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A comparison between the fourth order linear differential equation with its boundary value problem

In this paper, we study a fourth order linear differential equation. We found an upper bound for the solutions of this differential equation and also, we prove that all the solutions are in $L^4(0,\infty)$. By comparing these results we obtain that all the eigenfunction of the boundary value problem generated by this differential equation are bounded and in $L^4(0,\infty)$.

Keywords: linear differential equation, eigenvalue, eigenfunction, upper bound, linearly independent solution, $L^2(0,\infty)$, wrongskian, Gronwall inequality, Variation of parameters.

Introduction

The method of finding an upper bound for the solutions of a differential equation has been investigated by many authors. In papers [2, 4] by authors were investigated the solutions of the second order linear differential equation. They obtained some important properties of this equation such that all solutions of the differential equation are bounded and in the space $L^2(0,\infty)$. Here $L^2(0,\infty)$ is the space of all functions f which are continuous and satisfy the conditions:

$$\int_0^\infty |f(x)|^2 dx < \infty.$$

The estimate of upper bounds for the eigenfunctions of a boundary value problem was investigated by many authors. In papers [2–6, 10] by authors were investigated a second order differential equation of the form

$$y'' + q(x)\,y = \lambda^2 \rho(x)\,y, x \in [0,a].$$

They found a normalized eigenfunctions for this problem and an upper bound for this solution under a certain condions.
Methods of finding of general solution of a fourth order differential equation were studied by many authors, see: [1, 7–9, 11].

This paper is specified to study some important properties of solutions of a fourth order linear differential equation of the form:

$$y^{(4)}(x) + \{q(x) + r(x)\}\,y(x) = 0, \qquad 0 \le x < \infty, \tag{1}$$

where $r(x)$ is a function satisfying the condition:

$$\int_0^\infty |r(x)|\,dx < \infty. \tag{2}$$

We investigate whether the solutions of (1) are related to any general properties such as boundedness of the solutions of the differential equation

$$y^{(4)}(x) + q(x)\,y(x) = 0, \qquad 0 \le x < \infty. \tag{3}$$

Let $L^4(0,\infty)$ is the space of all continuous functions f for which satisfy the condition

$$\int_0^\infty |f(x)|^4 dx < \infty.$$

In this paper we show that all solutions of (1) are in $L^4(0,\infty)$. It is based on the fact that the solutions of (3) are in $L^4(0,\infty)$ under the condition (2). Moreover, we show that eigenfunctions of the boundary value problem which is generated by the differential equation $y^{(4)}(x)+\{\lambda+r(x)\}y(x)=0$ are bounded under a certain condion.

Let $f(x)$ and $g(x)$ be real-valued, continuous, and nonnegative in $[a,b]$ and suppose that $f(x)\le c+\int_a^x f(t)g(t)\,dt$, in $[a,b]$ where $c>0$ is a constant. Then,

$$f(x)\le c\, e^{\int_a^x g(t)dt}. \tag{4}$$

This is known as Gronwall inequality [2].

Expression for the solutions

In this section we found a general solutions for (1) by using the method of variation of parameter. We need some properties of the differential equations (1) and (3) which are immediate consequence of the results of chapter two in [2].

Lemma 1. There are solutions $\phi_j(x)$, $\{j=1,2,3,4\}$ of (3) such that $W(\phi_1,\phi_3,\phi_2,\phi_4)=1$ in $[0,\infty)$.

Proof. Let $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$be a fundamental system of solution of (3), then we obtain $W(\phi_1,\phi_3,\phi_2,\phi_4)=c\ in\ [0,\infty)$, where c is a non zero constant , we take $\phi_1(x)=y_1(x)$, $\phi_2(x)=y_2(x)$, $\phi_3(x)=y_3(x)$ and $\phi_4(x)=\frac{y_4(x)}{c}$, then we can easily establish that $W(\phi_1,\phi_3,\phi_2,\phi_4)=1$.

