

M.J. Mardanov¹, Y.A. Sharifov^{1,2}, K.E. Ismayilova³

¹*Institute of Mathematics and Mechanics, ANAS, Baku, Azerbaijan*

²*Baku State University, Baku, Azerbaijan*

³*Baku Engineering University, Khirdalan City, Azerbaijan*

(E-mail: *misirmardanov@yahoo.com, sharifov22@rambler.ru, keismayilova@beu.edu.az*)

Existence and uniqueness of solutions for the system of integro-differential equations with three-point and nonlinear integral boundary conditions

The paper examines a system of nonlinear integro-differential equations with three-point and nonlinear integral boundary conditions. The original problem demonstrated to be equivalent to integral equations by using Green function. Theorems on the existence and uniqueness of a solution to the boundary value problems for the first order nonlinear system of integro-differential equations with three-point and nonlinear integral boundary conditions are proved. A proof of uniqueness theorem of the solution is obtained by Banach fixed point principle, and the existence theorem then follows from Schaefer's theorem.

Keywords: three-point boundary conditions, nonlinear integral boundary value problems, existence and uniqueness of solutions, fixed point theorems.

Introduction

Multipoint boundary value problems for ordinary differential equations play a crucial role in various applications. It is epitomized the fact that, given a dynamical system with n degrees of freedom, there may exist exactly n states detected at n different times. A mathematical description of such a system results in an n -point boundary value problem. Another source of multipoint problems is the discretization of certain boundary value problems for partial differential equations over irregular domains with the method of lines. Multipoint problems for ordinary differential equations are a particular class of interface problems, and hence solvable with different techniques [1–4].

Integro-differential equations are encountered in many engineering and scientific disciplines, the problems can be represented as continuum phenomena and can be described approximately to partial differential equations. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aeroelastic phenomena, viscoelasticity, viscoelastic panel in supersonic gas flow, fluid dynamics, electrodynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors, and mathematical modeling of a hereditary phenomenon. For details, see [5–7] and the references therein. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermos-elasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [8]. For more details of nonlocal and integral boundary conditions, see [9–16] and references therein.

In the last few decades, the study of differential equations with nonlocal boundary conditions has been an interesting subject of mathematics, that has recently received the most significant attention of researchers; the reader is referred to [17–27]. It has been proposed by several authors that existence results for boundary value problems may be useful in real world problems. (see e.g., [28–30] and the references therein)

Problem statement and preliminaries

In this section, we set up problem statement and lemmas which are used throughout this paper. We denote by $C([0, T], R^n)$ the Banach space of all continuous functions from $[0, T]$ into R^n with the norm $\|x\| = \max \{|x(t)| : t \in [0, T]\}$, where $|\cdot|$ is the norm in space R^n .

We consider the existence, uniqueness of the system of nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x(t), (\chi x)(t)), \quad t \in [0, T], \tag{1}$$

subject to three-point and nonlinear integral boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = \int_0^T q(x(t)) dt, \tag{2}$$

where A, B, C are constant square matrices of order n such that $\det N \neq 0$, $N = A + B + C$; $f : [0, T] \times R^n \times R^n \rightarrow R^n$, $q : R^n \rightarrow R^n$, $g : [0, T] \times [0, T] \times R^n \rightarrow R^n$ are given functions, t_1 satisfies the condition $0 < t_1 < T$ and $(\chi x)(t) = \int_0^t g(t, s, x(s)) ds$.

For simplicity, the problem can be interpreted as solving the following problem:

Lemma 1. Suppose $\mu \in C([0, T], R^n)$ and $\det N \neq 0$. Then the unique solution of the following problem

$$\dot{x}(t) = \mu(t), \quad t \in [0, T] \tag{3}$$

with three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = \int_0^T \eta(s) ds, \tag{4}$$

is given by

$$x(t) = d + \int_0^T G(t, \tau)\mu(\tau)d\tau, \tag{5}$$

where

$$G(t, \tau) = \begin{cases} G_1(t, \tau), & t \in [0, t_1], \\ G_2(t, \tau), & t \in (t_1, T], \end{cases}$$

such that

$$G_1(t, \tau) = \begin{cases} N^{-1}A, & 0 \leq \tau \leq t, \\ -N^{-1}(B + C), & t < \tau \leq t_1, \\ -N^{-1}C, & t_1 < \tau \leq T, \end{cases}$$

and

$$G_2(t, \tau) = \begin{cases} N^{-1}A, & 0 \leq \tau \leq t_1, \\ N^{-1}(A + B), & t_1 < \tau \leq t, \\ -N^{-1}C, & t < \tau \leq T, \end{cases}$$

