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G.A. Yessenbayeva[1], A.N. Yesbayev[2], H. Poppell[3]

[1] *Ye.A. Buketov Karaganda State University, Kazakhstan;*
[2] *Nazarbayev Intellectual School, Nur-Sultan, Kazakhstan;*
[3] *Concordia International School Shanghai, Shanghai, China*
(E-mail: esenbaevagulsima@mail.ru)

On the inequality of different metrics for trigonometric polynomials

The article is devoted to the research question of inequalities for different metrics with trigonometric polynomials. The structure of this exploring, its main components and types, as well as its classical approaches are presented in this article. Nikolsky's inequalities in different metrics are well known for trigonometric polynomials. In this paper, inequalities of different metrics are proved in the Lorentz and Lebesgue spaces for trigonometric polynomials of one variable. A similar result is obtained for trigonometric polynomials of many variables. The article is focused mainly on mathematicians.

Keywords: Lebesgue spaces, Lorentz spaces, trigonometric polynomials, inequalities of different metrics.

Inequalities of different metrics play an important role among the essential attributes of mathematician's tools, exploring various mathematical structures. They are successfully used in many areas of modern theoretical and applied mathematics, so inequalities of different metrics have become an essential element of serious mathematical research, in particular research in functional analysis.

Denote by $L_p[0;2\pi)$ the space of functions $f(x)$, where the functions $f(x)$ are scalar-valued, measurable in the sense of Lebesgue on an interval $[0;2\pi)$ and integrable on $[0;2\pi)$ to the p-th degree

$$\|f\|_p = \left(\int\limits_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

for which the quantity C is finite provided that $1 \leq p < \infty$.

As usual, it is meant that in the limiting case $p = \infty$ the functions $f \in L_\infty[0;2\pi)$ are measurable and essentially limited with a finite essential maximum [1]

$$\|f\|_\infty = \sup_{x\in[0;2\pi)} vrai\, |f(x)| < \infty.$$

A function of type

$$T_m(z) = \sum_{k=-m}^{m} c_k e^{ikz}$$

is called a trigonometric polynomial of order m, where c_k $(k = -m, ..., m)$ are complex numbers and C is a variable.

The function

$$D_m(x) = \frac{1}{2} + \sum_{k=1}^{m} \cos kx = \frac{\sin\left(m + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}$$

is called the Dirichlet kernel.

Lorentz spaces $L_{p,q}$ are a more subtle scale of spaces than the scale of Lebesgue spaces L_p, and have a great use in the theory of Fourier series, in differential equations, in the theory of functional spaces.

Consider a space with a positive measure (U, μ). For a scalar-valued μ-measurable function f, which takes almost everywhere finite values, we introduce the distribution function $m(\sigma, f)$ by the formula

$$m(\sigma, f) = \mu \left\{ x : |f(x)| > \sigma \right\}.$$

For every measurable function f, we denote its non-increasing permutation by f^*, if f^* is defined by the following relation

$$f^*(t) = \inf\{\sigma : m(\sigma, f) \le t\}.$$

Lorentz spaces $L_{p,q}$ are defined as follows: a function f belongs to the space $L_{p,q}$, $1 \le p \le \infty$, if and only if

$$\|f\|_{L_{p,q}} = \left(\int_0^\infty \left(t^{\frac{1}{p}} \cdot f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,$$

when $1 \le q < \infty$,

$$\|f\|_{L_{p,\infty}} = \sup_t t^{\frac{1}{p}} \cdot f^*(t) < \infty,$$

when $q = \infty$ [2].

In the case $p = q$, the Lorentz spaces $L_{p,q}$ coincide with Lebesgue spaces L_p

$$L_{p,p} = L_p, \quad 1 \le p \le \infty.$$

Let E be a normalized functional space whose elements are defined up to equivalence with respect to Lebesgue measure. In other words, the elements of E are classes of equivalent functions, that is, almost everywhere coinciding functions. In the record $f \in E$, under the f designation we shall mean either a class of equivalent functions, or some function (a representative) of this class.

For the function $f \in E$, defined on the set $G \subset E$, the restriction f to $G^* \subset G$ is the function $f^* = f\,|_{G^*}$, defined on G^* by the equality

$$f^*(x) = f(x) \quad \forall x \in G^*.$$

Let E and F be two functional spaces. We say that E is embedded in F and write $E \subset_{\rightarrow} F$ if, firstly, all the elements of E (or of their restrictions to the domain of the elements of F) are contained in F and, secondly, there is a constant C independent of f such that the following inequality holds

$$\|f\|_F \le C\,\|f\|_E \quad \forall f \in E.$$

Theorem. Let $m \in \mathrm{N}$, $1 \le p < q \le \infty$, T_m be a trigonometric polynomial of order m, then the following inequality of different metrics holds

$$\|T_m\|_{L_{q,1}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}}, \tag{1}$$

where the parameter C is independent of m and T_m.

