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# Order of the trigonometric widths of the Nikol'skii-Besov classes with mixed metric in the metric of anisotropic Lorentz spaces

In this paper we estimate the order of the triginometric width of the Nikol'skii–Besov classes $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with mixed metric in the anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ when $\mathbf{1}<\mathbf{p}=(p_1,\ldots,p_n)<\mathbf{2}<\mathbf{q}=(q_1,\ldots,q_n)$. The concept of a trigonometric width in the one-dimensional case was first introduce by R.S. Ismagilov and he established his estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric width of Sobolev class $W_p^{\mathbf{r}}$, Nikol'skii class $H_p^{\mathbf{r}}$ in the space $L_q$ are established by E.S. Belinsky, V.E. Majorov, Yu. Makovoz, G.G. Magaril-Ilyaev, V.N. Temlyakov. This problem for the Besov class $B_{pq}^{\mathbf{r}}$ was investigated by A.S. Romanyuk, D.B. Bazarkhanov. The trigonometric width for the anisotropic Nikol'skii-Besov classes $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy.

*Keywords:* trigonometric widths, anisotropic Lorentz space, Nikol'skii–Besov class with mixed metric.

*Introduction*

Let $V\subset L_1(\mathbb{T}^n)$ be the normed space and $F\subset V$ be some functional class. The trigonometric width of the class $F$ in the space $V$ is defined as follows (see [1])

$$d_M^T(F,V)=\inf_{\Omega_M}\sup_{f\in F}\inf_{t(\Omega_M;\mathbf{x})}\|f(\cdot)-t(\Omega_M;\cdot)\|_V\,,$$

where $t(\Omega_M;\mathbf{x})=\sum_{j=1}^{M}c_j e^{i(\mathbf{k}_j,\mathbf{x})}$, $\Omega_M=\{\mathbf{k}_1,\ldots,\mathbf{k}_M\}$ is the set of vectors $\mathbf{k}_j=(k_1^j,\ldots,k_n^j)$ from the integer lattice $\mathbb{Z}^n$ and $c_j$ are some numbers $(j=1,\ldots,M)$.

The concept of a trigonometric width in the one-dimensional case was first introduced by R.S. Ismagilov [1] and he established its estimates for certain classes in the space of continuous functions. For a function of several variables exact orders of trigonometric widths of Sobolev class $W_p^{\mathbf{r}}$, Nikol'skii class $H_p^{\mathbf{r}}$ in the space $L_q$ are established by E.S. Belinsky [2], V.E. Majorov [3], Yu. Makovoz [4], G.G. Magaril-Ilyaev [5], V.N. Temlyakov [6]. This problem for the Besov class $B_{pq}^{\mathbf{r}}$ was investigated by A.S. Romanyuk [7], D.B. Bazarkhanov [8]. The trigonometric width for the anisotropic Nikol'skii-Besov classes $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ in the metric of the anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ was found by K.A. Bekmaganbetov and Ye. Toleugazy [9].

We study the problem of estimating the order of the trigonimetric width of the Nikol'skii-Besov classes $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with a mixed metric in the metric of anisotropic Lorentz spaces $L_{\mathbf{q}\theta}(\mathbb{T}^n)$.

*Preliniminaries and auxiliary results*

Let $f(\mathbf{x})=f(x_1,\ldots,x_n)$ be a measurable function defined by $\mathbb{T}^n$. Let multiindexes $\mathbf{1}\leq\mathbf{p}=(p_1,\ldots,p_n)<<\infty$. A Lebesgue space $L_{\mathbf{p}}(\mathbb{T}^n)$ with mixed metric is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^n)}=\left(\int_0^{2\pi}\left(\ldots\left(\int_0^{2\pi}|f(x_1,\ldots,x_n)|^{p_1}\mathrm{d}x_1\right)^{p_2/p_1}\ldots\right)^{p_n/p_{n-1}}\mathrm{d}x_n\right)^{1/p_n}.$$

Here, the expression $\left(\int_0^{2\pi}|f(t)|^p\mathrm{d}t\right)^{1/p}$ for $p=\infty$ is understood as $\sup_{0\leq t<2\pi}|f(t)|$.

