DOI 10.31489/2019M4/52-57

UDC 517.95

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Solving a nonhomogeneous integral equation with the variable lower limit

An nonhomogeneous integral equation with a singular kernel is considered. A feature of the equation under study is the incompressibility of the integral operator. In the study of the equation, an auxiliary simpler equation is used with the right-hand side equal to 1. The incompressibility of the integral operator for the equation under study is shown. Using the relations for an independent variable, the equation is equivalently reduced to a certain simplified equation. With the help of replacements for independent variables, the equation is reduced to an integral equation with a difference kernel. By applying the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. By using the inverse Laplace transform, a solution of the auxiliary integral equation is obtained in the form of a convergent series in some domain. The solution of the initial equation with an arbitrary right-hand side is written through the solution of the auxiliary equation.

Keywords: nonhomogeneous singular integral equation, auxiliary equation, Laplace transform, convergent series.

Introduction

The most complex and interesting objects of study of linear integral equations are irregular situations when the phenomenon of uniqueness of a solution is violated. One of these equations is an equation of the form:

$$\varphi(t) - \frac{1}{2\,a\sqrt{\pi}} \int_t^{\infty} \left[\frac{\tau+t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{ -\frac{(\tau+t)^2}{4a^2(\tau-t)} \right\} + \right.$$

$$\left. + \frac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{ -\frac{\tau-t}{4a^2} \right\} \right] \varphi(\tau)\, d\tau = f(t), \qquad (t>0). \tag{1}$$

For the kernel of equation (1):

$$K(\tau,t) = \frac{1}{2\,a\sqrt{\pi}} \left[\frac{\tau+t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{ -\frac{(\tau+t)^2}{4a^2(\tau-t)} \right\} + \frac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{ -\frac{\tau-t}{4a^2} \right\} \right], \tag{2}$$

we have [1]:

$$\lim_{t\to\infty} \int_t^{\infty} K(\tau,t)\, d\tau = \lim_{t\to\infty} \left(2e^{-\frac{2t}{a^2}} + 1 \right) = 1_{+0}.$$

Hence, the characteristic part of equation (1) is the second term of the kernel (2).

Than we have [1].

Theorem 1. For the singular integral Volterra equation (1) with the kernel (2) the norm of an integral operator acting in classes of continuous functions is equal to 3.

1 An auxiliary equation and reducing the integral equation to an equation with a difference kernel

Using relations:

$$\tau+t=2\tau-(\tau-t),\qquad \frac{(\tau+t)^2}{4a^2\,(\tau-t)}=\frac{\tau\,t}{a^2\,(\tau-t)}+\frac{\tau-t}{4a^2},$$

equation (1) will be rewritten as:

$$\varphi(t)-\int_t^\infty\frac{1}{2a\sqrt{\pi}}\Bigg\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}+$$
$$+\frac{1}{\sqrt{\tau-t}}\Big(1-\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}\Big)\Bigg\}\cdot\exp\left\{-\frac{\tau-t}{4a^2}\right\}\varphi(\tau)\,d\tau=f(t).$$

It is enough to find a solution to the «simplified» equation [2; 215]:

$$\psi(t)-\int_t^\infty k^*(t,\tau)\,\psi(\tau)\,d\tau\ =g(t), \tag{3}$$

where

$$k^*(t,\tau)=\frac{1}{2a\sqrt{\pi}}\Bigg\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}+\frac{1}{\sqrt{\tau-t}}\Big(1-\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}\Big)\Bigg\};$$

$$g(t)=\exp\left\{-\frac{t}{4a^2}\right\}\cdot f(t),\qquad \psi(t)=\exp\left\{-\frac{t}{4a^2}\right\}\cdot\varphi(t).$$

We consider an auxiliary equation with $g(t)=1$ in the (3):

$$\psi(t)-\frac{1}{2a\sqrt{\pi}}\int_t^\infty\Bigg\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}+$$
$$+\frac{1}{\sqrt{\tau-t}}\Big(1-\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}\Big)\Bigg\}\psi(\tau)\,d\tau=1. \tag{4}$$

Integral equation (4) is reduced to an equation with a difference kernel by means of replacements:

$$t=\frac{1}{t_1},\qquad \tau=\frac{1}{\tau_1}.$$

We have

$$\psi\left(\frac{1}{t_1}\right)-\frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{\sqrt{t_1}}{\tau_1^{3/2}\sqrt{t_1-\tau_1}}\left(1-\exp\left\{-\frac{1}{a^2\,(t_1-\tau_1)}\right\}\right)\psi\left(\frac{1}{\tau_1}\right)d\tau_1-$$
$$-\frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{2\,t_1^{3/2}}{\tau_1^{3/2}}\,(t_1-\tau_1)^{3/2}\exp\left\{-\frac{1}{a^2\,(t_1-\tau_1)}\right\}\psi\left(\frac{1}{\tau_1}\right)d\tau_1=1.$$

