

A.R. Yeshkeyev, G.A. Urken

*Ye.A. Buketov Karaganda State University, Kazakhstan  
(E-mail: guli\_1008@mail.ru)*

## Connection of Jonsson theory with some Jonsson polygons theories

The main result of this article is connected with some Jonsson theory of polygons and is devoted to obtaining a description of the characteristic of such polygons using invariants related to them. With this limited consideration (the group is a strong particular case of a monoid) were obtained for such a theory of the property of perfection and Jonsson property. Thus, for some existential complete and perfect Jonsson theories, there is the same Jonsson theory of polygons, in which two Jonsson theories are syntactically similar.

*Keywords:* Jonsson theory, semantic model, perfect theory, polygon, model companion, syntactic and semantic similarity of Jonsson theories.

In this article we will consider the connection of the Jonsson theory with a certain Jonsson theory of polygons. In particular, the Jonsson theory will be existentially complete and perfect. These restrictions are connected with the fact that the Jonsson theories are generally not complete and existential completeness is the minimum possible requirement for formulas with quantifiers. On another hand, in the imperfect case, we have very few algebraic examples to achieve any more descriptive results. The most striking example of imperfect Jonsson theory is the theory of groups, for which it has been proven that it does not have a model companion. All our previous achievements in the field of the study of Jonsson theories have been associated with the technique of working on a model companion of the theory under consideration [1–4]. The main idea that prompted us to write this article was the idea of transferring the syntactic similarity of the complete theories from [5] to the syntactic similarity of Jonsson theories [6–8]. In [5], it was shown that any complete theory is syntactically similar to a certain polygon (i.e. the theory of this polygon). In English literature, the term polygon over the  $S$  monoid usually uses the term  $S$  – acts [9–12]. In this article, we follow the terminology of Professor T.G. Mustafin, whom he first defined and formulated model-theoretical concepts and issues related to polygons [13–16].

We give a list of the necessary definitions of concepts and their necessary model-theoretical properties. The following definition belongs to T.G. Mustafin [5].

*Definition 1.* Let  $T_1$  and  $T_2$  be complete theories. We will speak, as  $T_1$  and  $T_2$  are syntactically similar, if  $f : F(T_1) \rightarrow F(T_2)$  exists bijection such that:

- 1) restriction  $f$  to  $F_n(T_1)$  is isomorphism of Boolean algebras  $F_n(T_1)$  and  $F_n(T_2)$ ,  $n < \omega$ ;
- 2)  $f(\exists v_{n+1}\varphi) = \exists v_{n+1}f(\varphi)$ ,  $\varphi \in F_{n+1}(T)$ ,  $n < \omega$ ;
- 3)  $f(v_1 = v_2) = (v_1 = v_2)$ .

Let  $T$  be an arbitrary Jonsson theory, then  $E(T) = \bigcup_{n < \omega} E_n(T)$ , where  $E_n(T)$  is a lattice of  $\exists$ -formulas with  $n$  free variables,  $T^*$  is a center of Jonsson theory  $T$ , i.e.  $T^* = Th(C)$ , where  $C$  is semantic model of Jonsson  $T$  theory in the sense of [14].

The following definition was introduced by first author of this article.

*Definition 2.* Let  $T_1$  and  $T_2$  are arbitrary Jonsson theories. We say, that  $T_1$  and  $T_2$  are Jonsson's syntactically similar, if exists a bijection  $f : E(T_1) \rightarrow E(T_2)$  such that:

- 1) restriction  $f$  to  $E_n(T_1)$  is isomorphism of lattices  $E_n(T_1)$  and  $E_n(T_2)$ ,  $n < \omega$ ;
- 2)  $f(\exists v_{n+1}\varphi) = \exists v_{n+1}f(\varphi)$ ,  $\varphi \in E_{n+1}(T)$ ,  $n < \omega$ ;
- 3)  $f(v_1 = v_2) = (v_1 = v_2)$ .

*Definition 3.* The Jonsson theory of  $T$  is called perfect if every semantic model of  $T$  is a saturated model of  $T^*$ .

*Definition 4.* By a polygon over a monoid  $S$  (or we called as  $S$  – acts) we mean a structure with only unary functions  $\langle A; f_\alpha : \alpha \in S \rangle$  such that:

- 1)  $f_e(a) = a$ ,  $\forall a \in A$ , where  $e$  is the unit of  $S$ ;
- 2)  $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ ,  $\forall \alpha, \beta \in S$ ,  $\forall a \in A$ .