$$d = N^{-1} \int_0^T \eta(s) ds.$$

Proof. If function $x = x(\cdot)$ is a solution of the differential equation (1), then for $t \in (0, T)$,

$$x(t) = x_0 + \int_0^t \mu(\tau)d\tau, \tag{6}$$

where x_0 is an arbitrary constant vector. Now we define x_0 so that, the function in equality (6) satisfies the condition (4)

$$x_0 = d - N^{-1}B \int_0^{t_1} \mu(t)dt - N^{-1}C \int_0^T \mu(t)dt. \quad (7)$$

Now we put the value x_0 determined from the equality (7) in (6) and obtain

$$x(t) = d - N^{-1}B \int_0^{t_1} \mu(t)dt - N^{-1}C \int_0^T \mu(t)dt + \int_0^t \mu(\tau)d\tau. \quad (8)$$

Assume that, $t \in [0, t_1]$. Then we can write the equality (8) as follows:

$$\begin{aligned} x(t) = d - N^{-1}B \left(\int_0^t \mu(\tau)d\tau + \int_t^{t_1} \mu(\tau)d\tau \right) - N^{-1}C \left(\int_0^t \mu(\tau)d\tau + \int_t^{t_1} \mu(\tau)d\tau \right) \\ - N^{-1}C \int_{t_1}^T \mu(t)dt + \int_0^t \mu(\tau)d\tau. \end{aligned} \quad (9)$$

We get (9) combining similar terms, and using the common technique for simplifying:

$$\begin{aligned} x(t) = d + (E - N^{-1}B - N^{-1}C) \int_0^t \mu(\tau)d\tau - (N^{-1}B + N^{-1}C) \int_t^{t_1} \mu(\tau)d\tau \\ - N^{-1}C \int_{t_1}^T \mu(t)dt = d + N^{-1}A \int_0^t \mu(\tau) d\tau \\ - N^{-1}(B + C) \int_t^{t_1} \mu(\tau)d\tau - N^{-1}C \int_{t_1}^T \mu(t)dt, \end{aligned} \quad (10)$$

where E is an identity matrix.

Define new function as follows:

$$G_1(t, \tau) = \begin{cases} N^{-1}A, & 0 \leq \tau \leq t, \\ -N^{-1}(B + C), & t < \tau \leq t_1, \\ -N^{-1}C, & t_1 < \tau \leq T. \end{cases}$$

Equality (10) can be rewritten as integral equation below:

$$x(t) = d + \int_0^T G_1(t, \tau)\mu(\tau)d\tau.$$

Now assume that, $t \in (t_1, T]$. Then we can write the equality (8) as follows:

$$x(t) = d - N^{-1}B \int_0^{t_1} \mu(t)dt - N^{-1}C \int_0^{t_1} \mu(t)dt - N^{-1}C \left(\int_{t_1}^t \mu(\tau)d\tau + \int_t^T \mu(\tau)d\tau \right)$$

$$\begin{aligned}
 + \int_0^{t_1} \mu(t) dt + \int_{t_1}^t \mu(\tau) d\tau &= d + (E - N^{-1}B - N^{-1}C) \int_0^{t_1} \mu(t) dt + (E - N^{-1}C) \int_{t_1}^t \mu(\tau) d\tau \\
 - N^{-1}C \int_t^T \mu(\tau) d\tau &= d + N^{-1}A \int_0^{t_1} \mu(t) dt + N^{-1}(A + B) \int_{t_1}^t \mu(\tau) d\tau - N^{-1}C \int_t^T \mu(\tau) d\tau.
 \end{aligned}$$

We establish a new function as follows:

$$G_2(t, \tau) = \begin{cases} N^{-1}A, & 0 \leq \tau \leq t_1, \\ N^{-1}(A + B), & t_1 < \tau \leq t, \\ -N^{-1}C, & t < \tau \leq T. \end{cases}$$

Hence, if $t \in (t_1, T]$, then we can write the equality (8) as follows:

$$x(t) = d + \int_0^T G_2(t, \tau) \mu(\tau) d\tau.$$

Thus, the solution of the boundary value problem (3)-(4) can be shown as follows:

$$x(t) = d + \int_0^T G(t, \tau) \mu(\tau) d\tau.$$

We showed that the argument given above is valid (5). Proof is completed.