Proof. Applying the Jung – O.Neil inequality [3]

$$\|f * g\|_{L_{q,s}} \le C \cdot \|f\|_{L_{p,t_1}} \cdot \|g\|_{L_{r,t_2}},$$

where

$$\frac{1}{s} = \frac{1}{t_1} + \frac{1}{t_2}, \qquad \frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1, \tag{2}$$

and putting in it

$$s = 1, \quad t_1 = \infty, \quad t_2 = 1,$$

we receive the inequality

$$\|f * g\|_{L_{q,1}} \le C \cdot \|f\|_{L_{p,\infty}} \cdot \|g\|_{L_{r,1}}. \tag{3}$$

Since

$$S_M(f) = f * D_M; \tag{4}$$

$$(D_M)^*(t) \le C \cdot \min\left(\frac{1}{t}, M\right), \tag{5}$$

where S_M is the partial sum of the Fourier series, D_M is the Dirichlet kernel, and $(D_M)^*(t)$ is a non-increasing permutation [4], then denoting

$$g = D_M$$

and using estimate (5), we obtain the following relation

$$\|g\|_{L_{r,1}} = \int_0^\infty t^{\frac{1}{r}} \cdot g^*(t)\frac{dt}{t} \le \int_0^\infty t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t} =$$

$$= \int_0^{\frac{1}{M}} t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t} + \int_{\frac{1}{M}}^\infty t^{\frac{1}{r}} \cdot \min\left(\frac{1}{t},\, M\right)\frac{dt}{t}. \tag{6}$$

Taking into account that

$$\min_{0\le t\le \frac{1}{M}}\left(\frac{1}{t},\, M\right) = M, \quad \min_{\frac{1}{M}\le t\le \infty}\left(\frac{1}{t},\, M\right) = \frac{1}{t},$$

we transform the relation (6) to the form

$$\|g\|_{L_{r,1}} \le M \cdot \int_0^{\frac{1}{M}} t^{\frac{1}{r}-1}dt + \int_{\frac{1}{M}}^\infty t^{\frac{1}{r}-2}dt. \tag{7}$$

Since we have a condition $p < q$, from the relations (2) it follows that

$$\frac{1}{r} - 1 = \frac{1}{q} - \frac{1}{p} < 0,$$

and then the ratio (7) is converted to the following form

$$\|g\|_{L_{r,1}} \le M \cdot \int_0^{\frac{1}{M}} t^{\frac{1}{r}-1}dt + \int_{\frac{1}{M}}^\infty t^{\frac{1}{r}-2}dt = M \cdot \left.\frac{t^{\frac{1}{r}}}{\frac{1}{r}}\right|_0^{\frac{1}{M}} + \left.\frac{t^{\frac{1}{r}-1}}{\frac{1}{r}-1}\right|_{\frac{1}{M}}^\infty =$$

$$= rM \cdot \left(\frac{1}{M}\right)^{\frac{1}{r}} - \frac{r}{1-r}\left(\frac{1}{M}\right)^{\frac{1}{r}-1} =$$

$$= rM^{1-\frac{1}{r}} - \frac{r}{1-r}M^{1-\frac{1}{r}} = \frac{r^2}{r-1} \cdot M^{1-\frac{1}{r}} = C \cdot M^{1-\frac{1}{r}}$$

or

$$\|g\|_{L_{r,1}} \le C \cdot M^{1-\frac{1}{r}} = C \cdot M^{\frac{1}{p}-\frac{1}{q}}. \tag{8}$$

Using (4) and (8), from (3) we have an inequality of the form

$$\|f * g\|_{L_{q,1}} = \|S_M(f)\|_{L_{q,1}} \le C \cdot M^{\frac{1}{p}-\frac{1}{q}} \cdot \|f\|_{L_{p,\infty}}.$$

The last inequality holds for any functions$f \in L_{p,\infty}$.

Therefore, if we take f as a trigonometric polynomial of order m, that is, $f = T_m$, and, taking into account that $S_m(T_m) = T_m$, we obtain the sought-for inequality

$$\|T_m\|_{L_{q,1}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}}.$$

The theorem is proved.

Remark. We note that the inequality of different metrics from the theorem is more accurate than the classical Nikolsky's inequality of different metrics for trigonometric polynomials of order m [1]

$$\|T_m\|_{L_q} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}, \tag{9}$$

because

$$L_{q,1} \subset_{\rightarrow} L_{q,q} = L_q, \quad L_p = L_{p,p} \subset_{\rightarrow} L_{p,\infty},$$

that is,

$$\|T_n\|_{L_q} \le C \cdot \|T_n\|_{L_{q,1}}, \qquad \|T_m\|_{L_{p,\infty}} \le C \cdot \|T_m\|_{L_p}.$$

Corollary. Let $m \in$ N, $1 \le p < q \le \infty$, $1 \le r \le \infty$, then

$$\|T_m\|_{L_{q,r}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}, \tag{10}$$

where the parameter C is independent of m and T_m.