*Definition 5.* A theory of  $T$  is called model complete if, for any model of  $A$  and  $B$  of theory  $T$ , such that  $A \subseteq B \Rightarrow A \preceq B$ . Equivalently, every isomorphic embedding is an elementary embedding.

Consider the concepts of model completeness, model companion and model consistency theory  $T$  [17].

*Definition 6.*  $T$  and  $T_1$  is mutually model consistent, i.e. any model of the theory  $T$  is embedded in the model of the theory  $T_1$ . And simultaneously the inverse statement is true.

*Definition 7.* Let  $T, T^*$  be  $L$ -theories.  $T^*$  is a model completion of  $T$  if:

- (1)  $T$  and  $T^*$  are mutually model consistent;
- (2)  $T^*$  is model complete;
- (3) If  $\mathfrak{M}$ , then  $T^* \cup D(\mathfrak{M})$  is complete.

$T^*$  is a model companion of  $T$  if (1) and (2) hold.

*Theorem 1* [5]. For every  $T$  there is a polygon theory  $T_\Pi$  that if the signature  $T$  is finite, then  $T_\Pi$  is a hull of  $T$ , and otherwise - almost a hull  $T$ .

*Theorem 2* [18]. Let  $T$  be Jonsson theory. Then the following conditions are equivalent:

- 1)  $T$  is perfect;
- 2)  $T$  has a model companion.

In the article [19] polygons over  $S$  was considered, where  $S$  is a group. The main result of this article is obtaining the description of characterization of such polygons with the help of concerning invariants. Under this restricted consideration (a group is strong particular case of a monoid) was obtained the properties of perfectness and Jonssonness. By an occasion we paid attention from this work with the following lemma:

*Lemma* [19]. Let  $T$  be an  $\alpha$  - Jonsson theory and all the completions of  $T$  admit the elimination of quantifiers. Then

- 1)  $T$  is perfect;
- 2)  $T$  is 0-Jonsson.

And as a consequences of this lemma for us the point 1 from the following theorem is important.

*Theorem 3* [19]. 1) Each  $\alpha$  -Jonsson theory of polygons is perfect and it is a Jonsson theory for all  $\alpha$ ,  $0 \leq \alpha \leq \omega$ .

It is not surprised because from the article [20] follows that even for regular polygon there are non model complete regular polygons. It means that there are sufficient many non saturated semantic models of polygons.

Earlier the first author of this work have got the link between both kind of similarities for existentially complete and perfect two Jonsson theories:

*Theorem 4* [21]. Let  $T_1$  and  $T_2$  be  $\exists$  - complete perfect Jonsson theories. Then following conditions are equivalent:

- 1)  $T_1$  and  $T_2$  are Jonsson's syntactically similar;
- 2)  $T_1^*$  and  $T_2^*$  are syntactically similar (in the sense of [5]).

Essentially that the centers of Jonsson theories are complete theories and we can proceed to transfer the elementary properties of centers onto Jonsson theories. Eventually it is sufficient to check implementation of similarity for Jonsson theories over lattices of existentially formulas.

The main result of this article is the following theorem.

*Theorem 5.* Let  $T$  be  $\exists$  - complete perfect Jonsson theory  $\exists$  - complete perfect then there is  $T'$  -  $\exists$  - complete perfect Jonsson theory of polygons, such that:

Theory  $T$  will be J syntactically similar to theory  $T'$ .

*Proof.* For proof, we note the following facts:

1) If Jonsson theory  $T$  is perfect, its center  $T^* = Th(C)$  is Jonsson theory, where  $C$  is a semantic model of  $T$ . The proof can be extracted from the fact that Jonsson theory is perfect if and only if its center  $T^*$  is a model companion of  $T$ .

2) If two complete theories  $T_1$  and  $T_2$  are syntactically similar to each other in the sense of [5], then they are semantically similar to each other in the sense of [5]. The proof can be extracted from [5].

3) Semantically similar theories in the sense of [5] are invariant regarding to the first order semantic properties, which include the formula properties of elements and a subset of models of these theories.

Let us recall the definition of semantic property [5].