Lemma 2. Assume that $f : [0, T] \times R^n \times R^n \rightarrow R^n$, $q : R^n \rightarrow R^n$ are given functions. Then the function $x(t)$ is a solution of the boundary value problem (1)-(2) if and only if $x(t)$ is a solution of the integral equation

$$x(t) = D + \int_0^T G(t, \tau) f(\tau, x(\tau), (\chi x)(\tau)) d\tau, \quad (11)$$

where

$$D = N^{-1} \int_0^T q(x(t)) dt.$$

Proof. Let $x(t)$ be a solution of the boundary value problem (1)-(2). Proving statements similar to Lemmas 1 this lemma can be derived. By checking directly we identify the solution of integral equation (11) satisfies the boundary value problem (1)-(2). Lemma 2 is proved.

Existence results

Let P be an operator such that, $P : C([0, T], R^n) \rightarrow C([0, T], R^n)$ as

$$(Px)(t) = N^{-1} \int_0^T q(x(t)) dt + \int_0^T G(t, \tau) f(\tau, x(\tau), (\chi x)(\tau)) d\tau.$$

It is evident that, the problem (1)-(2) is equivalent to the fixed point problem $x = Px$. Thus, the problem (1)-(2) has a solution if and only if the operator P has a fixed point.

In Lemma 1, we use the most basic fixed point theorem named the contraction mapping principle and it uses the assumptions:

H1) There exist constants M_1, M_2 such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq M_1 |x_1 - y_1| + M_2 |x_2 - y_2|$$

for each $t \in [0, T]$ and all $x_1, x_2, y_1, y_2 \in R^n$.

H2) There exists a constant M_3 such that

$$|g(t, s, x) - g(t, s, y)| \leq M_3 |x - y|$$

for each $t, s \in [0, T]$ and all $x, y \in R^n$.

H3) There exists a constant M_4 such that

$$|q(x) - q(y)| \leq M_4 |x - y|$$

for all $x, y \in R^n$.

Theorem 1. Assume that, the assumptions H1)-H3) hold, and

$$L = \left[S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \|N^{-1}\| \right] < 1, \quad (12)$$

then the boundary-value problem (1)-(2) has a unique solution on $[0, T]$, where

$$S = \max_{[0, T] \times [0, T]} \|G(t, \tau)\|.$$

Proof. Setting $\max_{[0, T]} |f(t, 0, 0)| = M_f$, $\max_{[0, T]} |q(0)| = m_q$ and choosing $r \geq \frac{M_f T S + m_q T \|N^{-1}\|}{1-L}$ we show that $PB_r \subset B_r$ where

$$B_r = \{x \in C([0, T]R^n) : \|x\| \leq r\}.$$

For $x \in B_r$, we have

$$\begin{aligned} \|(Px)(t)\| &\leq \left\| N^{-1} \int_0^T (|q(x(t)) - q(0)| + |q(0)|) dt \right\| \\ &+ \int_0^T |G(t, \tau)| (|f(\tau, x(\tau), (\chi x)(\tau)) - f(\tau, 0, 0)| + |f(\tau, 0, 0)|) d\tau \\ &\leq M_4 T \|N^{-1}\| \|x\| + m_q T \|N^{-1}\| + S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) \|x\| + S T M_f \\ &\leq \left[M_4 T \|N^{-1}\| + S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) \right] r + S T M_f + m_q T \|N^{-1}\| \leq r. \end{aligned}$$

Now for any $x, y \in B_r$ we have

$$\begin{aligned} |(Px)(t) - (Py)(t)| &\leq \int_0^T \left| G(t, \tau) \left(f(\tau, x(\tau), \int_0^t g(t, s, x(s)) ds) - f(\tau, y(\tau), \int_0^t g(t, s, y(s)) ds) \right) \right| d\tau \\ &+ \left| N^{-1} \int_0^T (q(x(t)) - q(y(t))) dt \right| \\ &\leq S \int_0^T \left\{ M_1 |x(t) - y(t)| + M_2 \left| \int_0^t g(t, s, x(s)) ds - \int_0^t g(t, s, y(s)) ds \right| \right\} dt \\ &+ M_4 \|N^{-1}\| \int_0^T |x(t) - y(t)| dt \leq \left[S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \|N^{-1}\| \right] \|x - y\|, \end{aligned}$$

or

$$\|Px - Py\| \leq L \|x - y\|.$$

It is seen that, P is contraction by condition (12). So, the boundary-value problem (1)-(2) has a unique solution.

Theorem 2 (Schafer's fixed point theorem). Let X be a Banach space. Assume that, $G : X \rightarrow X$ is a completely continuous operator and the set $\rho = \{x \in X | x = \beta Gx, 0 < \beta < 1\}$ is bounded. Then G has a fixed point in X .