Lemma 2. If $\phi_j(x)$, $\{j=1,2,3,4\}$ are as in Lemma 1 and $\psi(x)$ is any solution of (1), then there are unique constants c_j for $j=1:4$ such that

$$\psi(x)=c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\psi_0(x), \tag{5}$$

where

$$\begin{aligned}\psi_0(x) \;=\; & \int_0^x [\phi_2(t)\phi_3'(t)\phi_4''(t)\phi_1(x)+\phi_2''(t)\phi_3(t)\phi_4'(t)\phi_1(x)+\phi_2'(t)\phi_3''(t)\phi_4(t)\phi_1(x)\\
& -\phi_2'(t)\phi_3(t)\phi_4''(t)\phi_1(x)-\phi_2(t)\phi_3''(t)\phi_4'(t)\phi_1(x)-\phi_2''(t)\phi_3'(t)\phi_4(t)\phi_1(x)\\
& -\phi_1(t)\phi_3'(t)\phi_4''(t)\phi_2(x)-\phi_1''(t)\phi_3(t)\phi_4'(t)\phi_2(x)-\phi_1'(t)\phi_3''(t)\phi_4(t)\phi_2(x)\\
& +\phi_1'(t)\phi_3(t)\phi_4''(t)\phi_2(x)+\phi_1(t)\phi_3''(t)\phi_4'(t)\phi_2(x)+\phi_1''(t)\phi_3'(t)\phi_4(t)\phi_2(x)\\
& +\phi_1(t)\phi_2'(t)\phi_4''(t)\phi_3(x)+\phi_1''(t)\phi_2(t)\phi_4'(t)\phi_3(x)+\phi_1'(t)\phi_2''(t)\phi_4(t)\phi_3(x)\\
& -\phi_1'(t)\phi_2(t)\phi_4''(t)\phi_3(x)-\phi_1(t)\phi_2''(t)\phi_4'(t)\phi_3(x)-\phi_1''(t)\phi_2'(t)\phi_4(t)\phi_3(x)\\
& -\phi_1(t)\phi_2'(t)\phi_3''(t)\phi_4(x)-\phi_1''(t)\phi_2(t)\phi_3'(t)\phi_4(x)-\phi_1'(t)\phi_2''(t)\phi_3(t)\phi_4(x)\\
& +\phi_1'(t)\phi_2(t)\phi_3''(t)\phi_4(x)+\phi_1(t)\phi_2''(t)\phi_3'(t)\phi_4(x)+\phi_1''(t)\phi_2'(t)\phi_3(t)\phi_4(x)]\\
& \times r(t)\psi(t)\,dt.\end{aligned}$$

Proof. If $\psi(x)$ is a solution of (1), then as we see in [2] by using variation of parameter there is unique constants $c_j$such that

$$\psi(x)=c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\psi_0(x), \tag{6}$$

where

$$\psi_0(x)=c_1(x)\phi_1(x)+c_2(x)\phi_2(x)+c_3(x)\phi_3(x)+c_4(x)\phi_4(x) \tag{7}$$

and

$$c_r(x) = \int_0^x \frac{W_r(\phi_1, \phi_3, \phi_2, \phi_4)(t)}{W(\phi_1, \phi_3, \phi_2, \phi_4)(t)} r(t)\psi(t)\,dt. \tag{8}$$

From Lemma 1 it follows that $W(\phi_1, \phi_3, \phi_2, \phi_4) = 1$.Therefore, (8) has the form

$$c_r(x) = \int_0^x W_r(\phi_1, \phi_3, \phi_2, \phi_4)(t) r(t)\psi(t)\,dt. \tag{9}$$

For $r = 1$, we have that

$$W_1(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \begin{vmatrix} 0 & \phi_2(t) & \phi_3(t) & \phi_4(t) \\ 0 & \phi_2'(t) & \phi_3'(t) & \phi_4'(t) \\ 0 & \phi_2''(t) & \phi_3''(t) & \phi_4''(t) \\ 1 & \phi_2'''(t) & \phi_3'''(t) & \phi_4'''(t) \end{vmatrix}$$

$$\begin{aligned} = \; & \phi_2(t)\phi_3'(t)\phi_4''(t) + \phi_2''(t)\phi_3(t)\phi_4'(t) + \phi_2'(t)\phi_3''(t)\phi_4(t) - \phi_2'(t)\phi_3(t)\phi_4''(t) \\ & -\phi_2(t)\phi_3''(t)\phi_4'(t) - \phi_2''(t)\phi_3'(t)\phi_4(t). \end{aligned}$$

