $(l_n + \lambda I)^{-1}$.

The proof of Lemma 2.7 follows from the representations (13), (15) and Lemma 2.5

3 On the existence of the resolvent. Proof of Theorem 1

Lemma 2.7 implies that

$$u_k(x, y) = \sum_{n=-k}^k (l_n + \lambda I)^{-1} f_n(y) e^{inx} \tag{17}$$

is the solution to the problem

$$(L + \lambda I)u_k(x, y) = f_k(x, y),$$

$$u_k^{(i)}(-\pi, y) = u_k^{(i)}(\pi, y), i = 0, 1, 2$$

where $f_k(x, y) \xrightarrow{L_2(R)} f(x, y)$, $f_k(x, y) = \sum_{n=-k}^k f_n(y) \cdot e^{inx}$, $i^2 = -1$, $(l_n + \lambda I)^{-1}$ is the inverse operator to the operator $l_n + \lambda I$. Using the inequality (3), we obtain

$$\|u_k(x, y)\|_2 \leq \frac{1}{(\delta_0 + \lambda)} \|f_k(x, y)\|_2. \tag{18}$$

Since $f_k(x, y) \xrightarrow{L_2(R)} f(x, y)$, then from (18) we find that

$$\|u_k - u_m\|_2 \leq \frac{1}{\delta_0} \|f_k - f_m\|_2 \rightarrow 0, \quad k, m \rightarrow \infty.$$

Hence, by the completeness of $L_2(\Omega)$, it follows that there exist a unique function $u \in L_2(\Omega)$ such that

$$u_k(x, y) \rightarrow u(x, y) \text{ as } k \rightarrow \infty. \tag{19}$$

(17) and (19) imply that

$$u(x, y) = (L + \lambda I)^{-1} f(x, y) = \sum_{n=-\infty}^{\infty} (l_n + \lambda I)^{-1} f_n(y) e^{inx} \tag{20}$$

is a strong solution for:

$$(L + \lambda I)u = f, \tag{21}$$

$$u^{(i)}(-\pi, y) = u^{(i)}(\pi, y), \quad i = 0, 1, 2. \tag{22}$$

for any $f \in L_2(\Omega)$.

Let us recall the definition of a strong solution. The function $u \in L_2(\Omega)$ is called a strong solution of the problem (21)-(22), if there exists a $\{u_k\}_{k=1}^{\infty} \subset C_{0,\pi}^{\infty}(\Omega)$ such that

$$\|u_k - u\|_{L_2(\Omega)} \rightarrow 0, \quad \|(L + \lambda I)u_k - f\|_{L_2(\Omega)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Now, it is not difficult to verify that the formula (20) is the inverse operator to the closed operator $L + \lambda I$.

Lemma 2.8. [13]. Let the operator $L + \lambda_0 I$ ($\lambda_0 > 0$) is boundedly invertible in $L_2(\Omega)$ and the estimate $\|(L + \lambda I)u\|_{L_2(\Omega)} \geq \|u\|_{L_2(\Omega)}$, $u \in D(L + \lambda I)$ holds for $\lambda \in [0, \lambda_0]$. Then the operator $L : L_2(\Omega) \rightarrow L_2(\Omega)$ is also boundedly invertible.

This and Lemma 2.1 implies that Theorem 1 holds for all $\lambda \geq 0$. Theorem 1 is completely proved.

4 On the separability of the operator

First, we give the lemmas that reduce the question of separability of an operator with unbounded coefficients to the case of an operator with periodic coefficients.

Lemma 2.9. Let $z(y) \in D(l_n + \lambda I)$ and $z(y) = u(y) + i\vartheta(y)$, then $in^3 R_2(y)z(y) \in L_2(R)$ if and only if $n^3 R_2(y)u(y) \in L_2(R)$ and $n^3 R_2(y)\vartheta(y) \in L_2(R)$.

Proof. Necessity. Let $in^3 R_2(y)z(y) \in L_2(R)$. Then

$$\begin{aligned} \|in^3 R_2(y)z(y)\|_2^2 &= \int_{-\infty}^{\infty} |in^3 R_2(y)z(y)|^2 dy = \int_{-\infty}^{\infty} |n^6 R_2^2(y)||z(y)|^2 dy = \\ &= \int_{-\infty}^{\infty} |n^6 R_2^2(y)|(|u|^2 + |\vartheta|^2) dy = \|n^3 R_2(y)u\|_2^2 + \|n^3 R_2(y)\vartheta\|_2^2 \end{aligned}$$

It follows that $n^3 R_2(y)u(y) \in L_2(R)$ and $n^3 R_2(y)\vartheta(y) \in L_2(R)$.