For the function $f \in L_{\mathbf{p}}(\mathbb{T}^n)$ we denote

$$\Delta_{\mathbf{s}}(f,\mathbf{x}) = \sum_{\mathbf{k}\in\rho(\mathbf{s})} a_{\mathbf{k}}(f) e^{i(\mathbf{k},\mathbf{x})},$$

where $\{a_{\mathbf{k}}(f)\}_{\mathbf{k}\in\mathbb{Z}^n}$ are Fourier coefficients of the function $f$ with respect to the multiple trigonometric system $\rho(\mathbf{s}) = \{\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{Z}^n : 2^{s_i-1} \le |k_i| < 2^{s_i}, i = 1,\ldots,n\}$, $(\mathbf{k},\mathbf{x}) = \sum_{j=1}^n k_j x_j$ – inner product.

Let $\mathbf{0} < \alpha = (\alpha_1,\ldots,\alpha_n) < \infty$, $\mathbf{0} < \tau = (\tau_1,\ldots,\tau_n) \le \infty$. The class of Nikol'skii-Besov $B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$ with a mixed metric is the set of functions $f$ from $L_{\mathbf{p}}(\mathbb{T}^n)$ for which the following inequality holds

$$\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} = \left\| \left\{ \mathbf{2}^{(\alpha,\ \mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\mathbb{Z}_+^n} \right\|_{l_\tau} \le 1,$$

where $\|\cdot\|_{l_\tau}$ is the norm of discrete Lebesgue space $l_\tau$ with a mixed metric.

Let $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ be a measurable function defined on $\mathbb{T}^n$. We denote by $f^*(\mathbf{t}) = f^{*_1,\ldots,*_n}(t_1,\ldots,t_n)$ the function obtained by applying to the first nonicreasing permutation, successively with respect to the variables $x_1,\ldots,x_n$ for fixed other variables.

Let multiindexes $\mathbf{q} = (q_1,\ldots,q_n)$, $\theta = (\theta_1,\ldots,\theta_n)$ satisfy the conditions: if $0 < q_j < \infty$, then $0 < \theta_j \le \infty$, if $q_j = \infty$, then $\theta_j = \infty$ for every $j = 1,\ldots,n$. An anisotropic Lorentz space $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ is the set of functions for which the following quantity is finite

$$\|f\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} = \left( \int_0^{2\pi} \ldots \left( \int_0^{2\pi} \left( t_1^{1/q_1} \ldots t_n^{1/q_n} f^{*_1,\ldots,*_n}(t_1,\ldots,t_n) \right)^{\theta_1} \frac{dt_1}{t_1} \right)^{\theta_2/\theta_1} \ldots \frac{dt_n}{t_n} \right)^{1/\theta_n}.$$

Let $\Omega_M$ be a set containing at most $M$ vectors $\mathbf{k} = (k_1,\ldots,k_n) \in \mathbb{Z}^n$.

*Lemma 1* [10]. Let $\mathbf{2} \le \mathbf{q} < \infty$. Then for any trigonometric polynomial

$$P(\Omega_M,\mathbf{x}) = \sum_{j=1}^{M} e^{i(\mathbf{k}^j,\mathbf{x})}$$

and for any number $N \le M$ there exists a trigonometric polynomial $P(\Omega_N,\mathbf{x})$ containing at most $N$ harmonics and such that

$$\|P(\Omega_M,\cdot) - P(\Omega_N,\cdot)\|_{L_{\mathbf{q}}(\mathbb{T}^n)} \le CMN^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$ and all coefficients $P(\Omega_N,\mathbf{x})$ are the same and do not exceed $MN^{-1}$.

*Corollary 1* [11]. Let $\mathbf{2} < \mathbf{q} = (q_1,\ldots,q_n) \le \infty$, $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) \le \infty$. Then for any trigonometric polynomial

$$P(\Omega_M,\mathbf{x}) = \sum_{j=1}^{M} e^{i(\mathbf{k}^j,\mathbf{x})}$$

and for any number $N \le M$ there exists a trigonometric polynomial $P(\Omega_N,\mathbf{x})$ containing at most $N$ harmonics and such that

$$\|P(\Omega_M,\cdot) - P(\Omega_N,\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \le CMN^{-1/2},$$

moreover $\Omega_N \subset \Omega_M$ and all coefficients $P(\Omega_N,\mathbf{x})$ are the same and do not exceed $MN^{-1}$.