That is

$$\begin{aligned} W_1(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \; & \phi_2(t)\phi_3'(t)\phi_4''(t) + \phi_2''(t)\phi_3(t)\phi_4'(t) + \phi_2'(t)\phi_3''(t)\phi_4(t) \\ & -\phi_2'(t)\phi_3(t)\phi_4''(t) - \phi_2(t)\phi_3''(t)\phi_4'(t) - \phi_2''(t)\phi_3'(t)\phi_4(t). \end{aligned}$$

For $r = 2, 3, 4$,applying the same way, we obtain

$$\begin{aligned} W_2(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \; & -\phi_1(t)\phi_3'(t)\phi_4''(t) - \phi_1''(t)\phi_3(t)\phi_4'(t) - \phi_1'(t)\phi_3''(t)\phi_4(t) \\ & +\phi_1'(t)\phi_3(t)\phi_4''(t) + \phi_1(t)\phi_3''(t)\phi_4'(t) + \phi_1''(t)\phi_3'(t)\phi_4(t), \end{aligned}$$

$$\begin{aligned} W_3(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \; & \phi_1(t)\phi_2'(t)\phi_4''(t) + \phi_1''(t)\phi_2(t)\phi_4'(t) + \phi_1'(t)\phi_2''(t)\phi_4(t) \\ & -\phi_1'(t)\phi_2(t)\phi_4''(t) - \phi_1(t)\phi_2''(t)\phi_4'(t) - \phi_1''(t)\phi_2'(t)\phi_4(t), \end{aligned}$$

$$\begin{aligned} W_4(\phi_1, \phi_2, \phi_3, \phi_4)(t) = \; & -\phi_1(t)\phi_2'(t)\phi_3''(t) - \phi_1''(t)\phi_2(t)\phi_3'(t) - \phi_1'(t)\phi_2''(t)\phi_3(t) \\ & +\phi_1'(t)\phi_2(t)\phi_3''(t) + \phi_1(t)\phi_2''(t)\phi_3'(t) + \phi_1''(t)\phi_2'(t)\phi_3(t). \end{aligned}$$

Substituting these values of W_r in (9) and then (9) in (7), we get the result.

Bounded solution

In this section we obtain that all solutions of (1) are bounded. It is based on boundednees of solutions of (3) and condition (2).

Theorem 1. Let that all solutions and their derivatives up to order three of (3) be bounded in $[0, \infty)$ and the condition (2) is hold, then all the solutions of (1) are bounded in $[0, \infty)$.

Proof. Let $\phi_1(x), \phi_2(x), \phi_3(x)$ and $\phi_4(x)$ be four linearly independent solutions of (3) such that $W(\phi_1, \phi_2, \phi_3, \phi_4) = 1$ and let $\psi(x)$ be any solution of (1), then by Lemma 2 there are constants c_1, c_2, c_3 and c_4 such that

$$\begin{aligned}
\psi(x) &= c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\int_0^x[\phi_2(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_1(x) \\
&+\phi_2^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_1(x)+\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_1(x)-\phi_2^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_1(x) \\
&-\phi_2(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_1(x)-\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_1(x)-\phi_1(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_2(x) \\
&-\phi_1^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_2(x)-\phi_1^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_2(x)+\phi_1^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_2(x) \\
&+\phi_1(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_2(x)+\phi_1^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_2(x)+\phi_1(t)\phi_2^{'}(t)\phi_4^{''}(t)\phi_3(x) \\
&+\phi_1^{''}(t)\phi_2(t)\phi_4^{'}(t)\phi_3(x)+\phi_1^{'}(t)\phi_2^{''}(t)\phi_4(t)\phi_3(x)-\phi_1^{'}(t)\phi_2(t)\phi_4^{''}(t)\phi_3(x) \\
&-\phi_1(t)\phi_2^{''}(t)\phi_4^{'}(t)\phi_3(x)-\phi_1^{''}(t)\phi_2^{'}(t)\phi_4(t)\phi_3(x)-\phi_1(t)\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(x) \\
&-\phi_1^{''}(t)\phi_2(t)\phi_3^{'}(t)\phi_4(x)-\phi_1^{'}(t)\phi_2^{''}(t)\phi_3(t)\phi_4(x)+\phi_1^{'}(t)\phi_2(t)\phi_3^{''}(t)\phi_4(x) \\
&+\phi_1(t)\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(x)+\phi_1^{''}(t)\phi_2^{'}(t)\phi_3(t)\phi_4(x)]r(t)\psi(t)\,dt, \qquad (10)
\end{aligned}$$

