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A note on the second order of accuracy difference scheme for elliptic-parabolic equations in Hölder spaces

The present paper is devoted to the study of a second order of accuracy difference scheme for a solution of the elliptic-parabolic equation with nonlocal boundary condition. The well-posedness of the second order of accuracy difference scheme in Hölder spaces is established. Coercivity estimates in Hölder norms for an approximate solution of a nonlocal boundary value problem for elliptic-parabolic differential equation are obtained. Results of numerical experiments are presented in order to support the aforementioned theoretical statements.

Keywords: difference scheme, elliptic-parabolic equation, Hölder spaces, well-posedness, coercivity inequalities.

Introduction

In the last decades, boundary value problems with nonlocal boundary conditions have been an important research topic in many natural phenomena. Methods and theories of solutions of the nonlocal boundary value problems for elliptic, parabolic, and mixed type differential equations have been studied extensively in a large cycle of papers (see, for example, [1–20] and the references given therein).

In paper [1], the well-posedness of the nonlocal boundary value problem

$$\begin{cases} \frac{-d^2u(t)}{dt^2} + Au(t) = g(t), 0 \leq t \leq 1, \\ \frac{du(t)}{dt} - Au(t) = f(t), -1 \leq t \leq 0, \\ u(0+) = u(0-), u^{'}(0+) = u^{'}(0-), u(1) = u(-1) + \mu \end{cases} \tag{1}$$

in Hölder spaces was determined. Furthermore, the coercivity inequalities for solutions of the nonlocal boundary value problem for elliptic-parabolic equations were obtained.

In article [2], the first order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + Au_k = g_k, g_k = g(t_k), t_k = k\tau, 1 \leq k \leq N-1, \\ \frac{u_k-u_{k-1}}{\tau} - Au_{k-1} = f_k, f_k = f(t_{k-1}), t_k = (k-1)\tau, -(N-1) \leq k \leq -1, \\ u_1 - u_0 = u_0 - u_{-1}, u_N = u_{-N} + \mu \end{cases}$$

for an approximate solution of problem (1) was constructed. Also the well-posedness of the difference scheme in Hölder spaces was proven. Moreover, coercivity estimates in Hölder norms for the solutions of difference scheme for elliptic-parabolic equations were derived.

In this study, the well-posedness of the following second order of accuracy difference scheme

$$\begin{cases} -\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+Au_k=g_k, g_k=g(t_k), \\ t_k=k\tau, 1\leq k\leq N-1, N\tau=1; \\ \frac{u_k-u_{k-1}}{\tau}-\frac{1}{2}(Au_k+Au_{k-1})=f_k, f_k=f(t_{k-\frac{1}{2}}); \\ t_{k-\frac{1}{2}}=(k-\frac{1}{2})\tau, -(N-1)\leq k\leq 0; \\ u_2-4u_1+3u_0=-3u_0+4u_{-1}-u_{-2},\ u_N=u_{-N}+\mu \end{cases} \tag{2}$$

for the approximate solution of nonlocal boundary value problem (1) in Hölder spaces is presented. In addition coercivity inequalities for solutions of difference schemes are obtained.

The rest of this paper is organized as follows. In section 2, the main theorem on well-posedness of the difference scheme (2) will be presented. In section 3, an application of the main theorem will be given. In section 4, the numerical results are presented. Finally, in section 5, the conclusion will be given.

Well-posedness of the difference scheme (2)

Throughout this paper, we have adopted the following symbols. H denotes a Hilbert space and $A=\delta I$, where $\delta>\delta_0>0$, is a self-adjoint positive definite operator. I is an identity operator, $B=\frac{1}{\tau}(\tau A+\sqrt{A(4+\tau^2A)}$ is a given self-adjoint positive definite operator and $B\geq\delta^{\frac{1}{2}}I$. In addition, $R=(I+\tau B)^{-1}$ is a bounded operator defined on the whole space H. The following operators

$$P=(I-\frac{\tau A}{2}), G=(I+\frac{\tau A}{2})^{-1}, K=(I+2\tau A+\frac{5}{4}(\tau A)^2)^{-1}$$

exist and are bounded for the self-adjoint positive operator A.