After that dividing both sides of the last equality by $t_1^{3/2}$, we introduce the following notation:

$$y\,(t_1)=\frac{1}{t_1^{3/2}}\cdot\psi\left(\frac{1}{t_1}\right).$$

As a result, we obtain the equation:

$$t_1\cdot y_1(t_1)-\frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{1}{(t_1-\tau_1)^{1/2}}\Big(1-\exp\Big\{-\frac{1}{a^2(t_1-\tau_1)}\Big\}\Big)y(\tau_1)\,d\tau_1-$$
$$-t_1\cdot\frac{1}{2a\sqrt{\pi}}\int_0^{t_1}\frac{2}{(t_1-\tau_1)^{3/2}}\exp\Big\{-\frac{1}{a^2(t_1-\tau_1)}\Big\}y(\tau_1)\,d\tau_1=\frac{1}{\sqrt{t_1}}. \tag{5}$$

2 Solving the equation with a difference kernel

Applying the Laplace transform to the equation (5) we obtain the operator equation:

$$-\overline{y}^{'}(p)-\frac{1}{2a\sqrt{p}}\left(1-\exp\left(-\frac{2\sqrt{p}}{a}\right)\right)\overline{y}(p)+\left\{\exp\left(-\frac{2\sqrt{p}}{a}\right)\overline{y}(p)\right\}^{'}_{p}=\frac{\sqrt{\pi}}{\sqrt{p}}.$$

After simple transformations we finally get

$$\overline{y}^{'}(p)+\frac{1}{2a\sqrt{p}}\frac{ch\frac{\sqrt{p}}{a}}{sh\,\frac{\sqrt{p}}{a}}\overline{y}(p)=-\frac{\sqrt{\pi}}{\sqrt{p}\left(1-\exp\left(-\frac{2\sqrt{p}}{0}\right)\right)}. \tag{6}$$

The solution of the differential equation (6) is the following function:

$$\overline{y}(p)=\frac{C}{sh\frac{\sqrt{p}}{a}}-a\sqrt{\pi}\frac{\exp\left(\frac{\sqrt{p}}{a}\right)}{sh\frac{\sqrt{p}}{a}}. \tag{7}$$

We rewrite (7) in the form:

$$\overline{y}(p)=\frac{2C}{\exp\left(\frac{\sqrt{p}}{a}\right)-\exp\left(-\frac{\sqrt{p}}{a}\right)}-\frac{2a\sqrt{\pi}}{1-\exp\left(-\frac{2\sqrt{p}}{a}\right)},$$

or in the form

$$\overline{y}(p)=\frac{2C}{\exp\left(\frac{\sqrt{p}}{a}\right)-\exp\left(-\frac{\sqrt{p}}{a}\right)}-2a\sqrt{\pi}\sum_{n=0}^{\infty}\exp\left(-\frac{2n\sqrt{p}}{a}\right). \tag{8}$$

To (8) we apply the inverse Laplace transform [2] we get the solution to equation (5):

$$y(t_1)=-C\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};a^2t_1\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}\frac{n}{t_1^{\frac{3}{2}}}\exp\left(-\frac{n^2}{a^2t_1}\right),$$

where

$$\widehat{\theta}_0\,(\nu;t)=\frac{1}{\sqrt{\pi x}}\left\{\sum_{n=0}^{\infty}\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)-\sum_{n=-1}^{-\infty}n\cdot\exp\left(-\frac{1}{x}\left(\nu+n+\frac{1}{2}\right)^2\right)\right\}$$

is the modified theta function.

3 Solving the «simplified» equation

Returning to the original variables, we get

$$\psi(t)=-\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}n\cdot\exp\left(-\frac{n^2}{a^2}t\right). \tag{9}$$

(9) is the solution of the auxiliary equation (4) with the right-hand side $g(t)=1$.

We denote (9) by

$$\omega(t)=-\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0}-2\sum_{n=1}^{\infty}n\cdot\exp\left(-\frac{n^2}{a^2}t\right). \tag{10}$$

Then [1; 546] the solution of the «simplified»- equation (3) with an arbitrary right-hand side $g(t)$ is expressed in terms of $\omega(t)$ using the formula

$$\psi\,(t)=g(0)\,\omega(t)+\int_t^{\infty}\omega(t-\tau)\,g'(\tau)\,d\tau.$$

In view of notation (10), we obtain a solution to the equation (3)

$$\psi\,(t) = -g(0)\,\left(\frac{C}{t^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t}\right)\right]_{\nu=0} + 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}t\right)\right) -$$

$$-\int_t^{\infty}\left(\frac{C}{(t-\tau)^{\frac{3}{2}}}\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};\frac{a^2}{t-\tau}\right)\right]_{\nu=0} + 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}(t-\tau)\right)\right) g'(\tau)\,d\tau. \tag{11}$$