By our hypothesis, there are constants k_0, k_1, k_2 such that
$|\phi_j(x)| \le k_0$, $\left|\phi_j^{'}(x)\right| \le k_1$, $\left|\phi_j^{''}(x)\right| \le k_2$ in $[0,\infty]$.
Hence from 10 it follows that

$$|\psi(x)| \le (|c_1|+|c_2|+|c_3|+|c_4|)\,k_0 + 18k_0^2k_1k_2\int_0^x |r(t)|\,|\psi(t)|\,dt.$$

Then, using Gronwall's Inequality, we obtain

$$|\psi(x)| \le (|c_1|+|c_2|+|c_3|+|c_4|)\,k_0 e^{18k_0^2k_1k_2\int_0^x |r(t)|dt}.$$

Since by our hypothesis $\int_0^x |r(t)|\,dt$ is bounded in $[0,\infty)$, then $\psi(x)$ is bounded in $[0,\infty)$. which it completed the proof.

$L^4(0,\infty)$ property of the solution

In this section we obtain that all solutions of (1) are $L^4(0,\infty)$ when the solutions of (3) are in $L^4(0,\infty)$ and $r(x)$ satisfy the condition (2).

Theorem 2. Suppose that all solutions and their derivatives up to order three of (3) be in $L^4(0,\infty)$ and $r(x)$ is bounded in $[0,\infty)$. Then all the solutions of (1) are in $L^4(0,\infty)$.

Proof. Let $\phi_1(x), \phi_2(x), \phi_3(x)$ and $\phi_4(x)$ be four linearly independent solutions of (3) such that $W(\phi_1,\phi_2,\phi_3,\phi_4) = 1$ and let $\psi(x)$ be any solution of (1), then by Lemma 2 there are constants c_1, c_2, c_3 and c_4 such that

$$\begin{aligned}
\psi(x) &= c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+c_4\phi_4(x)+\int_0^x[\phi_2(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_1(x) \\
&+\phi_2^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_1(x)+\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_1(x)-\phi_2^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_1(x) \\
&-\phi_2(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_1(x)-\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_1(x)-\phi_1(t)\phi_3^{'}(t)\phi_4^{''}(t)\phi_2(x) \\
&-\phi_1^{''}(t)\phi_3(t)\phi_4^{'}(t)\phi_2(x)-\phi_1^{'}(t)\phi_3^{''}(t)\phi_4(t)\phi_2(x)+\phi_1^{'}(t)\phi_3(t)\phi_4^{''}(t)\phi_2(x) \\
&+\phi_1(t)\phi_3^{''}(t)\phi_4^{'}(t)\phi_2(x)+\phi_1^{''}(t)\phi_3^{'}(t)\phi_4(t)\phi_2(x)+\phi_1(t)\phi_2^{'}(t)\phi_4^{''}(t)\phi_3(x) \\
&+\phi_1^{''}(t)\phi_2(t)\phi_4^{'}(t)\phi_3(x)+\phi_1^{'}(t)\phi_2^{''}(t)\phi_4(t)\phi_3(x)-\phi_1^{'}(t)\phi_2(t)\phi_4^{''}(t)\phi_3(x) \\
&-\phi_1(t)\phi_2^{''}(t)\phi_4^{'}(t)\phi_3(x)-\phi_1^{''}(t)\phi_2^{'}(t)\phi_4(t)\phi_3(x)-\phi_1(t)\phi_2^{'}(t)\phi_3^{''}(t)\phi_4(x) \\
&-\phi_1^{''}(t)\phi_2(t)\phi_3^{'}(t)\phi_4(x)-\phi_1^{'}(t)\phi_2^{''}(t)\phi_3(t)\phi_4(x)+\phi_1^{'}(t)\phi_2(t)\phi_3^{''}(t)\phi_4(x) \\
&+\phi_1(t)\phi_2^{''}(t)\phi_3^{'}(t)\phi_4(x)+\phi_1^{''}(t)\phi_2^{'}(t)\phi_3(t)\phi_4(x)]r(t)\psi(t)\,dt. \qquad (11)
\end{aligned}$$