Now we apply Schafer's fixed point theorem and it uses the following assumption:

Theorem 3. Assume that the functions $f : [0, T] \times R^n \times R^n \rightarrow R^n$ and $q : R^n \rightarrow R^n$ are continuous and there exist functions $\rho, \lambda \in C([0, T], R^+)$ such that $|f(t, x(t), (\chi x)(t))| \leq \rho(t)$, $|q(x(t))| \leq \lambda(t)$, $\forall t \in [0, T]$, $x \in C([0, T], R^n)$ and with $\sup_{t \in [0, T]} |\rho(t)| = \|\rho\|$, $\sup_{t \in [0, T]} |\lambda(t)| = \|\lambda\|$. Then the boundary value

problem (1)-(2) has at least one solution on $[0, T]$.

Proof. Let P be the operator defined in (12). We use Schaefer's fixed point theorem to prove that P has a fixed point. The proof will be given in several steps.

Step 1: Here we prove that P is continuous. Let $\{x_n\}$ be a sequence such that $x_n \rightarrow x$ in $C([0, T]; R^n)$. Then, for each $t \in [0, T]$

$$\begin{aligned} & |(Px)(t) - (Px_n)(t)| \\ &= \left| N^{-1} \int_0^T (q(x(t)) - q(x_n(t))) dt + \int_0^T G(t, \tau) (f(\tau, x(\tau), (\chi x)(\tau)) - f(\tau, x_n(\tau), (\chi x_n)(\tau))) d\tau \right| \\ &\leq \left[S \left(M_1 T + \frac{M_2 M_3 T^2}{2} \right) + M_4 T \|N^{-1}\| \right] \|x(t) - x_n(t)\| \leq L \|x - x_n\|. \end{aligned}$$

From here we get $\|(Px)(t) - (Px_n)(t)\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that the operator P is continuous.

Step 2: P maps bounded sets into bounded sets in $C([0, T]; R^n)$. Indeed, it is enough to show that for any $\eta > 0$ there exists a positive constant ω such that for each $x \in B_\eta = \{x \in C([0, T]; R^n) : \|x\| \leq \eta\}$ we have $\|P(x)\| \leq \omega$. We have for each $t \in [0, T]$

$$|(Px)(t)| \leq T \|N^{-1}\| \|\lambda\| + TS \|\rho\|.$$

This implies that

$$\|(Px)(t)\| \leq T \|N^{-1}\| \|\lambda\| + TS \|\rho\|.$$

Step 3. The operator P maps bounded sets into equicontinuous sets of $C([0, T], R^n)$. Let $\tau_1, \tau_2 \in [0, T]$, $\tau_1 < \tau_2$, B_η be a bounded set of $C([0, T]; R^n)$ as in Step 2, and let $x \in B_\eta$.

Case 1. Let be $\tau_1, \tau_2 \in [0, t_1]$. Then,

$$\begin{aligned} (Px)(\tau_2) - (Px)(\tau_1) &= N^{-1} A \int_0^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau)) d\tau \\ &\quad - N^{-1} (B + C) \int_{\tau_2}^{t_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau - N^{-1} A \int_0^{\tau_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau \\ &\quad + N^{-1} (B + C) \int_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau = \int_{\tau_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau)) d\tau. \end{aligned}$$

Case 2: Let be $\tau_1 \in [0, t_1]$ and $\tau_2 \in (t_1, T]$. Then

$$\begin{aligned} (Px)(\tau_2) - (Px)(\tau_1) &= N^{-1}A \int_0^{\tau_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau \\ &+ N^{-1}(A+B) \int_{t_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau)) d\tau - N^{-1}C \int_{\tau_2}^T f(\tau, x(\tau), (\chi x)(\tau)) d\tau \\ &- N^{-1}A \int_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau + N^{-1}(B+C) \int_{\tau_1}^{t_1} f(\tau, x(\tau), (\chi x)(\tau)) d\tau \\ &+ N^{-1}C \int_{t_1}^T f(\tau, x(\tau), (\chi x)(\tau)) d\tau = \int_{\tau_1}^{\tau_2} f(\tau, x(\tau), (\chi x)(\tau)) d\tau. \end{aligned}$$

Apparently, in both cases

$$|(Px)(\tau_2) - (Px)(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |f(\tau, x(\tau), (\chi x)(\tau))| d\tau.$$

As $\tau_2 \rightarrow \tau_1$, the right hand side of the preceding inequality tends to zero. Taking into account that the mapping P is continuous and equivalently continuous, we conclude that the mapping $P : C([0, T], R^n) \rightarrow C([0, T], R^n)$ is completely continuous by Arzela-Ascoli theorem.

