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New exact particular analytical solutions of the triangular restricted three-body problem

The triangular restricted three-body problem is studied in special non-inertial central reference frame with origin at forces centre of this problem. Masses are arbitrary values. We studied the solutions of dimensionless differential equations of motion of the triangular restricted three-body problem in rotating reference frame in the pulsating variables. For the non-circular planar restricted three-body problem we have found out new exact analytical solutions. In these solutions, all the three bodies form an isosceles triangle with variable height. Also, we have found new class of analytical solutions of the planar circular restricted three-body problem in the form of non-isosceles triangle. The basis of this non-isosceles triangle is distance between the primary bodies, the ratio of sides of non-isosceles triangle is constant and infinitesimal small body is at vertex of this non-isosceles triangle. Obtained exact particular analytical solutions can be used for topological analysis of the general three-body problem.

Keywords: restricted three-body problem, non-inertial reference frame, invariant of center of forces, exact particular analytical solutions.

Introduction

We considered the restricted three-body problem with constant masses m_1, m_2, m_3. The condition of the restricted three-body problem statement is $m_2 \ll m_3,\ m_2 \ll m_1,\ m_3 \geq m_1$. It is widely known that at random masses of the primary bodies m_1 и m_2, the restricted three-body problem has the exact particular solutions - Lagrange solutions, when all the three bodies form an equilateral triangle [1–3]. Also there exist the solution in the form of isosceles triangle when masses of the primary bodies are equal to each other [1–3]. The problem has various applications, but the general analytical solution of this problem in finite form is not found. Due-to this, lot aspects of this problem are studied by different methods and there are plenty publications on this problem. In [4], there have been done orbit classification with numerical computation of the planar restricted three-body problem. In the work [5], good review on resonance of the Lidov-Kozai. In the work [6], there have been considered various applications of the restricted three-body problem to the Earth-Moon system and the Pluto-Charon system. The libration point orbits of the system the Earth and the Moon is described in the work [7].

In the work [8], the perturbing planar circular restricted three-body problem is used to study the restricted n-body problem. In the work [9], the elliptical restricted three-body problem is investigated and energy analysis has been conducted. In the work [10], the short-term capture of an asteroid is studied in the system Sun-Moon in the framework of the restricted four-body problem. In the work [11], based on the planar elliptical restricted three-body problem, calculation method of energy variation for one and two-impulse powered swing-by of spacecraft proposed for the Earth-Moon system. In the work [12], the charged restricted three-body problem is studied, linear stability of planar solutions is investigated and resonance curves are analyzed. In the work [13], the very long-term evolution of the hierarchical restricted three-body problem with the Lidov-Kozai cycles. In the work [14], the invariant manifold structures of the collinear libration points of the restricted three-body problem are investigated. In the work [15], through numerical simulation of the restricted elliptical three-body problem the borders of stable regions around the secondary body found. In the work [16], the existence and stability of the non-collinear libration points in the restricted three-body problem when both the primaries are ellipsoid with equal mass and identical shape are investigated. The two planet three-body problem composed of a central star and two massive planets is investigated and the authors show that secular dynamics of this system can be described using only two parameters, the ratios of the semi-major axes and the planetary masses [17].

In the paper [18], the stability of the equilibrium points under the influence of the small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation pressures of the primaries is investigated. in the work [19], the elliptic isosceles restricted three-body problem with consecutive collision is investigated and the existence of many families of periodic solutions has been proved. In the paper [20], circumbinary accretion discs in the framework of the restricted three-body problem is investigated through numerical solutions of viscous hydrodynamics equations and implicit changes of behavior of the disc near some mass ratio.

Above mentioned analysis of publications shows that the search for new exact particular analytical solutions for random masses m_1 and m_3 is important task. This work is a continuation of our research done in the paper [21]. In this work, we study analytically the triangular restricted three-body problem, when three bodies form triangle during all the time of motion. The problem is studied in the special non-inertial central reference frame with the origin at the center of forces [2, 21] through using invariant of center of forces.