Then by hypothesis there are constants C, k_0, k_1, k_2 such that $|r(x)| \leq C$ in $[0,\infty)$, and $\int_0^\infty |\phi_j(x)|^4 dx \ \leq k_0,\ \int_0^\infty \left|\phi_j'(x)\right|^4 dx \ \leq k_1,\ \int_0^\infty \left|\phi_j''(x)\right|^4 dx \ \leq k_2$ for $j = 1,2,3,4$.

Now, applying the Holder's inequality for integral, we get

$$
\begin{aligned}
&\Big|\int_0^x [\phi_2(t)\phi_3'(t)\phi_4''(t)\phi_1(x)+\phi_2''(t)\phi_3(t)\phi_4'(t)\phi_1(x)+\phi_2'(t)\phi_3''(t)\phi_4(t)\phi_1(x)\\
&-\phi_2'(t)\phi_3(t)\phi_4''(t)\phi_1(x)-\phi_2(t)\phi_3''(t)\phi_4'(t)\phi_1(x)-\phi_2''(t)\phi_3'(t)\phi_4(t)\phi_1(x)\\
&-\phi_1(t)\phi_3'(t)\phi_4''(t)\phi_2(x)-\phi_1''(t)\phi_3(t)\phi_4'(t)\phi_2(x)-\phi_1'(t)\phi_3''(t)\phi_4(t)\phi_2(x)\\
&+\phi_1'(t)\phi_3(t)\phi_4''(t)\phi_2(x)+\phi_1(t)\phi_3''(t)\phi_4'(t)\phi_2(x)+\phi_1''(t)\phi_3'(t)\phi_4(t)\phi_2(x)\\
&+\phi_1(t)\phi_2'(t)\phi_4''(t)\phi_3(x)+\phi_1''(t)\phi_2(t)\phi_4'(t)\phi_3(x)+\phi_1'(t)\phi_2''(t)\phi_4(t)\phi_3(x)\\
&-\phi_1'(t)\phi_2(t)\phi_4''(t)\phi_3(x)-\phi_1(t)\phi_2''(t)\phi_4'(t)\phi_3(x)-\phi_1''(t)\phi_2'(t)\phi_4(t)\phi_3(x)\\
&-\phi_1(t)\phi_2'(t)\phi_3''(t)\phi_4(x)-\phi_1''(t)\phi_2(t)\phi_3'(t)\phi_4(x)-\phi_1'(t)\phi_2''(t)\phi_3(t)\phi_4(x)\\
&+\phi_1'(t)\phi_2(t)\phi_3''(t)\phi_4(x)+\phi_1(t)\phi_2''(t)\phi_3'(t)\phi_4(x)+\phi_1''(t)\phi_2'(t)\phi_3(t)\phi_4(x)]\\
&\times r(t)\psi(t)\,dt\Big|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 6C|\phi_1(x)|(k_0k_1k_2)^{\frac{1}{4}}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac{1}{4}} + 6C|\phi_2(x)|(k_0k_1k_2)^{\frac{1}{4}}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac{1}{4}}\\
&+6C|\phi_3(x)|(k_0k_1k_2)^{\frac{1}{4}}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac{1}{4}} + 6C|\phi_4(x)|(k_0k_1k_2)^{\frac{1}{4}}\left[\int_0^x |\psi(t)|^4 dt\right]^{\frac{1}{4}}\\
=\ & 6C(k_0k_1k_2)^{\frac{1}{4}}\left(|\phi_1(x)|+|\phi_2(x)|+|\phi_3(x)|+|\phi_4(x)|\right)\Psi^{\frac{1}{4}}(x).
\end{aligned}
$$