Lemma 1. The following necessary estimates for P^k, R^k and T_τ are satisfied in [3] and [4]:

$$||P^k||_{H\to H}\leq 1,\ \ ||G||_{H\to H}\leq 1,\ \ k\tau||AP^kG^2||_{H\to H}\leq M,\ k\geq 1,\ \delta>0; \tag{3}$$

$$||R^K||_{H\to H}\leq M(1+\tau B)^{-k},\ \ k\tau||BR^K||_{H\to H}\leq M,\ k\geq 1,\ \delta>0; \tag{4}$$

where A is a self-adjoint positive operator and M is independent of τ.

From these estimates it follows that

$$||(I+B^{-1}A(I+\tau A+\frac{\tau}{2}G^{-2})K(I-R^{2N-1})+K(I-\frac{\tau^2A}{2})G^{-2}R^{2N-1}- \tag{5}$$

$$-K(I-\frac{\tau^2A}{2})G^{-2}(2I+\tau B)R^NP^N)^{-1})^{-1}||_{H\to H}\leq M.$$

Here, we will study well-posedness of (2) in Hölder space. Consider $F_\tau(H)=F([a,b]_\tau,H)$ as the linear space of mesh functions $\varphi^\tau=\{\varphi_k\}_{N_a}^{N_b}$ defined on $[a,b]_\tau=\{t_k=kh, N_a\leq k\leq N_b, N_a\tau=a, N_b\tau=b\}$ with values in Hilbert space H.

Let $C([a,b]_\tau,H), C^\alpha([-1,1]_\tau,H), \tilde{C}^\alpha([-1,1]_\tau,H), \tilde{C}^{\frac{\alpha}{2}}([-1,1]_\tau,H), \tilde{C}^\alpha([0,1]_\tau,H)$ be Banach spaces with the norms

$$||\varphi^\tau||_{C([a,b]_\tau,H)}=\max_{N_a\leq k\leq N_b}||\varphi_k||_H,$$

$$||\varphi^\tau||_{C^\alpha([-1,1]_\tau,H)}=||\varphi^\tau||_{C([-1,1]_\tau,H)}+\sup_{-N\leq k<k+r\leq 0}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\frac{\alpha}{2}}+$$

$$+\sup_{1\leq k<k+r\leq N-1}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\alpha},$$

$$||\varphi^\tau||_{\tilde{C}^\alpha([-1,1]_\tau,H)}=||\varphi^\tau||_{C([-1,1]_\tau,H)}+\sup_{-N\leq k<k+2r\leq 0}||\varphi_{k+2r}-\varphi_k||_H(2r\tau)^{-\frac{\alpha}{2}}+$$

$$+\sup_{1\leq k<k+r\leq N-1}||\varphi_{k+r}-\varphi_k||_H(r\tau)^{-\alpha},$$

$$||\varphi^\tau||_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} = ||\varphi^\tau||_{C([-1,0]_\tau,H)} + \sup_{-N\le k<k+2r\le 0} ||\varphi_{k+2r}-\varphi_k||_H (2r\tau)^{-\frac{\alpha}{2}},$$

$$||\varphi^\tau||_{\tilde{C}^{\alpha}([0,1]_\tau,H)} = ||\varphi^\tau||_{C([0,1]_\tau,H)} + \sup_{1\le k<k+r\le N-1} ||\varphi_{k+r}-\varphi_k||_H (r\tau)^{-\alpha}.$$

Recall that the Banach space $E_\alpha = E_\alpha(B,H)$, where $0<\alpha<1$ consists of $v\in H$, for which the following norm is finite [5]

$$||v||_{E_\alpha} = \sup_{z>0} z^\alpha ||B(z+B)^{-1}v||.$$

The following holds for all $\beta<\alpha$:

$$D(B)\subset E_\alpha(B,H)\subset E_\beta(B,H)\subset H.$$

Theorem 1. Assume that $(I+\tau B)(f_{-N+1}+g_{N-1})\in E_\alpha$, $(I+\tau B)(f_0+g_1)\in E_{\frac{\alpha}{2}}$, and $A\mu\in E_\alpha$. Then, the solution of difference problem (2) obeys the coercivity inequalities