2 Equations of motion of the restricted three-body problem in different reference frames and invariants of center of forces

2.1. Classical equations of motion of the restricted three-body problem in absolute reference frame.

In an absolute reference frame $OX^*Y^*Z^*$ the differential equations of motion of the restricted three-body problem with constant masses m_1,m_2 and m_3, can be written in the following way [1–3]

$$\ddot{\vec{R}}_1^* = \vec{F}_1^* = fm_3\frac{\vec{R}_3^* - \vec{R}_1^*}{R_{13}^{*3}}, \quad \ddot{\vec{R}}_3^* = \vec{F}_3^* = fm_1\frac{\vec{R}_1^* - \vec{R}_3^*}{R_{31}^{*3}}, \tag{1}$$

$$\ddot{\vec{R}}_2^* = \vec{F}_2^* = f\left(m_1\frac{\vec{R}_1^* - \vec{R}_2^*}{R_{21}^{*3}} + m_3\frac{\vec{R}_3^* - \vec{R}_2^*}{R_{23}^{*3}}\right), \tag{2}$$

In these equations $\vec{R}_i^*$ - radius-vector, $\vec{R}_{ij}^*\,(i \neq j)$ – distances between the bodies. Differentiation in time t is denoted by dot over symbol. The system of differential equations (1) describes the two-primary bodies problem with masses m_1, m_3. From this differential equations system, one can obtain the well-known relation

$$m_1\vec{R}_1^* + m_3\vec{R}_3^* = \vec{a}^*t + \vec{b}^*, \quad \vec{a}^* = \overrightarrow{const}, \quad \vec{b}^* = \overrightarrow{const}\,. \tag{3}$$

The equation of motion (2) describes motion of infinitely small body m_2 in the Newtonian gravity field of the two primary bodies m_1, m_3 - the classical restricted three-body problem.

2.2. Differential equations of the restricted three-body problem in the special non-inertial central reference frame and invariants of center of forces.

Then we go to the special non-inertial central reference frame through the formulas $\vec{R}_i^* = \vec{R}_G + \vec{r}_i$, $i = 1, 2, 3$. where $\vec{R}_G$ - radius-vector of the forces center G in the absolute reference frame, $\vec{r}_i$- radius-vectors of the bodies in the special reference frame. The axes of the new reference frame are $Gxyz$ parallel to the corresponding axes of the absolute reference frame $OX^*Y^*Z^*$. The differential equations of the restricted three-body problem (2) in the special non-inertial central reference frame $Gxyz$ have been obtained in the work [21]

$$\ddot{\vec{r}}_2 - \vec{F}_2 = \vec{W}, \tag{4}$$

$$\vec{F}_2 = f\left(m_1\frac{\vec{r}_1 - \vec{r}_2}{\Delta_{21}^3} + m_3\frac{\vec{r}_3 - \vec{r}_2}{\Delta_{23}^3}\right), \tag{5}$$

$$\vec{W} = \vec{W}\,(t) = -f\frac{(m_1 - km_3)}{k+1}\frac{\vec{r}_{31}}{r_{31}^3} + 2\dot{\vec{r}}_{31}\frac{d}{dt}\left(\frac{1}{1+k}\right) + \vec{r}_{31}\frac{d^2}{dt^2}\left(\frac{1}{1+k}\right) \qquad (6)$$

where the dimensionless parameter of the problem is denoted by

$$\frac{r_3}{r_1} = k = k(t) > 0, \qquad (7)$$

Δ_{ij} - distances between infinitesimal small body and the primary bodies

$$\Delta_{21} = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} = \Delta_{12},$$

$$\Delta_{23} = [(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2]^{1/2} = \Delta_{32},$$

$\vec{r}_{31}\,(x_{31}, y_{31}, z_{31})$ - solution of the differential equations of the two primary bodies system, which can be obtained from the (1)

$$\ddot{\vec{r}}_{31} = -f\frac{m_3 + m_1}{r_{31}^3}\vec{r}_{31}.$$

From the integral

$$\vec{r}_{31} \times \dot{\vec{r}}_{31} = \vec{c}_{31} = \overrightarrow{const} \neq 0 \qquad (8)$$

one can see that the orbit is planar and the orbit is on the plane $Gxyz$. The equation (3) can be rewritten in the following form

$$(m_1 + m_3)\,\vec{R}_G + m_1\vec{r}_1 + m_3\vec{r}_3 = \vec{a}^*t + \vec{b}^*, \qquad (9)$$

in order to define the origin of the special non-inertial reference frame, one needs to know the dimensionless variable k. If one obtains k, then taking into account (7)