For any $\mathbf{s} \in \mathbb{Z}_+^n$ we consider a linear operator

$$\left(T_{N_{\mathbf{s}}} f\right)(\mathbf{x}) = f(\mathbf{x}) * \left( \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})} - t(\Omega_{N_{\mathbf{s}}},\mathbf{x}) \right),$$

where $t(\Omega_{N_{\mathbf{s}}},\mathbf{x})$ is a trigonometric polynomial from Corollary 1, which is approaching the «block» $t_{\mathbf{s}}(\mathbf{x}) = \sum\limits_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})}$.

*Lemma 2.* Let $\mathbf{1} < \mathbf{p} < \mathbf{2}$, the multiindex $\mathbf{q} = (q_1,\ldots,q_n)$ be such that $2 < q_j < p'$ for all $j = 1,\ldots,n$ and $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) \le \infty$. Then the norm operator $T_{N_{\mathbf{s}}}$ acting from $L_{\mathbf{p}}(\mathbb{T}^n)$ to $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ satisfies the following inequality

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{p}}(\mathbb{T}^n)\to L_{\mathbf{q}\theta}(\mathbb{T}^n)} \le C_1 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-(\mathbf{1/2}+\mathbf{1/p'})}.$$

*Proof.* Taking into account that the coefficients of the polynomial $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ are the same and do not exceed $2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1}$ by Parseval's equality we have

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{2}}(\mathbb{T}^n)\to L_{\mathbf{2}}(\mathbb{T}^n)} \le C_1 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1}. \tag{1}$$

Further, using the generalized Minkowski's inequalities and Corollary 1 we can write

$$\|T_{N_{\mathbf{s}}}f\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le \|f\|_{L_1(\mathbb{T}^n)} \left\| \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\cdot)} - t(\Omega_{N_{\mathbf{s}}},\cdot) \right\|_{L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le C_2 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1/2}\|f\|_{L_{\mathbf{1}}(\mathbb{T}^n)}.$$

Therefore, by definition, $\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}$ we find

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le C_2 2^{(\mathbf{1},\mathbf{s})}N_{\mathbf{s}}^{-1/2}. \tag{2}$$

Further, using the Riesz-Thorin interpolation theorem for Lebegue spaces and anisotropic Lorentz spaces, we obtain

$$\|T_{N_{\mathbf{s}}}\|_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)} \le \|T_{N_{\mathbf{s}}}\|^{1-\lambda}_{L_{\mathbf{2}}(\mathbb{T}^n)\to L_{\mathbf{2}}(\mathbb{T}^n)}\|T_{N_{\mathbf{s}}}\|^{\lambda}_{L_{\mathbf{1}}(\mathbb{T}^n)\to L_{\mathbf{q}^*\theta^*}(\mathbb{T}^n)}, \tag{3}$$

where $\mathbf{0} < \lambda < \mathbf{1}$ and $\mathbf{1}/\mathbf{p} = (\mathbf{1}-\lambda)/\mathbf{2} + \lambda/\mathbf{1}$, $\mathbf{1}/\mathbf{q}=(\mathbf{1}-\lambda)/\mathbf{2}+\lambda/\mathbf{q}^*$ and $\mathbf{1}/\theta = (\mathbf{1}-\lambda)/\mathbf{2} + \lambda/\theta^*$.

By substuting (1) and (2) to (3) and performing elementary transformations, we reccieve at the required estimate with the additional condition $\mathbf{0} < \theta = (\theta_1,\ldots,\theta_n) < \mathbf{p}' = (p',\ldots,p')$. For the remaining values of the parameters $\theta = (\theta_1,\ldots,\theta_n)$ the validity of the assertion follows from the embedding $L_{\mathbf{q}\theta_{\mathbf{1}}}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\theta_{\mathbf{2}}}(\mathbb{T}^n)$ for $\mathbf{0} < \theta_1 = (\theta_1^1,\ldots,\theta_n^1) \le \theta_2 = (\theta_1^2,\ldots,\theta_n^2) \le \infty$.

Let us formulate a special case of the embedding theorem from E.D. Nursultanov's paper ([12]) as a Lemma.