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} +$$
$$+\left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le$$
$$\le M_1\left\{||A\mu||_{E_\alpha} + ||(I+\tau B)(f_0+g_1)||_{E_{\frac{\alpha}{2}}} + ||(I+\tau B)(f_{-N+1}+g_{N-1})||_{E_\alpha} +\right.$$
$$\left.+\frac{1}{\alpha(1-\alpha)}\left[||f^\tau||_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]\right\}$$

and

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\{\tau^{-1}(u_k-u_{k-1})\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} +$$
$$+\left\|\{Au_k\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le$$
$$\le M_2\left\{||A\mu||_{E_\alpha} + ||(I+\tau B)(f_0+g_1)||_{E_{\frac{\alpha}{2}}} + \|(I+\tau B)(f_{-N+1}+g_{N-1})\|_{E_\alpha} +\right.$$
$$\left.+\frac{1}{\alpha(1-\alpha)}\left[\left\|\left(I+\frac{\tau A}{2}\right)f^\tau\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \|g^\tau\|_{C^\alpha([0,1]_\tau,H)}\right]\right\}.$$

Here M_1 and M_2 do not depend on f^τ, g^τ, μ, τ, and α.

Proof. By [6], we obtain

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{\tilde{C}^{\frac{\alpha}{2}}([-1,0]_\tau,H)} \le \tag{6}$$
$$\le M_1\left[\frac{1}{\alpha(1-\frac{\alpha}{2})}||f^\tau||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||Au_0||_{E_{\frac{\alpha}{2}}}\right]$$

and

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + \left\|\left\{\frac{1}{2}(Au_k+Au_{k-1})\right\}_{-N+1}^0\right\|_{C^{\frac{\alpha}{2}\le}([-1,0]_\tau,H)} \le \tag{7}$$
$$\le M_2\left[\frac{1}{\alpha(1-\frac{\alpha}{2})}\left\|\left(I+\frac{\tau A}{2}\right)f^\tau\right\|_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)} + ||Au_0||_{E_{\frac{\alpha}{2}}}\right]$$

for the solution of an inverse Cauchy difference problem

$$\begin{cases} \tau^{-1}(u_k-u_{k-1}) - \frac{A}{2}(u_k+u_{k-1}) = f_k, \\ -(N-1)\le k\le 0, \quad u_0 \text{ is given.} \end{cases}$$

By [3] and [7], we get

$$\left\|\{\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\}_1^{N-1}\right\|_{C^\alpha([0,1]_\tau,H)}+||\{Au_k\}_1^{N-1}||_{C^\alpha([0,1]_\tau,H)}\leq$$
$$\leq M\left[\frac{1}{\alpha(1-\alpha)}||g^\tau||_{C^\alpha([0,1]_\tau,H)}+||Au_0||_{E_\alpha}+||Au_N||_{E_\alpha}\right] \tag{8}$$

for the solution of boundary value problem

$$\begin{cases}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k=g_k,\\ 1\leq k\leq N-1,\ u_0,\ u_N \text{ are given.}\end{cases}$$

Then, the proof of Theorem 1 is based on coercivity inequalities (6)–(8) and estimates

$$||Au_0||_{E_\alpha}\leq M\left\{\frac{1}{\alpha(1-\frac{\alpha}{2})}\left[||f^\tau||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,H)}+|||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]+\right.$$
$$\left.+||A\mu||_{E_\alpha}+||(I+\tau B)(f_0+g_1)||_{E_\alpha}+||(I+\tau B)f_{-1}||_{E_\alpha}\right\} \tag{9}$$

and

$$||Au_N||_{E_\alpha}\leq M\left\{\frac{1}{\alpha(1-\frac{\alpha}{2})}\left[||f^\tau||_{C^\alpha([-1,0]_\tau,H)}+|||g^\tau||_{C^\alpha([0,1]_\tau,H)}\right]+\right.$$
$$\left.+||A\mu||_{E_\alpha}+||(I+\tau B)(f_0+g_1)||_{E_\alpha}\right\} \tag{10}$$

for the solution of the boundary value problem (2). Estimates (9) and (10) follow from the formulae

$$Au_0=\frac{1}{2}T\tau KG^{-2}\times\left\{(2I-\tau^2A)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu\right]-\right.\right.$$
$$-R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}(g_s-g_{N-1})\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}(g_s-g_1)\tau+$$
$$\left.+(I-R^{2N})AB^{-1}\sum_{s=1}^{N-1}R^{s-1}(g_s-g_1)\tau\}\right\}+$$
$$+(I-R^{2N})(I+\tau B)(\tau B^{-1}Ag_1-4GB^{-1}Af_0+PGB^{-1}Af_0+GB^{-1}Af_{-1})+$$
$$+(2I-\tau^2A)(2+\tau B)R^N(P^N-I)f_{-N+1}+$$
$$\left.+AB^2(R^{N-1}-I)\left[R^{N-1}g_{N-1}+(R^{2N}-R^{2N-1}-I)g_1\right]\right\}$$