*Definition 8.* A property (or a notion) of theories (or models, or elements of models) is called semantic if and only if it is invariant relative to semantic similarity.

For example from [5] it is known that:

*Proposition 1* [5]. The following properties and notions are semantic:

- (1) type;
  - (2) forking;
  - (3)  $\lambda$ -stability;
  - (4) Lascar rank;
  - (5) Strong type;
  - (6) Morley sequence;
  - (7) Orthogonality, regularity of types;
  - (8)  $I(\aleph_\alpha, T)$  - the spectrum function.
- 4) Being a Jonsson theory is a semantic property. The proof follows from [5, 15].

Since for any complete theory from Theorem 1 there follows the existence of some complete theory of polygons, we can consider in particular some complete Jonsson theory  $T_1^*$  that satisfies the condition of Theorem 4, i.e.  $T_1^*$  is the center of some  $\exists$ -complete perfect Jonsson theory  $T_1$ . Then  $T_1^*$  is syntactically similar in the sense of [5] to some complete theory of polygons  $T_\Pi$ . By virtue of the above facts 1–4 and Theorems 1, 4, we can conclude that  $T_\Pi$  is a complete and model complete Jonsson theory. Then there is some Jonsson theory  $T'_\Pi$  that satisfies the condition of Theorem 4 and  $T_\Pi$  is the center of the theory  $T'_\Pi$ . Then, by virtue of Theorem 4, we can conclude that  $T'_\Pi$  is the desired Jonsson theory of polygons satisfying Theorem 5, q.e.d.

All undetermined concepts in this article can be extracted from [21].

#### References

- 1 Yeshkeyev A.R. Strongly minimal Jonsson sets and their properties / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 80. — No. 4. — P. 47–51.
- 2 Poizat B. Positive Jonsson Theories / B. Poizat, A.R. Yeshkeyev // Logica Universalis. — 2018. — Vol. 12. — No. 1-2. — P. 101–127.
- 3 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 25–32.
- 4 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79. — No. 3. — P. 33–39.
- 5 Mustafin T.G. On similarities of complete theories / T.G. Mustafin // Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic. — Helsinki, 1990. — P. 259–265.
- 6 Ешкеев А.Р. О подобиях в йонсоновских теориях / А.Р. Ешкеев // Материалы 10-й Межвуз. конф. по математике и механике. — Алматы, 2005. — Т. 1. — С. 182–184.
- 7 Ешкеев А.Р. Подобие теорий предметных областей / А.Р. Ешкеев // Знания – Онтологии – Теории: (ЗОНТ-09) ИМ СО РАН: материалы Всерос. конф. — Новосибирск, 2009. — С. 207–212.
- 8 Yeshkeyev A.R. On similarities of  $\Delta - PJ$  - theories in enriched signature / A.R. Yeshkeyev // Bulletin of the Karaganda University. Mathematics series. — 2011. Vol. 62, — No. 4. — P. 54–57.
- 9 Гоулд В. Теоретико-модельные свойства свободных, проективных и плоских  $S$ -полигонов / В. Гоулд, А.В. Михалев, Е.А. Палютин, А.А. Степанова // Фундаментальная и прикладная математика. — 2008. — Т. 14, № 7. — С. 63–110.
- 10 Бунина Е.И. Элементарные свойства категории полигонов над моноидом / Е.И. Бунина, А.В. Михалев // Сиб. фонд «Алгебра и логика». — 2006. — Т. 45, № 6. — С. 687–709.
- 11 Бунина Е.И. Элементарная эквивалентность моноидов эндоморфизмов свободных полигонов / Е.И. Бунина, А.В. Михалев // Чебышевский сб. — 2005. — Т. 6, № 4. — С. 49–63.
- 12 Михалев А.В. Теоретико-модельные свойства регулярных полигонов / Е.В. Овчинникова, Е.А. Палютин, А.А. Степанова // Фундамент. и прикл. матем. — 2004. — Т. 10, № 4. — С. 107–157.
- 13 Мустафин Т.Г. Введение в прикладную теорию моделей / Т.Г. Мустафин, Т.А. Нурмагамбетов. — Караганда: Изд-во КарГУ, 1987. — 94 с.
- 14 Mustafin Y. Quelques proprietes des theories de Jonsson / Y. Mustafin // The Journal of Symbolic Logic. — 2002. — Vol. 67. — No. 2. — P. 528–536.