*Lemma 3* [12]. Let $\mathbf{1} \le \mathbf{p} = (p_1,\ldots,p_n) < \mathbf{q} = (q_1,\ldots,q_n) < \infty$, $\mathbf{0} < \tau = (\tau_1,\ldots,\tau_n) \le \infty$ and $\alpha = \mathbf{1}/\mathbf{p} - \mathbf{1}/\mathbf{q}$, then

$$B^{\alpha\tau}_{\mathbf{p}}(\mathbb{T}^n) \hookrightarrow L_{\mathbf{q}\tau}(\mathbb{T}^n).$$

Furthermore we need the following sets

$$Y^n(N,\gamma) = \left\{ \mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{Z}^n_+ : \sum_{j=1}^n \gamma_j s_j \ge N \right\},$$

$$\aleph^n(N,\gamma) = \left\{ \mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{Z}^n_+ : \sum_{j=1}^n \gamma_j s_j = N \right\}.$$

*Lemma 4 [13].* Let $n \in \mathbb{N}$, $n \ge 2$, $\mathbf{0} < \gamma' = (\gamma_1',\ldots,\gamma_n') \le \gamma = (\gamma_1,\ldots,\gamma_n) < \infty$, $\delta > 0$ and $\mathbf{0} < \varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \le \infty$. Then

$$\left\| \left\{ \mathbf{2}^{-\delta(\gamma,\ \mathbf{s})} \right\}_{\mathbf{s}\in Y^n(N,\ \gamma')} \right\|_{l_\varepsilon(\mathbb{Z}^n_+)} \le C 2^{-\delta\eta N} N^{\sum_{j\in A\setminus\{j_1\}} 1/\varepsilon_j},$$

where $\eta = \min\{\gamma_j/\gamma_j' : j = 1,\ldots,n\}$, $A = \{j : \gamma_j/\gamma_j' = \eta, j = 1,\ldots,n\}$, $j_1 = \min\{j : j \in A\}$.

*Lemma 5* [13]. Let $n \in \mathbb{N}$, $n \ge 2$, $\mathbf{0} < \gamma = (\gamma_1,\ldots,\gamma_n) < \infty$, $\delta \in \mathbb{R}$ and $\mathbf{0} < \varepsilon = (\varepsilon_1,\ldots,\varepsilon_n) \le \infty$. Then

$$\left\| \left\{ \mathbf{2}^{-\delta(\gamma,\ \mathbf{s})} \right\}_{\mathbf{s}\in \aleph^n(N,\ \gamma')} \right\|_{l_\varepsilon(\mathbb{Z}^n_+)} \asymp 2^{-\delta N} N^{\sum_{j=2}^n 1/\varepsilon_j}.$$

## *Main result*

The main result of this paper includes:

*Theorem 1.* Let $\mathbf{1} < \mathbf{p} = (p_1,\ldots,p_n) < \mathbf{2} < \mathbf{q} = (q_1,\ldots,q_n) < \mathbf{p_0'} = (p_0',\ldots,p_0')$, $p_0 = \max\{p_j : j = 1,\ldots,n\}$, $\mathbf{1} \le \tau = (\tau_1,\ldots,\tau_n), \theta = (\theta_1,\ldots,\theta_n)$ and $\alpha = (\alpha_1,\ldots,\alpha_n)$ be such that $\alpha_j > 1 + 1/p_j - 1/p_0$ for all $j = 1,\ldots,n$. Let $\zeta = \min\{\alpha_j - 1/p_j + 1/q_j : j = 1,\ldots,n\}$, $D = \{j = 1,\ldots,n : \alpha_j - 1/p_j + 1/q_j = \zeta\}$, $j_1 = \min\{j : j \in D\}$, $q_j = q_{j_1}$ for all $j \in D$ and $q_j \ge q_{j_1}$ for all $j \notin D$.

Then the following relation holds

$$d_M^T(B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n),L_{\mathbf{q}\theta}(\mathbb{T}^n)) \asymp M^{-(\alpha_{j_1}-1/p_{j_1}+1/2)}(\log M)^{(|D|-1)(\alpha_{j_1}-1/p_{j_1}+1/2)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}, \tag{4}$$

where $|D|$ is amount of elements of the set $D$, $a_+ = \max\{a;0\}$.

*Proof.* Let $f \in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$. For any natural number $M$ there exists the natural number $m$ such that $M \asymp 2^m m^{(|D|-1)}$. We will seek an approximating polynomial $P(\Omega_M, \mathbf{x})$ in the following form

$$P(\Omega_M;\mathbf{x}) = \sum_{(\gamma',\mathbf{s})} \triangle_{\mathbf{s}}(f,\mathbf{x}) + \sum_{m\le(\gamma',\mathbf{s})<\beta m} t(\Omega_{N_{\mathbf{s}}};\mathbf{x}) * \triangle_{\mathbf{s}}(f,\mathbf{x}), \tag{5}$$

where

$$\beta = \left(\alpha_{j_1} - 1/p_{j_1} + 1/2 - \frac{\log m}{m}\sum_{j\in D\setminus\{j_1\}}\Big((1/2-1/\tau_j)_+ - (1/\theta_j - 1/\tau_j)_+\Big)\right)/(\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1}),$$