and

$$Au_N=\frac{1}{2}P^NT\tau KG^{-2}\times\left\{(2I-\tau^2A)\left\{(2+\tau B)R^N\left[-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu\right]-\right.\right.$$
$$\left.-R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N-s}(g_s-g_{N-1})\tau+R^{N-1}AB^{-1}\sum_{s=1}^{N-1}R^{N+s}(g_s-g_1)\tau\right\}+$$
$$+(I-R^{2N})AB^{-1}\sum_{s=1}^{N-1}BR^{s-1}(g_s-g_1)\tau\}+$$
$$+(I-R^{2N})(I+\tau B)(\tau B^{-1}Ag_1-4GB^{-1}Af_0+PGB^{-1}Af_0+GB^{-1}Af_{-1})+$$
$$+(2I-\tau^2A)(2+\tau B)R^N(P^N-I)f_{-N+1}+$$
$$+AB^2(R^{N-1}-I)\{R^{N-1}g_{N-1}+(R^{2N}-R^{2N-1}-I)g_1\}\}-$$
$$-\tau\sum_{s=-N+1}^{0}AP^{s+N-1}G(f_s-f_{-N+1})+A\mu+(P^N-I)_{f_{-N+1}},$$

for the solution of problem (2) and estimates (3)–(5). The proof of Theorem 1 is complete.

An application of main theorem

In this section, an application of Theorem 1 is presented. Let Ω be a unit cube in the n-dimensional Euclidean space R^n $(0 < x_k < 1, 1 \le k \le n)$ with boundary S, $\bar{\Omega} = \Omega \cup S$, in $[-1,1] \times \Omega$. A nonlocal boundary value problem

$$\begin{cases} -u_{tt} - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = g(t,x), 0 < t < 1, x \in \Omega, \\ u_t + \sum\limits_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} = f(t,x), -1 < t < 0, x \in \Omega, \\ u(0+,x) = u(0-,x), u_t(0+,x) = u_t(0-,x), x \in \bar{\Omega}, \\ u(t,x) = 0, x \in S, -1 \le t \le 1, u(1,x) = u(-1,x), x \in \bar{\Omega} \end{cases} \tag{11}$$

is considered, where $a_r(x)$ $(x \in \Omega)$, $g(t,x)$ $(t \in (0,1), x \in \bar{\Omega})$, $f(t,x)$ $(t \in (-1,0), x \in \bar{\Omega})$ are given smooth functions and $a_r(x) \ge a \ge 0$ is a sufficiently large number.

The discretization of problem (11) is carried out in two steps. In the first step, the grid sets

$$\bar{\Omega}_h = \{x = x_m = (h_1 m_1, ..., h_n m_n), m = (m_1, m_2, ..., m_n),$$

$$0 \le m_r \le N, h_r N_r = 1, r = 1, ..., n\}, \Omega_h = \bar{\Omega}_h \cap \Omega, S_h = \bar{\Omega}_h \cap S$$

are defined.

We introduce the Hilbert space $L_{2h} = L_{2h}(\bar{\Omega})$ of the grid functions $\varphi^h(x) = \{\varphi(h_1 m_1, h_2 m_2, \cdots, h_n m_n)\}$ defined on $\bar{\Omega}_h$, equipped with the norm

$$||\varphi^h||_{L_{2h}} = \left(\sum_{x \in \bar{\Omega}_h} |\varphi^h(x)|^2 h_1 \cdots h_n \right)^{\frac{1}{2}},$$

and the Hilbert spaces $W_{2h}^1 = W_2^1(\bar{\Omega}_h)$, $W_{2h}^2 = W_2^2(\bar{\Omega}_h)$ defined on $\bar{\Omega}_h$, equipped with the norms

$$||\varphi^h||_{W_{2h}^1} = \left(\sum_{x \in \bar{\Omega}_h} \sum_{r=1}^{n} |(\varphi^h)_{x_r}|^2 h_1 \cdots h_n \right)^{\frac{1}{2}},$$

$$||\varphi^h||_{W_{2h}^2} = ||\varphi^h||_{L_{2h}} + \left(\sum_{x \in \bar{\Omega}_h} \sum_{r=1}^{n} |(\varphi^h)_{x_r \bar{x}_r, m_r}|^2 h_1 \cdots h_n \right)^{\frac{1}{2}}.$$