- 15 Мустафин Т.Г. Обобщенные условия Йонсона и описание обобщенно-йонсоновских теорий булевых алгебр / Т.Г. Мустафин // Матем. тр. — Новосибирск, 1998. — Т. 1. — № 1. — С. 135–197.
- 16 Мустафин Т.Г. О стабильной теории полигонов / Т.Г. Мустафин // Теория моделей и ее применения. — Новосибирск: Наука; Сиб. отд-ние, 1988. — С. 92–108.
- 17 Барвайс Дж. Справочная книга по математической логике: [В 4 ч.]. — Ч. 1. Теория моделей / Дж. Барвайс; пер с англ. — М.: Наука; Гл. ред. физ.-мат. лит., 1982. — 126 с.
- 18 Ешкеев А.Р. Йонсоновские теории. — Караганда: Изд-во КарГУ, 2009. — 250 с.
- 19 Мустафин Т.Г. Описание йонсоновских теорий полигонов над группой / Т.Г. Мустафин, Е.С. Нурхайдаров // Исследования в теории алгебраических систем: сб. науч. тр. (межвуз.). — Караганда: Изд-во КарГУ, 1995. — С. 67–73.
- 20 Овчинникова Е.В. Не линейно упорядоченный моноид, над которым класс регулярных полигонов полон, но не модельно полон / Е.В. Овчинникова // Алгебра и теория моделей. 3: сб. тр. — Новосибирск: Изд-во НГТУ, 2001. — С. 83–98.
- 21 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

А.Р. Ешкеев, Г.А. Уркен

## Кейбір йонсондық теория полигондарымен йонсондық теорияның байланысы

Мақаланың негізгі нәтижесі кейбір йонсондық теория полигонымен йонсондық теорияның байланысын және инварианттарға қатысты полигондардың көмегімен алынған сипаттамасы қарастырылды. Шектелген қарастыруда (группа моноидтың қатты дербес жағдайы болып табылады) теория үшін кемелділік және йонсондық қасиеттері алынды. Сонымен қатар экзистенциалды толық кемел йонсондық теориясы үшін кейбір экзистенциалды толық кемел йонсондық теорияның полигоны табылып, осы екі теорияның йонсондық синтаксистік ұқсас болатындығы көрсетілді.

*Клт сөздер:* йонсондық теория, семантикалық модель, кемел теория, полигон, модельді компаньон, йонсондық теориялардың синтаксистік және семантикалық ұқсастылығы.

А.Р. Ешкеев, Г.А. Уркен

## Связь йонсоновской теории с некоторой йонсоновской теорией полигонов

Основной результат данной статьи связан с некоторой йонсоновской теорией полигонов и посвящен получению описания характеристики таких полигонов с помощью относящихся к ним инвариантов. При этом ограниченном рассмотрении (группа является сильным частным случаем моноида) были получены для такой теории свойства совершенности и йонсоновости. Таким образом, для некоторых экзистенциально полных и совершенных йонсоновских теорий существует такая же йонсоновская теория полигонов, в которой две йонсоновские теории синтактически подобны.

*Ключевые слова:* йонсоновская теория, семантическая модель, совершенная теория, полигон, модельный компаньон, синтаксическое и семантическое подобия йонсоновских теорий.