Sufficiency. Let $n^3 R_2(y)u(y) \in L_2(R)$ and $n^3 R_2(y)\vartheta(y) \in L_2(R)$. Then

$$\|in^3 R_2(y)z(y)\|_2^2 = \|n^3 R_2(y)u\|_2^2 + \|n^3 R_2(y)\vartheta\|_2^2$$

Hence $in^3 R_2(y)z(y) \in L_2(R)$. Lemma is proved.

Remark. This Lemma is also true for $in^3 R_1(y)z(y)$.

By virtue of this Lemma, we consider the operator

$$(l_{n,j} + \lambda I)u = u'(y) + (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_0(y) + \lambda)u(y),$$

in the space $C_0^\infty(R)$ the set of infinitely differentiable, finite, and real-valued functions, where $R_{0,j}(y)$, $R_{1,j}(y)$, $R_{2,j}(y)$ are bounded periodic coefficients of the same period $\Delta_j = (j - 1, j + 1)$, $j = \pm 0, \pm 1, \pm 2, \dots$

Lemma 2.10. Let the condition i) be fulfilled. Then the estimates:

$$\|(l_{n,j} + \lambda I)u(y)\|_2 \geq R_0(y_j) \|u\|_2, \quad n = 0, \pm 1, \pm 2, \dots \tag{23}$$

where $R_0(y_j) = \min_{y \in \Delta_j} R_{0,j}(y)$;

$$\|(l_{n,j} + \lambda I)u(y)\|_2 \geq |n|R_1(\bar{y}_j) \|u\|_2, \quad n = 0, \pm 1, \pm 2, \dots \tag{24}$$

where $R_1(\bar{y}_j) = \min_{y \in \Delta_j} R_{1,j}(y)$;

$$\|(l_{n,j} + \lambda I)u(y)\|_2 \geq |n|^3 R_2(\bar{\bar{y}}_j) \|u\|_2 \quad n = 0, \pm 1, \pm 2, \dots \tag{25}$$

where $R_2(\bar{\bar{y}}_j) = \min_{y \in \Delta_j} |R_{2,j}(y)|$, $\|\cdot\|_2$ is the norm of $L_2(R)$.

Proof. Let $u(y) \in C_0^\infty(R)$. Given that $\int_{-\infty}^\infty u'(y)u(y)dy = 0$ and reproducing the computations used in the proof of Lemma 2.1, we have:

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| = \left| \int_{-\infty}^\infty (-in^3 R_{2,j}(y) + inR_{1,j}(y) + R_{0,j}(y) + \lambda)u^2 dy \right|, \tag{26}$$

From estimate (26) we find that

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| \geq \left| \int_{-\infty}^\infty (R_{0,j}(y) + \lambda)|u|^2 dy \right| \geq \min_{y \in \Delta_j} R_0(y) \|u\|_2^2, \tag{27}$$

Using the Cauchy-Bunyakovsky inequality, we obtain from (27) that

$$\|(l_{n,j} + \lambda I)u\|_2 \geq R_0(y_j) \|u\|_2, \tag{28}$$

where $R_0(y_j) = \min_{y \in \Delta_j} R_0(y)$.

The proof of the inequality (28) follows from the inequality (22) of lemma 2.10.

Further, from the inequality (26) we find

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| \geq \left| -i \int_{-\infty}^\infty (n^3 R_{2,j}(y) + nR_{1,j}(y))u^2 dy \right|. \tag{29}$$

Here we used the properties of complex numbers. From (29), by virtue of the condition i), we obtain the following inequalities:

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| \geq \left| \int_{-\infty}^\infty nR_{1,j}(y)u^2 dy \right| \geq |n|R_1(\bar{y}_j) \|u\|_2^2,$$

$$|\langle (l_{n,j} + \lambda I)u, u \rangle| \geq \left| \int_{-\infty}^\infty (-n^3 R_{2,j}(y))|u|^2 dy \right| \geq |n|^3 |R_2(\bar{\bar{y}}_j)| \|u\|_2^2,$$

where $R_1(\bar{y}_j) = \min_{y \in \Delta_j} R_{1,j}(y)$, $R_2(\bar{\bar{y}}_j) = \min_{y \in \Delta_j} |R_{2,j}(y)|$. From these inequalities and using the Cauchy-Bunyakovsky inequality, we find that:

$$\|(l_{n,j} + \lambda I)u\|_2 \geq |n|R_1(\bar{y}_j) \|u\|_2;$$

$$\|(l_{n,j} + \lambda I)u\|_2 \geq |n|^3 |R_2(\bar{\bar{y}}_j)| \|u\|_2.$$

Lemma 2.10 is proved.

Lemma 2.11. Let the condition i) be fulfilled and $\lambda \geq \lambda_0$, $\alpha = 0, 1, 2, 3$, $p(y)$ defined on R continuous function. Then the estimate

$$\| |p(y)|n|^\alpha (l_n + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)}^2 \leq c(\lambda) \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j) \rightarrow L_2(\Delta_j)}^2. \quad (30)$$

Proof. Let $f \in C_0^\infty(R)$. From the representation (16), taking the properties of the functions φ_j ($j \in Z$) into account, we have:

$$\begin{aligned} \| |p(y)|n|^\alpha (l_n + \lambda I)^{-1} f \|_{L_2(R)}^2 &= \| |p(y)|n|^\alpha K_\lambda (I - B_\lambda)^{-1} f \|_{L_2(R)}^2 = \\ &= \int_{-\infty}^{\infty} |p(y)|n|^\alpha \sum_{\{j\}} \varphi_j (l_{n,j} + \lambda I)^{-1} \varphi_j (I - B_\lambda)^{-1} f|^2 dy \end{aligned}$$

As it is known that on Δ_j ($j \in Z$) the interval only functions φ_{j-1} , φ_j , φ_{j+1} are nonzero, therefore

$$\begin{aligned} \| |p(y)|n|^\alpha (l_n + \lambda I)^{-1} f \|_{L_2(R)}^2 &\leq \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |p(y)|n|^\alpha \sum_{j-1}^{j+1} \varphi_j (l_{n,j} + \lambda I)^{-1} \varphi_j (I - B_\lambda)^{-1} f|^2 dy \leq \\ &\leq 9 \sum_{j=-\infty}^{\infty} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \varphi_j (I - B_\lambda)^{-1} f \|_{L_2(\Delta_j)}^2 \leq \\ &\leq 9 \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \cdot \sum_{j=-\infty}^{\infty} \int_{\Delta_j} |\varphi_j (I - B_\lambda)^{-1} f|^2 dy \leq \\ &\leq 9 \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \cdot \int_{-\infty}^{\infty} (\sum_j \varphi_j^2) |(I - B_\lambda)^{-1} f|^2 dy \end{aligned} \quad (31)$$

Since $(\sum_j \varphi_j^2) = 1$, then from (31) we have

$$\begin{aligned} \| |p(y)|n|^\alpha (l_n + \lambda I)^{-1} f \|_{L_2(R)}^2 &\leq 9 \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \cdot \int_{-\infty}^{\infty} |(I - B_\lambda)^{-1} f|^2 dy \leq \\ &\leq 9 \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \cdot \| (I - B_\lambda)^{-1} \|_{2 \rightarrow 2}^2 \cdot \| f \|_2^2. \end{aligned} \quad (32)$$

Lemma 2.5 implies that $\| I - B_\lambda \|_{2 \rightarrow 2}^2 < c(\lambda)$. From this and from (32) we obtain that

$$\| |p(y)|n|^\alpha (l_{n,j} + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)} \leq 9 \cdot c(\lambda) \sup_{j \in Z} \| |p(y)|n|^\alpha \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2.$$

Lemma 2.11 is proved.

Lemma 2.12. Let the conditions i)-ii) be fulfilled. Then the following estimates:

$$\| R_0(y) (l_n + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)} \leq C_0 < \infty; \quad (33)$$

$$\| R_1(y) |n| (l_n + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)} \leq C_1 < \infty; \quad (34)$$

$$\| R_2(y) |n|^3 (l_n + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)} \leq C_2 < \infty; \quad (35)$$

hold, where C_0 , C_1 , C_2 are independent of n ($n = 0, \pm 1, \pm 2, \dots$).