Proof. Since, under condition $r < r_1$, the relation is satisfied for Lorentz spaces

$$\|f\|_{L_{p,r}} \le C \cdot \|f\|_{L_{p,r_1}},$$

then from (1) we obtain (10)

$$\|T_m\|_{L_{q,r}} \le \|T_m\|_{L_{q,1}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,\infty}} \le$$

$$\le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_{p,p}} \le C \cdot m^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_m\|_{L_p}.$$

The required inequality is proved.

S.M. Nikolsky has obtained for any trigonometric polynomial from R^n the inequality, similar to the relation (9),

$$\|T_{m_1 \dots m_n}\|_q \le 2^n \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 \dots m_n}\|_p, \quad 1 \le p < q \le \infty,$$

where an arbitrary trigonometric polynomial of order $m_1, ..., m_n$ with variables $x_1, ..., x_n$ can be written in the form

$$T_{m_1 \dots m_n}(x_1, ..., x_n) = \sum_{k_1=-m_1}^{m_1} \dots \sum_{k_m=-m_n}^{m_n} c_{k_1 \dots k_n} e^{i \sum_{s=1}^{n} k_s x_s},$$

and L_p^* is the space of functions $f(x_1, ..., x_n)$, which are measurable in R^n, periodic with period 2π with respect to each of the variables x_i and integrable to the p-th power ($1 \le p < \infty$) on the period. Thus, for each function $f \in L_p^*$ the following relation takes place [1]

$$\|f\|_p^* = \left(\int_0^{2\pi} \dots \int_0^{2\pi} |f(x_1, ..., x_n)|^p \, dx_1 \dots dx_n\right)^{\frac{1}{p}} < \infty,$$

in the case of $p = \infty$, we have

$$\|f\|_\infty^* = \sup_{x_i} vrai \, |f(x_1, ..., x_n)|.$$

After conducting a similar proof, for any trigonometric polynomial in R^n, we obtain inequalities similar to relations (1) and (10),

$$\|T_{m_1 \dots m_n}\|_{L_{q,1}^*} \le 2^n \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 .. m_n}\|_{L_{p,\infty}^*}, \quad 1 \le p < q \le \infty;$$

$$\|T_{m_1 \dots m_n}\|_{L_{q,r}^*} \le C \cdot \left(\prod_{k=1}^{n} m_k\right)^{\frac{1}{p}-\frac{1}{q}} \cdot \|T_{m_1 .. m_n}\|_{L_p^*}, \quad 1 \le p < q \le \infty, \quad 1 \le r \le \infty,$$

where the parameter C does not depend on m and T_m.

Here, the space $L_{p,q}^*[0, 2\pi]^n$ is defined as the set of functions for which the inequality holds

$$\|f\|_{L_{p,q}^*} = \left(\int_0^{2\pi} \left(t^{\frac{1}{p}} \cdot f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

and $f^*(t) = f^{*_1, \dots, *_n}(t_1, ..., t_n)$ denotes a function obtained by applying a non-increasing permutation sequentially in variables $x_1, ..., x_n$ with fixed other variables and this function is called a non-increasing permutationof a measurable function $f(x_1, ... x_n)$ in $[0, 2\pi]^n$ [5].

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Г.А. Есенбаева, А.Н. Есбаев, Х. Поппелл

Тригонометриялық көпмүшеліктер үшін түрлі метрикадағы теңсіздіктер жайлы

Мақала тригонометриялық көпмүшеліктер үшін түрлі метриканың теңсіздіктерін зерттеуге арналған. Авторлар осы зерттеудің құрылымын берген, оның негізгі компонеттері мен түрлері, сонымен қатар оның классикалық тәсілдерін көрсеткен. Тригонометриялық көпмүшеліктер үшін Никольскийдің әртүрлі метрикадағы теңсіздіктері жақсы белгілі. Осы жұмыста Лоренц және Лебег кеңістіктеріндегі біртүрлі айнымалыдан тұратын тригонометриялық көпмүшеліктер үшін түрлі метрикалардың теңсіздіктері дәлелденген. Аналогиялық нәтиже ретінде бірнеше айнымалылардан тұратын тригонометриялық көпмүшеліктер үшін алынған. Мақала негізінен математиктерге арналған.

Кілт сөздер: Лебег кеңістігі, Лоренц кеңістігі, тригонометриялық көпмүшеліктер, түрлі метрикадағы теңсіздіктер.

Г.А. Есенбаева, А.Н. Есбаев, Х. Поппелл

О неравенстве разных метрик для тригонометрических полиномов

Статья посвящена вопросу исследования неравенств разных метрик для тригонометрических полиномов. Авторами представлены структура данного исследования, его основные компоненты и виды, а также его классические подходы. Для тригонометрических полиномов хорошо известны неравенства Никольского в разных метриках. В данной работе доказаны неравенства разных метрик в пространствах Лоренца и Лебега для тригонометрических полиномов одного переменного. Аналогичный результат получен для тригонометрических полиномов многих переменных. Статья ориентирована, главным образом, на математиков.

Ключевые слова: пространства Лебега, пространства Лоренца, тригонометрические полиномы, неравенства разных метрик.

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