$\gamma_j = (\alpha_j - 1/p_j + 1/q_j)/(\alpha_{j_1} - 1/p_{j_1} + 1/q_{j_1})$, $j = 1,\dots,n$, $\gamma'_j = \gamma_j$ for $j \in D$ and $1 < \gamma'_j < \gamma_j$ for $j \notin D$. The polynomials $t(\Omega_{N_{\mathbf{s}}}, \mathbf{x})$ are chosen for every "block" $t_{\mathbf{s}}(\mathbf{x}) = \sum\limits_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\mathbf{x})}$ according to Corollary 1 and numbers $N_{\mathbf{s}} = \left[2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m}2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right]$.

Note that according to Lemma 4

$$\sum_{m\le(\gamma',\mathbf{s})<\beta m} N_{\mathbf{s}} = 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m}\sum_{m\le(\gamma',\mathbf{s})<\beta m} 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})} \le$$

$$\le 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m}\left\|\left\{2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in Y^n(m,\gamma')}\right\|_{l_1} \le$$

$$\le 2^{(\alpha_{j_1}-1/p_{j_1}+1/p_0)m}2^{-(\alpha_{j_1}-1/p_{j_1}+1/p_0)m}m^{(|D|-1)} = 2^m m^{(|D|-1)} \asymp M,$$

so that $(\alpha_j - 1/p_j + 1/p_0 - 1)/(\alpha_{j_1} - 1/p_{j_1} + 1/p_0 - 1) > \gamma'_j$ at $j \notin D$.

Moreover according to equality (5) and Minkowski's inequality we have

$$\|f(\cdot) - P(\Omega_M;\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \le$$

$$\le C_1\left(\left\|\sum_{m\le(\gamma',\mathbf{s})<\beta m}(\triangle_{\mathbf{s}}(f,\cdot) - \triangle_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}};\cdot))\right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} + \left\|\sum_{(\gamma',\mathbf{s})\ge\beta m}\triangle_{\mathbf{s}}(f,\cdot)\right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}\right) =$$

$$= C_1(I_1(f) + I_2(f)). \tag{6}$$

Firstly we estimate $I_2(f)$. By Lemma 3 we have

$$I_2(f) \le C_2\left\|\left\{2^{(\mathbf{1/p}-\mathbf{1/q},\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')}\right\|_{l_\theta} \le I_3(f). \tag{7}$$

According to Hölder's inequality with parameters $\mathbf{1}/\theta = \mathbf{1}/\tau + \mathbf{1}/\varepsilon$, where $\mathbf{1}/\varepsilon = (\mathbf{1}/\theta - \mathbf{1}/\tau)_+$ and Lemma 4, taking into account that $\gamma' \le \gamma$ we find

$$I_3(f) = \left\|\left\{2^{(\alpha,\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \cdot 2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})(\gamma,\mathbf{s})}\right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')}\right\|_{l_\theta} \le$$

$$\le \left\|\left\{2^{(\alpha,\mathbf{s})}\|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)}\right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')}\right\|_{l_\tau} \times$$

$$\times\left\|\left\{2^{-(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1})(\gamma,\mathbf{s})}\right\}_{\mathbf{s}\in Y^n(\beta m,\gamma')}\right\|_{l_\varepsilon} \le$$

$$\leq C_3 \|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)} \cdot 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m} m^{\sum_{j\in D\setminus\{j_1\}} 1/\varepsilon_j} \leq$$

$$\leq C_3 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/\theta_j-1/\tau_j)_+}. \qquad (8)$$

Inserting (8) into (7) we have

$$I_2 \leq C_4 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/\theta_j-1/\tau_j)_+}.$$

Next by using $\beta$ we obtain

$$2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/q_{j_1}\right)\beta m} = 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/2-1/\tau_j)_+},$$

and consequently

$$I_2(f) \leq C_4 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)\beta m} m^{\sum_{j\in D\setminus\{j_1\}} (1/2-1/\tau_j)_+}.$$

Taking into account that $M \asymp 2^m m^{(|D|-1)}$ we have

$$I_2(f) \leq C_5 M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)} (\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}} (1/2-1/\tau_j)_+}. \qquad (9)$$