Proof. (30) shows that the operator $R_0(l_n + \lambda I)$ is bounded, if $\sup_{j \in Z} \| R_0(y) \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}$ is bounded.

Therefore, we will estimate the last expression

$$\begin{aligned} \| R_0(y) \varphi_j (l_n + \lambda I)^{-1} \|_{L_2(R) \rightarrow L_2(R)}^2 &\leq C(\lambda) \sup_{j \in Z} \| R_0(y) \varphi_j (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \leq \\ &\leq C(\lambda) \sup_{j \in Z} \max_{y \in \Delta_j} |R_0(y) \varphi_j|^2 \| (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2 \leq C(\lambda) \sup_{j \in Z} \max_{y \in \Delta_j} R_0^2(y) \| (l_{n,j} + \lambda I)^{-1} \|_{L_2(\Delta_j)}^2. \end{aligned}$$

Hence, taking the inequality (23) and the condition ii) into account, we find that

$$\|R_0(y)(l_n + \lambda I)^{-1}\|_{L_2(R) \rightarrow L_2(R)}^2 \leq C(\lambda) \sup_{|y-t| \leq 1} \frac{R_0^2(y)}{R_0^2(t)} < C(\lambda) \cdot \mu_0^2 \leq C_0^2 < \infty.$$

The last estimate proves the inequality (33) of Lemma 2.12.

Let us prove the inequality (34). Using Lemma 2.11, we have:

$$\begin{aligned} \|R_1(y)|n|(l_n + \lambda I)^{-1}\|_{L_2(R) \rightarrow L_2(R)}^2 &\leq C(\lambda) \sup_{j \in \mathbb{Z}} \|R_1(y)|n|\varphi_j(l_{n,j} + \lambda I)^{-1}\|_{L_2(\Delta_j) \rightarrow L_2(\Delta_j)}^2 \leq \\ &\leq C(\lambda) \sup_{j \in \mathbb{Z}} \max_{y \in \Delta_j} R_1^2(y)|n|^2 \|(l_{n,j} + \lambda I)^{-1}\|_{L_2(\Delta_j) \rightarrow L_2(\Delta_j)}^2. \end{aligned}$$

Hence, using (24), as well as considering condition ii), we find that

$$\|R_1(y)|n|(l_n + \lambda I)^{-1}\|_{L_2(R) \rightarrow L_2(R)}^2 \leq C(\lambda) \cdot \mu_1^2 \leq C_1^2 < \infty.$$

The inequality (34) is proved.

Let us prove the inequality (35). Reproducing the computations arguments used in the proof of the inequalities (33), (34) and using the inequality (25), we obtain

$$\begin{aligned} \|R_2(y)|n|^3(l_n + \lambda I)^{-1}\|_{L_2(R) \rightarrow L_2(R)}^2 &\leq C(\lambda) \sup_{j \in \mathbb{Z}} \|R_2(y)|n|^3\varphi_j(l_{n,j} + \lambda I)^{-1}\|_{L_2(\Delta_j) \rightarrow L_2(\Delta_j)}^2 \leq \\ &\leq C(\lambda) \cdot \mu_2^2 \leq C_2^2 < \infty. \end{aligned}$$

The inequality (35) is proved. Lemma 2.12 is completely proved.

5 Proof of Theorem 2

The representation (20) implies that

$$R_0(y)u(x, y) = R_0(y)(L + \lambda I)^{-1}f(x, y) = \sum_{n=-\infty}^{\infty} R_0(y)(l_n + \lambda I)^{-1}f_n(y) \cdot e^{inx}.$$

Calculate the norm $R_0(y)(L + \lambda I)^{-1}f$:

$$\begin{aligned} \|R_0(y)(L + \lambda I)^{-1}f\|_{L_2(\Omega)}^2 &= \left\| \sum_{n=-\infty}^{\infty} R_0(y)(l_n + \lambda I)^{-1}f_n(y) \cdot e^{inx} \right\|_{L_2(\Omega)}^2 \leq \\ &\leq 2\pi \sum_{n=-\infty}^{\infty} \|R_0(y)(l_n + \lambda I)^{-1}f_n(y) \cdot e^{inx}\|_{L_2(\Omega)}^2. \end{aligned}$$