Step 4. We show that a set $\Omega = \{x \in C([0, T], R^n) : x = \lambda P(x), \text{ for some } 0 < \lambda < 1\}$ is bounded. Assume that, $x = \lambda P(x)$ for some $0 < \lambda < 1$. Then for each $t \in [0, T]$, we can write

$$x(t) = \lambda N^{-1} \int_0^T q(x(t)) dt + \lambda \int_0^T G(t, \tau) f(\tau, x(\tau), (\chi x)(\tau)) d\tau.$$

From here we get

$$\|x\| \leq T \|N^{-1}\| \|\lambda\| + TS \|\rho\|.$$

Therefore, the set Ω is bounded. The conclusion of Theorem 2 applies and the operator P has at least one fixed point. So, there exists at least one solution for the problem (1)-(2) on $[0, T]$.

Example

Consider the following system of integro-differential equation

$$\begin{cases} \dot{x}_1 = \sin \alpha x_2, \\ \dot{x}_2 = \cos \left(\beta \int_0^t \frac{\sin(\gamma x_1)}{1+t^2} dt \right), \end{cases} \quad (A)$$

subject to

$$\begin{cases} x_1(0) + x_2(0) - x_2(\frac{1}{2}) = 1, \\ -x_1(\frac{1}{2}) + x_1(1) + x_2(1) = \int_0^1 \cos \delta x_2(t) dt. \end{cases} \quad (B)$$

Evidently,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A + B + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $t \in [0, \frac{1}{2}]$, we obtain

$$G_1(t, \tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \leq \tau \leq t, \\ \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, & t < \tau \leq \frac{1}{2}, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & \frac{1}{2} < \tau \leq 1, \end{cases}$$

and for $t \in (\frac{1}{2}, 1]$

$$G_2(t, \tau) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 0 \leq \tau \leq \frac{1}{2}, \\ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, & \frac{1}{2} < \tau \leq t, \\ \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}, & t < \tau \leq 1. \end{cases}$$

Obviously, $M_1 = |\alpha|$, $M_2 = |\beta|$, $M_3 = |\gamma|$, $M_4 = |\delta|$ and $\|S\| \leq 2$. If $L = 2 \left(|\alpha| + \frac{|\beta||\gamma|}{2} \right) + |\delta| < 1$, then boundary value problem (A)-(B) has a unique solution.

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М.Дж. Марданов, Я.А. Шарифов, К.Е. Исмаилова

Үш нүктелі және интегралдық шеттік шарттарымен берілген сызықты емес интегралды-дифференциалдық теңдеулер жүйесінің шешімінің бар болуы және жалғыздығы

Мақалада үш нүктелі және интегралдық шекаралық шарттарымен берілген сызықты емес интегралды-дифференциалдық теңдеулер жүйесі зерттелген. Бастапқыда, Грин функциясы арқылы эквивалентті интегралдық теңдеуге алып келді. Кейіннен, қозғалмайтын нүктелер туралы теореманы қолдана отырып, шеттік есептің шешімінің бар болуы және жалғыздығының жеткілікті шарты алынды. Шешімінің жалғыздығы теоремасының дәлелдемесі қозғалмайтын нүкте туралы Банах принципі бойынша алынды, содан кейін бар болуы теоремасы Шефер теоремасынан шығады.

Кілт сөздер: үш нүктелі шекаралық шарттар, сызықты емес интегралдық шеттік есептер, шешімінің бар болуы және жалғыздығы, қозғалмайтын нүкте туралы теорема.

М.Дж. Марданов, Я.А. Шарифов, К.Е. Исмаилова

Существование и единственность решений для систем интегро-дифференциальных уравнений с трёхточечными и нелинейными интегральными краевыми условиями

В статье исследована система нелинейных интегро-дифференциальных уравнений с трёхточечными и интегральными граничными условиями. Сначала с помощью функции Грина она приведена к эквивалентному интегральному уравнению. Далее, с использованием теоремы о неподвижных точках, получены достаточные условия существования и единственности решения краевой задачи. Доказательство теоремы единственности решения получено по принципу Банаха о неподвижной точке, а затем теорема существования следует из теоремы Шефера.

Ключевые слова: трёхточечные граничные условия, нелинейные интегральные краевые задачи, существование и единственность решений, теоремы о неподвижной точке.

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