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = \vec{r}_1 - r_3\,(-\vec{e}_1) = \vec{r}_1 - (kr_1)\,(-\vec{e}_1) = \vec{r}_1 + k\vec{r}_1 = (1+k)\,\vec{r}_1\,.$$

Therefore

$$\vec{r}_1 = \frac{1}{1+k}\vec{r}_{31}, \quad \vec{r}_3 = -\frac{k}{1+k}\vec{r}_{31}.$$

Then from the equation (9), it is possible to define the origin of special non-inertial central reference frame

$$(m_1 + m_3)\,\vec{R}_G = \vec{a}^*t + \vec{b}^* - (m_1\vec{r}_1 + m_3\vec{r}_3) = \vec{a}^*t + \vec{b}^* - \frac{m_1 - km_3}{1+k}\vec{r}_{31}. \qquad (10)$$

Thus, defining the origin of the special non-inertial central reference frame leads to the defining the parameter k. In accordance to the definition of special reference frame, the force$\vec{F}_2$ is directed to center of forces G all the time - to the beginning of the new reference frame. That is why

$$\vec{F}_2 \times \vec{r}_2 = 0. \qquad (11)$$

The equation (11) defines invariant of center of forces established in our work [21]. Invariant of forces center of the restricted three-body problem in the special non-inertial central reference frame in scalar form is

$$\left(\frac{m_3}{\Delta_{23}^3}r_3 - \frac{m_1}{\Delta_{21}^3}r_1\right)r_2\sin\alpha = 0, \qquad (12)$$

where α - is angle between the vectors $\vec{r}_1$ and $\vec{r}_2$. Thus, in the special non-inertial central reference frame, regardless of the primary bodies masses and properties of triangle formed by three bodies, the equation (12) is right for the restricted three-body problem during all time of motion.

3 The triangular restricted three-body problem.

From mathematical point of view, the equation (12) takes a place in several cases. In this work we study only one case

$$\frac{m_3}{\Delta_{23}^3}r_3 - \frac{m_1}{\Delta_{21}^3}r_1 = 0, \quad r_2\sin\alpha \neq 0. \qquad (13)$$

Other cases of fulfilment of forces center invariant (12) will be considered in another works. In the case (13) all three bodies form triangle during all time of motion. Size, shape and orientation of triangle changes over time. The equation (13) takes a place in the triangular restricted three-body problem. Taking into account the equation (7), the first equation in (13) can be rewritten as

$$\left(\frac{\Delta_{23}}{\Delta_{21}}\right)^3 = k\frac{m_3}{m_1}. \tag{14}$$

Thus in the special non-inertial central reference frame, regardless from primary bodies masses and properties of triangle formed by three bodies, the equation (14) is always right for the triangular restricted three-body problem.

In vector form, the invariant of forces center in the triangular restricted three-body problem can be written

$$m_1\Delta_{23}^3\vec{r}_1 + m_3\Delta_{21}^3\vec{r}_3 = 0. \tag{15}$$

Let us consider the equations of motion of the triangular restricted three-body problem (4)–(7) in the special non-inertial central reference frame, in the general case, when $k = k(t) \neq const$, $\vec{r}_2 = \vec{r}_2(x_2, y_2, z_2)$. The invariant of forces center of the triangular restricted three-body problem (14) or (15) can be rewritten as

$$\Delta_{21} = \left(\frac{m_1}{m_3 k}\right)^{1/3}\Delta_{23}. \tag{16}$$

In our work [21], using invariant of center of forces (16) and geometrical properties of triangle, the differential equations (4)–(6) can be rewritten as

$$\ddot{x}_2 + \frac{\mu_2 x_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = W_x\ , \quad \ddot{y}_2 + \frac{\mu_2 y_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = W_y \quad , \tag{17}$$

$$\ddot{z}_2 + \frac{\mu_2 z_2}{(r_2^2 + \sigma_2^2 r_{31}^2)^{3/2}} = 0. \tag{18}$$