Now, let us estimate the value $I_1(f)$. By using the Littlewood-Paley theorem (see [14]), we obtain

$$I_1(f) = \left\| \sum_{m\leq(\gamma',\mathbf{s})<\beta m} \left(\triangle_{\mathbf{s}}(f,\cdot) - \triangle_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}};\cdot)\right) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \leq$$

$$\leq C_6 \left( \left\| \sum_{m\leq(\gamma',\mathbf{s})<\beta m} \left(\triangle_{\mathbf{s}}(f,\cdot) - \triangle_{\mathbf{s}}(f,\cdot) * t(\Omega_{N_{\mathbf{s}}};\cdot)\right) \right\|^2_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right)^{1/2} \leq$$

$$\leq C_6 \left\| \left\{ \left\| \triangle_{\mathbf{s}}(f,\cdot) * \left( \sum_{\mathbf{k}\in\rho(\mathbf{s})} e^{i(\mathbf{k},\cdot)} - t(\Omega_{N_{\mathbf{s}}},\cdot) \right) \right\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_2} =$$

$$= C_6 \left\| \left\{ \|T_{\mathbf{s}} \triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{q}\theta}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_2}, \qquad (10)$$

where $\aleph^n(m,\beta m,\gamma') = \{\mathbf{s}\in\mathbb{Z}^n_+ : m \leq (\gamma',\mathbf{s}) < \beta m\}$.

By using Lemma 2 and inequality of different metric for trigonometric polynomials in the Lebesgue spaces with mixed metric (see [14]) for $1 < p_j < p_0$ $(j = 1,\ldots,n)$, from (10) we have

$$I_1(f) \leq C_7 \left\| \left\{ 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p_0}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_2} \leq$$

$$\leq C_8 \left\| \left\{ 2^{(\mathbf{1},\mathbf{s})} N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)} 2^{(\mathbf{1/p}+\mathbf{1/p_0},\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_2} =$$

$$= C_8 \left\| \left\{ N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)} 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})} \cdot 2^{(\alpha,\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_2}. \qquad (11)$$

According to Hölder's inequality with parameters $\mathbf{1/2} = \mathbf{1}/\tau + \mathbf{1}/\varepsilon$, where $\mathbf{1}/\varepsilon = (\mathbf{1/2} - \mathbf{1}/\tau)_+$ and by (11) we find

$$I_1(f) \leq C_8 \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\triangle_{\mathbf{s}}(f,\cdot)\|_{L_{\mathbf{p}}(\mathbb{T}^n)} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_\tau} \times$$

$$\times \left\| \left\{ N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)} 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})} \right\}_{\mathbf{s}\in\aleph^n(m,\beta m,\gamma')} \right\|_{l_\varepsilon} \leq$$

$$\le C_8\|f\|_{B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)}\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}\le$$

$$\le C_8\left\|\left\{N_{\mathbf{s}}^{-\left(1/2+1/p_0'\right)}2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon} \tag{12}$$

for any function $f\in B_{\mathbf{p}}^{\alpha\tau}(\mathbb{T}^n)$.

By continuing (12), according to the Lemma 4 we have

$$I_1(f)\le C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\times$$

$$\times\left\|\left\{2^{\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\cdot 2^{-(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}=$$

$$=C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\times$$

$$\times\left\|\left\{2^{-\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in\aleph^n\left(m,\beta m,\gamma'\right)}\right\|_{l_\varepsilon}\le$$

$$\le C_8 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}\left\|\left\{2^{-\left(1/2+1/p_0'\right)(\alpha-\mathbf{1/p}+\mathbf{1/p_0}-\mathbf{1},\mathbf{s})}\right\}_{\mathbf{s}\in Y^n\left(m,\gamma'\right)}\right\|_{l_\varepsilon}\le$$

$$\le C_9 2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-1/p_{j_1}+1/p_0\right)m}2^{-\left(1/2+1/p_0'\right)\left(\alpha_{j_1}-\frac{1}{p_{j_1}}+\frac{1}{p_0}-1\right)m}m^{\sum_{j\in D\setminus\{j_1\}}1/\varepsilon_j}=$$

$$=C_9 2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+},$$

as $\left(\alpha_j-1/p_j+1/p_0-1\right)/\left(\alpha_{j_1}-1/p_{j_1}+1/p_0-1\right)>\gamma_j'$ at $j\notin D$.