It is known that the differential expression

$$A_h^x u^h = -\sum_{r=1}^{n} (a_r(x) u_{\bar{x}_r}^h)_{x_{r,m_r}} \tag{12}$$

defines a positive operator A_h^x acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$, for all $x \in S_h$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$\begin{cases} -\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = g^h(t,x), 0 < t < 1, x \in \Omega_h, \\ \frac{du^h(t,x)}{dt} - A_h^x u^h(t,x) = f^h(t,x), -1 < t < 0, x \in \Omega_h, \\ u^h(0+,x) = u^h(0-,x), \frac{du^h(0+,x)}{dt} = \frac{du(0-,x)}{dt}, x \in \bar{\Omega}_h, \\ u^h(1,x) = u^h(-1,x), x \in \bar{\Omega}_h \end{cases} \tag{13}$$

for an infinite system of ordinary differential equations. In the second step problem (13) is replaced by the difference scheme (2) (see [7]).

$$\begin{cases} -\frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+A_h^x u_k^h(x)=g_k^h(x), \\ g_k^h(x)=g^h(t_k,x), t_k=k\tau, 1\le k\le N-1, N\tau=1, x\in\Omega_h, \\ \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau^2}-\frac{A_h^x}{2}(u_k^h(x)+u_{k-1}^h(x))=f_k^h(x), \\ f_k^h(x)=f^h(t_{k-\frac{1}{2}},x_n), t_{k-\frac{1}{2}}=(k-\frac{1}{2})\tau, -N+1\le k\le 0, x\in\Omega_h, \\ -u_2^h(x)+4u_1^h(x)-3u_0^h(x)=3u_0^h(x)-4u_{-1}^h(x)+u_{-2}^h(x), x\in\bar\Omega_h, \\ u_N^h(x)=u_{-N}^h(x), x\in\bar\Omega_h. \end{cases}$$

Theorem 2. Let τ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (11) obeys the coercivity stability estimates

$$||\{\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}+||\{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+$$
$$+||\{u_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,W_{2h}^2)}+||\left\{\frac{1}{2}(u_k^h+u_{k-1}^h)\right\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h}^2)}\le$$
$$\le M_3\Big\{||f_0^h+g_1^h||_{W_{2h}^1}+\tau||f_0^h+g_1^h||_{W_{2h}^2}+||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^1}+\tau||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^2}+$$
$$+\frac{1}{\alpha(1-\alpha)}\left[||\{f_k^h\}_{-N+1}^0||_{\tilde C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+||\{g_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}\right]\Big\},$$
$$||\{\tau^{-2}(u_{k+1}^h-2u_k^h+u_{k-1}^h)\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}+||\{\tau^{-1}(u_k^h-u_{k-1}^h)\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+$$
$$+||\{u_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,W_{2h}^2)}+||\{\frac{1}{2}(u_k^h+u_{k-1}^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h})}\le$$
$$\le M_4\Big\{||f_0^h+g_1^h||_{W_{2h}^1}+\tau||f_0^h+g_1^h||_{W_{2h}^2}+||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^1}+\tau||f_{-N+1}^h+g_{N-1}^h||_{W_{2h}^2}+$$
$$+\frac{1}{\alpha(1-\alpha)}\left[||\{f_k^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,L_{2h})}+\tau||\{f_k^h\}_{-N+1}^0||_{C^{\frac{\alpha}{2}}([-1,0]_\tau,W_{2h}^2)}+||\{g_k^h\}_1^{N-1}||_{C^\alpha([0,1]_\tau,L_{2h})}\right]\Big\},$$

where M_3 and M_4 do not depend on τ, h, α, f_k^h, $-N+1\le k\le 0$, and $g_k^h(x)$, $1\le k\le N-1$.

Applying the symmetry properties of the difference operator A_h^x acting in the space of grid functions $u^h(x)$, Theorem 1, and the theorem on coercivity of elliptic difference problem [8] conclude the proof of Theorem 2.