From this and from Lemma 2.12 we have that

$$\|R_0(y)(L + \lambda I)^{-1}f\|_{L_2(\Omega)}^2 \leq 2\pi \sum_{n=-\infty}^{\infty} \|R_0(y)(l_n + \lambda I)^{-1}\|_{L_2(\Omega)}^2 \cdot \|f_n(y)\|_{L_2(\Omega)}^2 \leq C_0^2 \|f\|_{L_2(\Omega)}^2.$$

From the last estimate, it follows that

$$\|R_0(y)u(x, y)\|_{L_2(\Omega)} = \|R_0(y)(L + \lambda I)^{-1}f(x, y)\|_{L_2(\Omega)} \leq C_0 \|(L + \lambda I)^{-1}u\|_{L_2(\Omega)} \tag{36}$$

where $(L + \lambda I)u = f$.

Similarly, by virtue of the estimate (34) we have:

$$\left\| R_1(y) \frac{\partial u}{\partial x} \right\|_{L_2(\Omega)} \leq C_1 \|(L + \lambda I)^{-1}u\|_{L_2(\Omega)}. \tag{37}$$

Similarly, repeating the above calculations, we get that

$$\left\| R_1(y) \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(\Omega)}^2 \leq C_2 \|(L + \lambda I)^{-1} u\|_{L_2(\Omega)}. \quad (38)$$

Now the inequalities (36)-(38) imply that

$$\begin{aligned} \left\| \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} &= \left\| (L + \lambda I) - R_2(y) \frac{\partial^3 u}{\partial x^3} - R_1(y) \frac{\partial u}{\partial y} - R_0(y)u - \lambda u \right\|_{L_2(\Omega)} \leq \|(L + \lambda I)\|_{L_2(\Omega)} + \\ &+ \left\| R_2(y) \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(\Omega)} + \left\| R_1(y) \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} + \|R_0(y)u\|_{L_2(\Omega)} + \|\lambda u\|_{L_2(\Omega)} \leq \|(L + \lambda I)u\|_{L_2(\Omega)} + \\ &+ C_2 \|(L + \lambda I)u\|_{L_2(\Omega)} + C_1 \|(L + \lambda I)u\|_{L_2(\Omega)} + C_0 \|(L + \lambda I)u\|_{L_2(\Omega)} + \lambda \|(L + \lambda I)u\|_{L_2(\Omega)} \leq \\ &\leq C(\lambda) \|(L + \lambda I)u\|_{L_2(\Omega)}. \end{aligned} \quad (39)$$

We obtain from the inequalities (36)-(39) that

$$\left\| \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} + \left\| R_2(y) \frac{\partial^3 u}{\partial x^3} \right\|_{L_2(\Omega)} + \left\| R_1(y) \frac{\partial u}{\partial y} \right\|_{L_2(\Omega)} + \|R_0(y)u\|_{L_2(\Omega)} \leq C(\lambda)(\|Lu\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}).$$

Theorem 2 is completely proved.

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Кортевег-де Фриз типті сызықты сингулярлы операторлардың бір класының бар болуы және бөліктенуі туралы

Мақалада шексіз облыста коэффициенттері шексіздікте жылдам өсетін Кортевег-де Фриза типті сызықты оператор зерттелген. Жоғарыдағы көрсетілген дифференциалдық оператор үшін төмендегі сұрақтар қарастырылған:

- резольвентаның бар болуы;
- оператордың бөліктенуі, яғни оператордың анықталу облысындағы функциялардың тегістігі зерттелген.

Кілт сөздер: резольвента, Кортевег-де Фриз типті сингулярлы оператор, бөліктену.

М.В. Муратбеков, А.О. Сулеймбекова

О существовании резольвенты и делимости одного класса линейных сингулярных операторов типа Кортевега-де Фриза

В статье исследован линейный дифференциальный оператор типа Кортевега-де Фриза в случае неограниченной области с сильно растущими коэффициентами на бесконечности. Для указанного выше оператора изучены следующие вопросы:

- существование резольвенты;
- делимость оператора, т.е. гладкость функций из области определения изучаемого оператора.

Ключевые слова: резольвента, сингулярный оператор типа Кортевега-де Фриза, делимость.

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