Taking into account that $M\asymp 2^m m^{(|D|-1)}$ we find

$$I_2(f)\le C_{10}M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)}(\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}. \tag{13}$$

Inserting (9) and (13) into (6) we obtain the inequality, which gives the upper estimate in (4).

For the proof of the lower estimate we consider the following value

$$e_M(F)_V=\sup_{f\in F}\inf_{\{b_j,\mathbf{k}_j\}_{j=1}^M}\left\|f-\sum_{j=1}^M b_j e^{i(\mathbf{k}_j,\mathbf{x})}\right\|_V,$$

which is called the best $M$-term appoximation of the class $F$ in metric space $V$.

Moreover, by the definition, the following inequality holds

$$e_M(F)_V\le d_M^T(F,V).$$

By using the condition $2<q_j$ $(j=1,\ldots,n)$ we have

$$e_M(f)_{L_{\mathbf{2}}(\mathbb{T}^n)}\le C_{11}e_M(f)_{L_{\mathbf{q}\theta}(\mathbb{T}^n)}.$$

For the proof of the lower estimate we will use double relation, which follows from the general results of S.M. Nikol'skii (see [15]). According to this relation for any function $f\in L_2(\mathbb{T}^n)$ the following equality holds

$$e_M(f)_{L_2(\mathbb{T}^n)}=\inf_{\Omega_M}\sup_{P\in\mathcal{L}^\perp,\|P\|_{L_2(\mathbb{T}^n)}\le 1}\left|\int_{\mathbb{T}^n}f(\mathbf{x})P(\mathbf{x})d\mathbf{x}\right|, \tag{14}$$

where $\mathcal{L}$ is a linear span of a system of functions $\{e^{i(\mathbf{k},\mathbf{x})}\}_{\mathbf{k}\in\Omega_M}$.

We consider the function

$$f(\mathbf{x})=m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{m\le\left(\gamma',\mathbf{s_0}\right)<m+n}\prod_{j=1}^n 2^{-(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}e^{i(\mathbf{k},\mathbf{x})},$$

where $D'=\{j\in D:\ 2<\tau_j\}\cup\{j_1\}$, $\mathbf{s_0}=(s_1^0,\ \ldots,\ s_n^0)$, $s_j^0=s_j$ at $j\in D'$ and $s_j^0=0$ at $j\notin D'$.

In the paper [16] it was proved that the function $C_{12}f(\mathbf{x})$ belongs to the class $B^{\alpha\tau}_{\mathbf{p}}(\mathbb{T}^n)$.

Let us construct the function $P(\mathbf{x})$ satisfying the condition (14).

Let
$$u(\mathbf{x})=\sum_{(\gamma',\mathbf{s_0})\le m}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}e^{i(\mathbf{k},\mathbf{x})}, \tag{15}$$
and $\Omega_M$ be an arbitrary collection of integer vectors $\mathbf{k}=(k_1,\dots,k_n)$.

We denote by
$$v(\mathbf{x})=\sum_{(\gamma',\mathbf{s_0})\le m}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})\cap\Omega_M}e^{i(\mathbf{k},\mathbf{x})}$$
the function, containing only those terms of (15), for which $\mathbf{k}\in\Omega_M$. By Minkowski's inequality and Parseval's equality for function $w(\mathbf{x})=u(\mathbf{x})-v(\mathbf{x})$ we have
$$\|\omega\|_{L_2(\mathbb{T}^n)}\le C_{13}M^{1/2}.$$

We consider the function $P(\mathbf{x})=C_{13}^{-1}M^{-1/2}w(\mathbf{x})$, then $\|P\|_{L_2(\mathbb{T}^n)}\le 1$. Since the function $w(\mathbf{x})=u(\mathbf{x})-v(\mathbf{x})$ does not contain the harmonics from $\Omega_M$, then function $P\in\mathcal{L}^{\perp}$. Thus, the function $P(\mathbf{x})$ satisfies the conditions from (14).