with the following designation

$$\mu_2 = f\frac{(m_3^{2/3} + m_1^{2/3}k^{1/3})^{3/2}}{(1+k)^{1/2}} > 0, \quad \sigma_2^2 = \frac{k}{(k+1)^2} > 0. \tag{19}$$

$$W_x = B_2\frac{x_{31}}{r_{31}^3} + D_2\dot{x}_{31} + E_2 x_{31}\ , \quad W_y = B_2\frac{y_{31}}{r_{31}^3} + D_2\dot{y}_{31} + E_2 y_{31}\ , \tag{20}$$

$$D_2 = 2\frac{d}{dt}\left(\frac{1}{1+k}\right), \quad E_2 = \frac{d^2}{dt^2}\left(\frac{1}{1+k}\right), \quad B_2 = -f\frac{m_1 - km_3}{k+1}. \tag{21}$$

The equations (17), (18), in accordance to the solution of the two-body problem, in the case (8), describes the elliptical (in particular circular), hyperbolic or parabolic triangular restricted three-body. The forces center invariant (16) can be rewritten as

$$\left(x_2 - \frac{1}{1+k}x_{31}\right)^2 + \left(y_2 - \frac{1}{1+k}y_{31}\right)^2 + \left(z_2 - \frac{1}{1+k}z_{31}\right)^2 =$$

$$= \left(\frac{m_1}{km_3}\right)^{2/3}\left[\left(x_2 + \frac{k}{1+k}x_{31}\right)^2 + \left(y_2 + \frac{k}{1+k}y_{31}\right)^2 + \left(z_2 + \frac{k}{1+k}z_{31}\right)^2\right]. \tag{22}$$

The system of equations (17)–(21) and (22) have four scalar values x_2, y_2, z_2, k, that is why these four scalar equations represent closed system of equations describing the triangular restricted three-body problem.

4 The differential equations of the triangular restricted three-body problem, in the rotating special non-inertial central reference frame in the pulsating variables

Let us consider the problem in the rotating special non-inertial central reference frame $G\xi\eta\zeta$ in the dimensionless pulsating variables. The new axe ξ go through the bodies with masses m_3 and m_1. The transition formulas are following [1–3, 21]

$$x_2 = r \cdot \xi \cos\theta - r \cdot \eta \sin\theta, \quad y_2 = r \cdot \xi \sin\theta + r \cdot \eta \cos\theta, \quad z_2 = r \cdot \zeta, \tag{23}$$

$$d\theta = \frac{c}{r^2} dt, \quad r_2^2 = \xi^2 + \eta^2 + \zeta^2 = r^2 \rho^2. \tag{24}$$

In the analytical expressions (23), (24), the values $r = r\,(t) = r_{31}$ and $\theta = \theta\,(t) = \theta_{31}$ are defined by solution of the two-body problem. In the rotating special non-inertial central reference frame in dimensionless pulsating variables, the differential equations of the triangular restricted three-body problem are [21]

$$\xi'' - 2\eta' - \frac{1}{1 + e\cos\theta} \left(1 - \frac{A}{\left(\rho^2 + \sigma_2^2\right)^{3/2}} \right) \xi = \frac{1}{1 + e\cos\theta} B + s'', \tag{25}$$

$$\eta'' + 2\xi' - \frac{1}{1 + e\cos\theta} \left(1 - \frac{A}{\left(\rho^2 + \sigma_2^2\right)^{3/2}} \right) \eta = 2s', \tag{26}$$

$$\zeta'' + \frac{1}{1 + e\cos\theta} \left(e\cos\theta + \frac{A}{\left(\rho^2 + \sigma_2^2\right)^{3/2}} \right) \zeta = 0, \tag{27}$$

where dimensionless variables are

$$A = \frac{\left[1 + \nu^{2/3} k^{1/3}\right]^{3/2}}{(1+k)^{1/2}(1+\nu)} = \frac{\left(s^{1/3} + \nu^{2/3}\,(1-s)^{1/3}\right)^{3/2}}{1+\nu} > 0, \quad s = \frac{1}{1+k}, \tag{28}$$

$$B = \frac{k - \nu}{(k+1)(1+\nu)} = \frac{1 - s\,(1+\nu)}{1+\nu}, \quad \nu = \frac{m_1}{m_3} = const > 0, \quad \sigma_2^2 = s - s^2. \tag{29}$$