Numerical results

We have not been able to obtain a sharp estimate for the constants figuring in the inequalities in order to support theoretical statements. So, we will give the following results of numerical experiments of the following nonlocal boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(1-2\pi^2)e^t\sin\pi x\sin\pi y, 0<t<1,\ 0<x,y<1; \\ \frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(1-2\pi^2)e^t\sin\pi x\sin\pi y,\ \ -1<t\le 0,\ 0<x,y<1; \\ u\,(1,x,y)-u\,(-1,x,y)=(e-e^{-1})\sin\pi x\sin\pi y,\ x,y\in[0,1] \end{cases} \tag{14}$$

for a two dimensional elliptic-parabolic equation with the following Dirichlet conditions

$$\begin{cases} u\,(0-,x,y)=u\,(0+,x,y)\,,u_t\,(0-,x,y)=u_t\,(0+,x,y)\,; \\ u\,(t,0,y)=u\,(t,1,y)=0,\ y\in[0,1],\ t\in[0,1]; \\ u(t,x,0)=u(t,x,1)=0,\ x\in[0,1],\ t\in[0,1]. \end{cases} .$$

The exact solution of problem (14) is $u(t,x,y) = e^t \sin \pi x \sin \pi y$.

Now, we give the results of the numerical analysis in order to compare and conclude the accuracy of solutions for the first and second order of accuracy difference schemes. The numerical solutions are recorded for different values of N and M and $u_{n,m}^k$ represents the numerical solutions of these difference schemes at $u(t_k, x_n, y_m)$.

Table is constructed for $N = M = 10, 20, 30$, respectively and the error is computed by the following formula

$$E = \max_{-N \le k \le N, 1 \le n,m \le M-1} |u(t_k, x_n, y_m) - u_{n,m}^k|.$$

The results of the exact and numerical solutions are given in the following Table.

Table

Error analysis

Method	N=M=10	N=M=20	N=M=30
1^{st} order of accuracy d. s.	0.0938	0.0459	0.0237
2^{nd} order of accuracy d. s.	0.0122	0.0031	0.0014

Therefore, the results confirm that the second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme.

Conclusion

In the present work, the second order of accuracy difference scheme for the approximate solution of problem (1) has been presented. Also, the theorem on well-posedness of this problem in Hölder spaces has been established and the coercivity estimates for the solution of the second order difference schemes for the approximate solution of the nonlocal boundary value elliptic-parabolic problem have been constructed. Furthermore, the numerical experiments have been given. Some of results of the present article were presented in the conference proceedings [20] and [29] as extended abstracts without proofs and without numerical results of error analysis, respectively.

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Гёльдер кеңістіктеріндегі эллипстік-параболалық түрдегі теңдеулер үшін дәлдігі екінші ретті айырымдық схемалар жөнінде ескертпе

Мақала шеттік шарттары локалдық емес эллипстік-параболалық түрдегі теңдеулерді шешу үшін дәлдігі екінші ретті айырымдық схемаларды зерттеуге арналған. Дәлдігі екінші ретті айырымдық схеманың Гельдер кеңістіктерінде орнықты болатындығы көрсетілген. Шеттік шарттары локалдық емес эллипстік-параболалық түрдегі теңдеудің жуық шешімі үшін Гельдер нормасында коэрцитивті бағалаулар алынған. Теориялық тұжырымдар жұмыста келтірілген сандық есептеулермен расталды.

Кілт сөздер: айырымдық схема, эллипстік-параболалық түрдегі теңдеу, Гельдер кеңістіктері, коэрцитивті теңсіздіктер.

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Замечание о разностной схеме второго порядка точности для эллиптико-параболических уравнений в пространствах Гёльдера

Статья посвящена изучению разностной схемы второго порядка точности для решения эллиптико-параболического уравнения с нелокальным граничным условием. Установлена корректность разностной схемы второго порядка точности в пространствах Гёльдера. Получены оценки коэрцитивности в нормах Гёльдера для приближенного решения нелокальной краевой задачи для эллиптико-параболического дифференциального уравнения. Результаты численных экспериментов представлены для поддержки упомянутых выше теоретических утверждений.

Ключевые слова: разностная схема, эллиптико-параболическое уравнение, пространства Гёльдера, коэрцитивные неравенства.