As

$$-\left[\frac{\partial}{\partial\nu}\,\widehat{\theta}_0\left(\frac{\nu}{2};x\right)\right]_{\nu=0} =$$

$$= \frac{1}{2\sqrt{\pi}x^{\frac{3}{2}}}\left\{\sum_{n=0}^{+\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right) - \sum_{n=-1}^{-\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right)\right\} =$$

$$= \frac{1}{\sqrt{\pi}x^{\frac{3}{2}}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4x}\right),$$

then equality (11) transforms to the form

$$\psi\,(t) = g(0)\,\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}t\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}t\right)\right) +$$

$$+\int_t^{\infty}\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{(2n+1)^2}{4a^2}(t-\tau)\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{n^2}{a^2}(t-\tau)\right)\right) g'(\tau)\,d\tau. \tag{12}$$

The following theorem is proved:

Theorem 2. The integral equation (3) in the class of essentially bounded functions at $g(t)\in L_\infty(0<t<+\infty)$ has the solution defined by the formula (12).

4 Main result

The solution of the integral equation (1) taking into account the obtained expression (12) and [2; 215] has the explicit form:

$$\varphi\,(t) = f(0)\,\left(\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}t\right) - 2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{4n^2-1}{4a^2}t\right)\right) +$$

$$+\int_t^{\infty}\left[\frac{C}{a^3\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\left(-\frac{n^2+n}{a^2}(t-\tau)\right) -\right.$$

$$\left.-2\sum_{n=1}^{\infty} n\cdot\exp\left(-\frac{4n^2-1}{4a^2}(t-\tau)\right)\right]\left(f'(\tau) - \frac{1}{4a^2}\,f(\tau)\right)\,d\tau. \tag{13}$$

5 Main result

Theorem 3. The solution of the integral equation (1) with the singular kernel (2) in the class of essentially bounded functions at $t\geq t_0>0$ has an explicit form defined by the formula (13).

Remark. Singular homogeneous integral equations with kernels of Volterra type were considered in works [3–5]. Their kernels were also «incompressible». The weight classes of the solution existence were found. We also note that boundary value problems for a spectrally loaded parabolic equation reduce to this kind of singular integral equations, when the load line moves according to the law $x=t$ [6–11] and problems for essentially loaded equation of heat conduction [12–16].

In works [17, 18] it is shown that the homogeneous Volterra integral equation of the second kind, to which the homogeneous boundary value problem of heat conduction in the degenerating domain is reduced, has a nonzero solution.

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М.Т. Космакова, Д.М. Ахманова, Ж.М. Тулеутаева, Л.Ж. Қасымова

Бір біртекті емес айнымалы төменгі шекті интегралдық теңдеудің шешілуі

Сингулярлы ядролы біртекті емес интегралдық теңдеу қарастырылған. Зерттелетін теңдеудің ерекшелігі интегралдық оператордың сығылмайтындығы болып табылады. Теңдеуді зерттеу кезінде оң жағы 1-ге тең қарапайым қосалқы теңдеу қолданылды. Тәуелсіз айнымалы үшін қатынастар пайдаланып, теңдеу эквивалентті қандайда бір ықшам теңдеуге келтірілді. Тәуелсіз айнымалылар үшін ауыстырулар қолданылып, теңдеу айырымдық ядролы интегралдық теңдеуге сәйкестендірілді. Алынған теңдеу Лаплас түрлендіруін қолдану арқылы бірінші ретті кәдімгі дифференциалдық (сызықтық) теңдеуге келтірілді. Оның шешуі табылды. Лапластың кері түрлендіруі көмегімен қосымша интегралдық теңдеудің жинақты қатар түріндегі қандайда бір облыстағы шешуі алынды. Кез келген оң жағымен берілген бастапқы теңдеудің шешуі көмекші теңдеудің шешуі арқылы жазылды.

Кілт сөздер: біртекті емес сингулярлы интегралдық теңдеу, қосалқы теңдеу, Лаплас түрлендіруі, жинақты қатар.

М.Т. Космакова, Д.М. Ахманова, Ж.М. Тулеутаева, Л.Ж. Касымова

Решение одного неоднородного интегрального уравнения с переменным нижним пределом

Рассмотрено неоднородное интегральное уравнение с сингулярным ядром. Особенностью исследуемого уравнения является несжимаемость интегрального оператора. При исследовании уравнения использовано вспомогательное более простое уравнение с правой частью, равной 1. Используя соотношения для независимой переменной, уравнение эквивалентно сводится к некоторому упрощенному уравнению. С помощью замен для независимых переменных уравнение сводится к интегральному уравнению с разностным ядром. Применением преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. С помощью обратного преобразования Лапласа получено решение вспомогательного интегрального уравнения в виде сходящегося ряда в некоторой области. Выписано решение исходного уравнения с произвольной правой частью через решение вспомогательного уравнения.

Ключевые слова: неоднородное сингулярное интегральное уравнение, вспомогательное уравнение, преобразование Лапласа, сходящийся ряд.