In the equations (25)–(27) and further, differentiation in θ is denoted by stroke. Invariant of the center of forces of the triangular restricted three-body problem (22) in the pulsating variables ξ,η,ζ with the denotations (28), (29), can be written as

$$(\xi - \xi_1)^2 + \eta^2 + \zeta^2 = \left(\frac{\nu s}{1-s} \right)^{2/3} \left[(\xi - \xi_3)^2 + \eta^2 + \zeta^2 \right], \quad \xi_1 = s, \quad \xi_3 = -\,(1-s)\,. \tag{30}$$

Let us denote that the three differential equations (25)–(27) and one algebraic equation (30) consist four variables ξ,η,ζ и s that is why the system is closed.

The differential equations of motion (25)–(27) of the triangular restricted three-body problem in general case corresponding to the parameter $s = s\,(t) \neq const$ ($k = k\,(t) \neq const$) in the special non-inertial central reference frame in the pulsating variables are convenient for establishing exact particular analytical solutions. The mass parameter ν can be included into these equations in accordance to (28), (29).

While studying the solutions of differential equations of motion of the triangular restricted three-body problem in the special non-inertial central reference frame (25)–(27) and (30), it is convenient to distinguish the three possible cases:

$$1. \quad k = m_1/m_3 = \nu = const \tag{31}$$

$$2. \quad k = const \neq \nu = m_1/m_3 = const \tag{32}$$

$$3. \quad k = k(t) \neq const. \tag{33}$$

In each case it is needed to define the required four scalar values, uniquely satisfying the system of equations (25)–(27) and (30). Let us consider each case.

5 The first case - the isosceles non-circular restricted three-body problem.

Let consider particular and important case of the triangular restricted three-body problem (31), when the equations (25)–(27), (30) can get significantly simplified. Let the following condition take a place

$$k = m_1/m_3 = \nu = const > 0. \tag{34}$$

Let us note that in the case (34), the values of masses m_1 и m_3 are completely different. In this case, from (30)

$$\Delta_{21} = \Delta_{23} = \Delta. \tag{35}$$

From the equation (10) it is seen that, at $k = m_1/m_3$ the special non-inertial central reference frame $Gxyz$ transforms into the barycentric reference frame G_0xyz. It is well-known that the barycentric reference frame is inertial reference frame. At that, radius-vector of the barycenter is defined by the relation (10) in the absolute reference frame. Accordingly taking into account (35), the vector form of the forces center invariant to be transformed into the invariant of masses center

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0.$$

In this case from (35) it comes that the triangle formed by three bodies is isosceles during all time of motion and at vertex of this triangle is massless body. This case is studied by us in the works [22–24], but from different point of view. The isosceles restricted three-body problem is described in general case and it can be elliptical (circular in particular), parabolic, hyperbolic and rectilinear isosceles restricted three-body problem. Let us consider the most interesting case, when the following conditions can take a place in the equations (25)–(27), (30)

$$e \neq 0, \quad \zeta = 0, \quad k = m_1/m_3 = \nu = const > 0. \tag{36}$$

In this particular case we have the planar isosceles non-circular restricted three-body problem and some variables in the differential equations (25)–(27) will get simply

$$\sigma_2^2 = \sigma^2 = \frac{m_1 m_3}{(m_1 + m_3)^2}, \quad A = 1, \quad B = 0.$$

In the barycentric rotating reference frame in pulsating variables, the differential equations of motion of the planar isosceles non-circular ($e \neq 0$) restricted three-body problem is

$$\xi'' - 2\eta' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{1}{(\rho^2 + \sigma^2)^{3/2}}\right)\xi = 0\,, \tag{37}$$

$$\eta'' + 2\xi' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{1}{(\rho^2 + \sigma^2)^{3/2}}\right)\eta = 0. \tag{38}$$