## References

- 1 Yeshkeyev, A.R. (2015). Strongly minimal Jonsson sets and their properties. *Bulletin of the Karaganda University. Mathematics series*, Vol. 80, 4, 47–51.
- 2 Poizat, B., & Yeshkeyev, A.R. (2018). Positive Jonsson Theories. *Logica Universalis*, Vol. 12, 1-2, 101–127.
- 3 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series*, Vol. 79, 3, 25–32.
- 4 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series*, Vol. 79, 3, 33–39.
- 5 Mustafin, T.G. (1990). On similarities of complete theories. *Logic Colloquium '90. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic*. Helsinki.
- 6 Yeshkeyev, A.R. (2005). O podobiiakh v ionsonovskikh teoriiakh [On similarities in Jonsson theories]. *Materialy 10-i Mezhvuzovskoi konferentsii po matematike i mekhanike*, Vol. 1. (pp. 182–184) [in Russian].
- 7 Yeshkeyev, A.R. (2009). Podobie teorii predmetnykh oblastei [The similarity of theories of subject areas]. *Znaniia – Ontologii – Teorii: (ZONT-09) IM SO RAN: Materialy Vseros. konf.* (pp. 207–212). Novosibirsk [in Russian].
- 8 Yeshkeyev, A.R. (2011). On similarities of  $\Delta - PJ$  – theories in enriched signature. *Bulletin of the Karaganda University. Mathematics series*, Vol. 62, 2, 54–57.
- 9 Gould, V., Mikhalev, A.V., Palyutin, E.A., & Stepanova, A.A. (2008). Teoretiko-modelnye svoistva svobodnykh, proektivnykh i ploskikh  $S$ -polihonov [Model-theoretic properties of free, projective, and flat  $S$ -acts]. *Fundamentalnaia i prikladnaia matematika*, Vol. 14, 7, 63–110 [in Russian].
- 10 Bunina, E.I., & Mikhalev, A.V. (2006). Elementarnye svoistva katehorii polihonov nad monoidom [Elementary properties of the category of polygons over a monoid]. *Sib. fond «Algebra i lohika»*, Vol. 45, 6, 687–709 [in Russian].
- 11 Bunina, E.I., & Mikhalev, A.V. (2005). Elementarnaia ekvivalentnost monoidov endomorfizmov svobodnykh polihonov [Elementary equivalence of monoids of endomorphisms of free polygons]. *Chebyshevskii sbornik*, Vol. 6, 4, 49–63 [in Russian].
- 12 Mikhalev, A.V., Ovchinnikova, E.V., Paliutin, E.A., & Stepanova, A.A. (2004). Teoretiko-modelnye svoistva rehuliarnykh polihonov [Model-theoretic properties of regular polygons]. *Fundamentalnaia i prikladnaia matematika*, 10, 4, 107–157 [in Russian].
- 13 Mustafin, T.G., & Nurmagambetov, T.A. (1987). Vvedenie v prikladnuiu teoriiu modelei [Introduction to applied Model Theory]. Karaganda: Izdatelstvo KarHU [in Russian].
- 14 Mustafin, Y. (2002). Quelques proprietes des theories de Jonsson. *The Journal of Symbolic Logic*, Vol. 67, 2, 528–536.
- 15 Mustafin, T.G. (1998). Obobshchennye uslovia Yonsona i opisaniie obobshchenno-ionsonovskikh teorii bulevykh alhebr [Generalized Jonsson's conditions and a description of generalized Jonsson theories of Boolean algebras]. *Matematicheskie trudy – Mathematical Works*, Vol. 1, 1, 135–197. Novosibirsk [in Russian].
- 16 Mustafin, T.G. (1988). O stabilnoi teorii polihonov [On the stable theory of polygons]. *Teoriia modelei ee primeneniia*. Novosibirsk: Nauka; Sibirskaiia otdelenie [in Russian].
- 17 Barwise J. (Ed.) (1982). *Spravochnaia kniha po matematicheskoi lohike. Chast 1. Teoriia modelei [Handbook of mathematical logic. Part 1. Model theory]*. Moscow: Nauka [in Russian].
- 18 Yeshkeyev, A.R. (2009). Ionsonovskie teorii [Jonsson theory]. Karaganda: Izdatelstvo KarHU [in Russian].
- 19 Mustafin, T.G., & Nurkhaidarov, E.S. (1995). Opisanie ionsonovskikh teorii polihonov nad hruppoi [Description of the Jonsson polygon theories over the group]. *Issledovaniia v teorii alhebraicheskikh sistem. Sbornik nauchnykh trudov (mezhvuzovskii)*. Karaganda: Izdatelstvo KarHU [in Russian].
- 20 Ovchinnikova, E.V. (2001). Ne lineino uporiadochennyi monoid, nad kotorym klass rehuliarnykh polihonov polon, no ne modelno polon [Non linearly ordered monoid over which the class of regular polygons is complete but not model is complete]. *Algebra i teoriia modelei*. Novosibirsk: Izdatelstvo NHTU [in Russian].
- 21 Yeshkeyev, A.R., & Kasymetova, M.T. (2016). *Ionsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models]*. Karaganda: Izdatelstvo KarHU [in Russian].