Now, from the equation (11) it follows that

$$
\begin{aligned}
&|\psi(x)| \leq |c_1|\ |\phi_1(x)|+|c_2|\ |\phi_2(x)|+|c_3|\ |\phi_3(x)|+|c_4|\ |\phi_4(x)|\\
&+6C(k_0k_1k_2)^{\frac{1}{4}}(|\phi_1(x)|+|\phi_2(x)|+|\phi_3(x)|+|\phi_4(x)|)\Psi^{\frac{1}{4}}(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
|\psi(x)|^4 \ \leq\ & (|c_1|\ |\phi_1(x)|+|c_2|\ |\phi_2(x)|+|c_3|\ |\phi_3(x)|+|c_4|\ |\phi_4(x)|\\
&+6C(k_0k_1k_2)^{\frac{1}{4}}\left(|\phi_1(x)|+|\phi_2(x)|+|\phi_3(x)|+|\phi_4(x)|\right)\Psi^{\frac{1}{4}}(x))^4. \qquad (12)
\end{aligned}
$$

Using the elementary inequality for any two real numbers x, y

$$(x+y)^4 \leq 8\left(x^4+y^4\right)$$

and equation (12), we get

$$
\begin{aligned}
|\psi(x)|^4 \ \leq\ & (|c_1|\ |\phi_1(x)|+|c_2|\ |\phi_2(x)|+|c_3|\ |\phi_3(x)|+|c_4|\ |\phi_4(x)|)^4\\
&+1296C^4(k_0k_1k_2)(|\phi_1(x)|+|\phi_2(x)|+|\phi_3(x)|+|\phi_4(x)|)^4\Psi(x). \qquad (13)
\end{aligned}
$$

Using the elementary inequality for any real numbers a, b, c, d

$$(a+b+c+d)^4 \leq 64\left(a^4+b^4+c^4+d^4\right)$$

and equation (13), we obtain

$$
\begin{aligned}
|\psi(x)|^4 \ \leq\ & 64\left(|c_1|^4\ |\phi_1(x)|^4+|c_2|^4\ |\phi_2(x)|^4+|c_3|^4\ |\phi_3(x)|^4+|c_4|^4\ |\phi_4(x)|^4\right)\\
&+82944C^4(k_0k_1k_2)\left(|\phi_1(x)|^4+|\phi_2(x)|^4+|\phi_3(x)|^4+|\phi_4(x)|^4\right)\\
&\times\Psi(x). \qquad (14)
\end{aligned}
$$

Integrating (14) over $[0, X]$, we can write

$$\begin{aligned}
\int_0^X |\psi(x)|^4 dx \quad &\leq \quad 64(|c_1|^4 \int_0^X |\phi_1(x)|^4 dx + |c_2|^4 \int_0^X |\phi_2(x)|^4 dx \\
&\quad + |c_3|^4 \int_0^X |\phi_3(x)|^4 dx \; + |c_4|^4 \int_0^X |\phi_4(x)|^4 dx) \\
&\quad + 82944 C^4 (k_0 k_1 k_2) \int_0^X (\, |\phi_1(x)|^4 + \; |\phi_2(x)|^4 \\
&\quad + |\phi_3(x)|^4 + \; |\phi_4(x)|^4) \Psi(x) \, dx \\
&\leq \quad 64 k_0 \left(|c_1|^4 \; + |c_2|^4 \; + |c_3|^4 \; + |c_4|^4 \right) \\
&\quad + 82944 C^4 (k_0 k_1 k_2) \int_0^X (\, |\phi_1(x)|^4 + \; |\phi_2(x)|^4 \\
&\quad + |\phi_3(x)|^4 + \; |\phi_4(x)|^4) \Psi(x) \, dx.
\end{aligned}$$