Taking into account (36) and (28)–(29), from forces center invariant (30) expressed in pulsating variables, one can obtain

$$\xi = \xi^* = \frac{m_3 - m_1}{2\,(m_1 + m_3)} = const \neq 0. \tag{39}$$

Thus in equations of the planar isosceles non-circular ($e \neq 0$) restricted three-body problem (37)–(38) the axe ξ is defined. This is constant value and defined by the formula (39). Taking into account (39), the equations of motion (37)–(38) will get more simply and we obtain dynamical system with one degree of freedom

$$\eta' = -\frac{\xi^*}{2\,(1 + e\cos\theta)}\left(1 - \frac{1}{(\eta^2 + 1/4)^{3/2}}\right), \tag{40}$$

$$\eta'' = \frac{\eta}{1 + e\cos\theta}\left(1 - \frac{1}{(\eta^2 + 1/4)^{3/2}}\right) \tag{41}$$

From the differential equations system (40), (41), one can obtain integrals identifying new trajectory in the planar non-circular ($e \neq 0$) isosceles restricted three-body problem

$$\xi^*\eta' + \eta^2 = c_1, \quad c_1 = \xi^*\eta_0' + \eta_0^2 = const, \quad \xi^* \neq 0. \tag{42}$$

The differential equation (42), depending on the value c_1, has three types of solutions. Let us emphasize that the particular case of the equations system (40), (41) when

$$e = 0, \quad r_{31} = a = const, \quad c_{31} = const \neq 0 \tag{43}$$

is studied by us in details in the works [25, 26]. In these works, the case (43) is studied through different methods.

6 The second case. Reduction to quadrature of solutions of the planar circular triangular restricted three-body problem.

In the case (32)

$$k = const \neq m_1/m_3 > 0, \quad (k \neq \nu) \tag{44}$$

all three bodies form non-isosceles triangle during all time of motion

$$\xi'' - 2\eta' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\xi = B \quad , \quad B \neq 0\,, \tag{45}$$

$$\eta'' + 2\xi' - \frac{1}{1 + e\cos\theta}\left(1 - \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\eta = 0, \tag{46}$$

$$\zeta'' + \frac{1}{1 + e\cos\theta}\left(e\cos\theta + \frac{A}{(\rho^2 + \sigma_2^2)^{3/2}}\right)\zeta = 0, \tag{47}$$

with

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2, \quad \nu = \frac{m_1}{m_3} = const > 0, \quad \sigma_2^2 = \frac{k}{(1+k)^2} = const > 0,$$

$$A = \frac{\left[1 + \nu^{2/3}k^{1/3}\right]^{3/2}}{(1+k)^{1/2}(1+\nu)} = const > 0, \quad B = \frac{k - \nu}{(k+1)(1+\nu)} = const \neq 0.$$

Taking into account (28), (29) and the condition (44) the forces center invariant (30) can be written as

$$(\xi - \xi_1)^2 + \eta^2 + \zeta^2 = (\nu/k)^{2/3}\left[(\xi - \xi_3)^2 + \eta^2 + \zeta^2\right], \tag{48}$$

$$\xi_1 = \frac{1}{1+k} = const, \quad \xi_3 = -\frac{k}{1+k} = const.$$

Based on the obtained equations, we can establish exact particular analytical solutions of the planar triangular circular restricted three-body problem. Let take a place the following condition in the equations (45)–(47)

$$e = 0, \quad \zeta = 0, \quad k = const \neq m_1/m_3 > 0. \tag{49}$$

Taking into account (49), from the center of forces invariant (48)

$$\eta^2 + \xi^2 = E_1\xi + E_0, \tag{50}$$

$$E_1 = \frac{2(1 + \varepsilon\nu)}{(1-\varepsilon)(1+\nu)} = const, \quad E_0 = -\frac{(1 - \varepsilon\nu^2)}{(1-\varepsilon)(1+\nu)^2} = const.$$

$$1 - \varepsilon = 1 - (\nu/k)^{2/3} = const \neq 0$$

.

From the equation of motion (45)-(47), the Jacobi integral can be derived

$$\frac{1}{2}(\xi'^2 + \eta'^2) - \frac{1}{2}(\xi^2 + \eta^2) - \frac{A}{(\rho^2 + \sigma_2^2)^{1/2}} - B\xi = C = const. \tag{51}$$

The existing of the two equations (50) and (51), in the case (49) allows us to reduce to quadrature the solution of the problem.