According to (14) and by Lemma 5 we obtain
$$e_M(f)_{L_2(\mathbb{T}^n)}\ge C_{14}M^{-1/2}\left|\int_{\mathbb{T}^n}f(\mathbf{x})\,\omega(\mathbf{x})d\mathbf{x}\right|\ge$$
$$\ge C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=m}\prod_{j=1}^{n}2^{-(\alpha_j+1-1/p_j)s_j^0}\sum_{\mathbf{k}\in\rho(\mathbf{s_0})}1=$$
$$\ge C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\sum_{(\gamma',\mathbf{s_0})=m}\prod_{j=1}^{n}2^{-(\alpha_j-1/p_j)s_j^0}=$$
$$=C_{14}M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\left\|\left\{2^{-\left(\alpha_{j_1}-1/p_{j_1}\right)(\mathbf{1},\mathbf{s})}\right\}_{\aleph^{|D|}(\mathbf{1},\mathbf{s})}\right\|_{l_1}\asymp$$
$$\asymp M^{-1/2}m^{-\sum_{j\in D'\setminus\{j_1\}}1/\tau_j}\cdot 2^{-\left(\alpha_{j_1}-1/p_{j_1}\right)m}m^{(|D|-1)}, \tag{16}$$
where $\bar{\mathbf{s}}=(s_{j_1},\dots,s_{j_{|D|}})$.

Taking into account that $M\asymp 2^m m^{(|D|-1)}$ from (16) we have
$$e_M(f)_{L_2(\mathbb{T}^n)}\ge C_{15}2^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)m}m^{\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}=$$
$$=C_{16}M^{-\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)}(\log M)^{(|D|-1)\left(\alpha_{j_1}-1/p_{j_1}+1/2\right)+\sum_{j\in D\setminus\{j_1\}}(1/2-1/\tau_j)_+}. \tag{17}$$

By (17) lower estimate in (4) follows.

*Remark.* Note, that for $\mathbf{p}=(p,\dots,p)$, $\tau=(\tau,\dots,\tau)$ and $\mathbf{q}=\theta=(q,\dots,q)$ the statement of the theorem coincides with the corresponding result of A.S. Romanyuk [7].

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## Анизотропты Лоренц кеңістігіндегі метрикасындағы аралас метрикалы Никольский-Бесов класындағы тригонометриялық көлденеңнің реті

Тригонометриялық көлденең ұғымын бірөлшемді жағдайда алғаш рет Р.С. Исмагилов енгізді және үзіліссіз функциялар кеңістігінде бірқатар кластар үшін оларға бағалаулар белгіледі. Көп айнымалы функциялар үшін $L_q$ кеңістігінде Соболевтің $W_p^r$, Никольскийдің $H_p^r$ кластарындағы тригонометриялық көлденеңдер үшін дәл бағалауларды Э.С.Белинский, В.Е. Майоров, Ю. Маковоз, Г.Г. Магарил-Ильяев, В.Н. Темляков анықтады. Бұл есепті $B^{\mathbf{r}}_{pq}$ Бесов класы үшін А.С. Романюк, Д.Б. Базарханов зерттеді. Анизотропты Никольский-Бесов $B^{\alpha\tau}_{\mathbf{pr}}(\mathbb{T}^n)$ класы үшін тригонометриялық көлденең анизотропты Лоренц кеңістіктері метрикасында К.А. Бекмағанбетов және Е. Төлеуғазымен табылды.

*Кілт сөздер:* тригонометриялық көлденең, анизотропты Лоренц кеңістіктері, аралас метрикалы Никольский-Бесов класы.

К.А. Бекмаганбетов, К.Е. Кервенев, Е. Толеугазы

# Порядок тригонометрического поперечника класса Никольского-Бесова со смешанной метрикой в метрике анизотропного пространства Лоренца

Понятие тригонометрического поперечника в одномерном случае впервые введено Р.С. Исмагиловым и им были установлены оценки для некоторых классов в пространстве непрерывных функций. Для функций многих переменных точные порядки тригонометрических поперечников класса Соболева $W_p^r$, Никольского $H_p^r$ в пространстве $L_q$ установлены Э.С.Белинским, В.Е. Майоровым, Ю. Маковозом, Г.Г. Магарил-Ильяевым, В.Н. Темляковым. Эта задача для класса Бесова $B_{pq}^{\mathbf{r}}$ исследована А.С. Романюком и Д.Б. Базархановым. Тригонометрический поперечник для анизотропного класса Никольского-Бесова $B_{\mathbf{pr}}^{\alpha\tau}(\mathbb{T}^n)$ в метрике анизотропных пространств Лоренца $L_{\mathbf{q}\theta}(\mathbb{T}^n)$ был найден К.А. Бекмаганбетовым и Е. Толеугазы.

*Ключевые слова:* тригонометрический поперечник, анизотропные пространства Лоренца, класс Никольского-Бесова со смешанной метрикой.

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