That means

$$\begin{aligned}
&\Psi(x) \leq 64 k_0 \left(|c_1|^4 \; + |c_2|^4 \; + |c_3|^4 \; + |c_4|^4 \right) \\
&+ 82944 C^4 (k_0 k_1 k_2) \int_0^X \left(\, |\phi_1(x)|^4 + \; |\phi_2(x)|^4 + \; |\phi_3(x)|^4 + \; |\phi_4(x)|^4 \right) \Psi(x) \, dx.
\end{aligned}$$

Then, using the Gronwall's Inequality, we obtain

$$\begin{aligned}
\Psi(X) \quad &\leq \quad 64 k_0 \left(|c_1|^4 \; + |c_2|^4 \; + |c_3|^4 \; + |c_4|^4 \right) \\
&\quad \times e^{82944 C^4 (k_0 k_1 k_2) \int_0^X \left(\, |\phi_1(x)|^4 + \; |\phi_2(x)|^4 + \; |\phi_3(x)|^4 + \; |\phi_4(x)|^4 \right) dx}.
\end{aligned}$$

This means that $\Psi(x)$ is a bounded as $X \to \infty$. Thus we get $\psi(x) \in L^4(0, \infty)$.

Corollary 1. Let λ be a complex parameter and there be a value λ_0 such that all solution and their derivatives up to order three of the equation

$$y^{(4)} + \{\lambda - Q(x)\} y(x) = 0 \tag{15}$$

are in $L^4(0, \infty)$ when $\lambda = \lambda_0$. Then all solutions of the equation are in $L^4(0, \infty)$ for every λ.

Proof. We can write

$$\lambda - Q(x) = \lambda_0 + Q(x) + (\lambda - \lambda_0).$$

Then the differential equation has the following form

$$y^{(4)} + \{\lambda_0 - Q(x) + (\lambda - \lambda_0)\} y(x) = 0.$$

Comparing with (1) and (3), we obtain $q(x) = \lambda_0 - Q(x)$ and $r(x) = \lambda - \lambda_0$. This means that $r(x)$ is a constant function which is bounded in $[0, \infty)$. Then by using the Theorem 3 we obtain that all solutions of (15) are in $L^4(0, \infty)$ for every λ.

Conclusion

In the present paper, we study some properties of a general linear differential equation of fourth order in infinite interval of the form: $y^{(4)}(x) + \{q(x) + r(x)\} y(x) = 0, \qquad 0 \leq x < \infty$, where $r(x)$ is a function which satisfies the condition: $\int_0^\infty |r(x)| \, dx < \infty$. A simple application of this result is provided.

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Карван Х.Ф. Жвемер, Рандо Р.К. Расул

Төртінші ретті сызықты дифференциалдық теңдеуді оның шеттік есебімен салыстыру

Мақалада төртінші ретті сызықты дифференциалдық теңдеу қарастырылған. Авторлар бұл дифференциалдық теңдеудің жоғарғы бағамын, сонымен қатар барлық шешімі $L^4(0,\infty)$ табылатындығын дәлелдеген. Алынған нәтижелерді салыстыра келе, осы дифференциалдық теңдеуден туындаған шеттік есептің барлық меншікті функциялары шектелген және $L^4(0,\infty)$ орналасқан болып табылады.

Кілт сөздер: сызықты дифференциалдық теңдеу, меншікті мән, меншікті функция, жоғарғы бағамы, сызықты тәуелсіз шешімі, $L^2(0,\ infty)$, вронскиан, Гронуолла теңсіздігі, тұрақтыны вариациялау.

Карван Х.Ф. Жвемер, Рандо Р.К. Расул

Сравнение линейного дифференциального уравнения четвертого порядка с его краевой задачей

В статье изучено линейное дифференциальное уравнение четвертого порядка. Авторами найдена верхняя оценка для решений этого дифференциального уравнения, а также доказано, что все решения находятся в $L^4(0,\infty)$. Сравнивая эти результаты, авторы пришли к выводу, что все собственные функции краевой задачи, порожденные этим дифференциальным уравнением, ограничены и находятся в $L^4(0,\infty)$.

Ключевые слова: линейное дифференциальное уравнение, собственное значение, собственная функция, верхняя оценка, линейно независимое решение, $L^2(0,\ infty)$, вронскиан, неравенство Гронуолла, вариация постоянных (параметров).

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