The parameter k, in accordance to the inequality (44), is defined from the condition of defining of possible motion region.

7 The third case

The general case (33) is most interesting and sophisticated, that is why it shall be investigated in another work.

8 Conclusion

In this work, the triangular restricted three-body problem is investigated analytically in the special non-inertial central reference frame with the origin at the center of forces. The solutions of differential equations of the triangular restricted three-body problem is in the rotating special non-inertial central reference frame in dimensionless pulsating variables. New exact particular solutions have been obtained.

In the planar triangular non-circular restricted three-body problem ($e \neq 0$) there have been found out new exact particular solutions of differential equations of motion in the form of isosceles triangle with variable height for arbitrary values of masses. There have been obtained new exact particular analytical solutions of differential equations of motion of the planar triangular circular restricted three-body problem ($e = 0$) in the form of non-isosceles triangle at arbitrary values of masses of the primary bodies. The basis of this non-isosceles triangle is distance between the primary bodies, and the ratio of lateral sides is permanent. A massless body is on vertex of this triangle.

We plan to perform detailed analysis of the equations of motion and to investigate stability of obtained new solutions of the triangular restricted three-body problem. The obtained exact particular analytical solutions can be effectively used for topological analysis of the general solution.

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Шектелген үшбұрышты үш дене мәселесінің жаңа нақты дербес аналитикалық шешімдері

Үшбұрышты шектелген үш дене мәселесі аналитикалық жолмен арнайы инерциалды емес централды санақ жүйесінде қарастырылған. Бұл санақ жүйенің басы күштер центрінде орналасады. Негізгі екі дене массалары кез-келген шама. Айналмалы санақ жүйесінің пульсирленген айнымалыларында үшбұрышты шектелген үш дене мәселесінің өлшемсіз дифференциалдық теңдеулері зерттелді. Шеңберлік емес жазық шектелген үш дене мәселесінде жаңа нақты дербес теңбүйірлі биіктігі айналмалы үшбұрыш түрінде аналитикалық шешімдер анықталды. Және теңбүйірлі емес үшбұрыш түріндегі жазық шеңберлік шектелген үш дене мәселесінің жаңа аналитикалық теңдеулер классы табылды. Теңбүйірлі емес үшбұрыштың негізін екі негізгі денелер арақашықтығы құрайды, теңбүйірлі емес үшбұрыштың бүйір қабырғаларының қатынасы тұрақты шама және осы теңбүйірлі емес үшбұрыштың төбесінде массасы шексіз аз дене орналасады. Табылған нақты дербес шешімдерді жалпы мәселені зерттеу үшін топологиялық талдауға қолдануға болады.

Кілт сөздер: шектелген үш дене мәселесі, инерциалды емес санақ жүйесі, күштер центрінің инварианты, нақты дербес аналитикалық шешімдер.

М.Дж. Минглибаев, Т.М. Жумабек

Новые точные частные аналитические решения треугольной ограниченной задачи трех тел

Аналитически исследована треугольная ограниченная задача трех тел в специальной неинерциальной центральной системе координат с началом в центре сил исследуемой задачи. При этом массы основных тел произвольные. Изучены решения безразмерных дифференциальных уравнений движения треугольной ограниченной задачи трех тел во вращающейся системе координат в пульсирующих переменных. В некруговой плоской ограниченной задаче трех тел установлены новые точные аналитические частные решения, в виде равнобедреннего треугольника переменной высоты. Также аналитически найден новый класс решений плоской круговой ограниченной задачи трех тел в виде неравнобедренного треугольника. Основанием неравнобедренного треугольника является расстояние между основными телами, отношение боковых сторон неравнобедренного треугольника постоянное, и на вершине этого неравнобедренного треугольника находится тело малой массы. Установленные точные частные аналитические решения можно эффективно использовать для топологического анализа общего решения проблемы.

Ключевые слова: ограниченная задача трех тел, неинерциальная система координат, инвариант центра сил, точные частные аналитические решения.

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