UDC 512.54

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On the automorphism groups of relatively free groups of infinite rank: a survey

The paper is intended to be a survey on some topics within the framework of automorphisms of a relatively free groups of infinite rank. We discuss such properties as *tameness, primitivity, small index, Bergman property*, and so on.

Key words: variety of groups, relatively free group, countably infinite rank, automorphism group, tame automorphism, small index property, cofinality, Bergman property.

Introduction

Let F_∞ be a free group of infinite rank, in particular, let F_ω be a free group of countably infinite rank. In further of the paper $X_\omega = \{x_1, ..., x_i, ...\}$ be a basis of F_ω, and $X_n = \{x_1, ..., x_n\}$ be a basis of a free group F_n of any finite rank n. Thus F_n is naturally embedded into F_ω, and F_n is naturally embedded into every group F_m where $m \geq n$. More generally, let Λ be a set (finite or infinite) and F_Λ be the free group of rank $|\Lambda|$ with basis $X_\Lambda = \{x_\lambda : \lambda \in \Lambda\}$. Then for each subset Ξ of Λ free group F_Ξ is a subgroup of F_Λ, and for every $\Psi, \Xi \subseteq \Psi \subseteq \Lambda$, free group F_Ξ is a subgroup of F_Ψ.

For any variety of groups $\mathcal{C}$, let $V = \mathcal{C}(F_\Lambda)$ denote the verbal subgroup of F_Λ corresponding to $\mathcal{C}$ (see [1] for information on varieties and related concepts.). Then $G_\Lambda = F_\Lambda/V$ is the free group of rank $|\Lambda|$ in $\mathcal{C}$. In particular, G_ω is the free group of countably infinite rank in $\mathcal{C}$. Write $\bar{x}_i = x_i V$ for $i = 1, ..., i, ...$. Then $X_V = \{\bar{x}_1, ..., \bar{x}_i, ...\}$ is a basis of G_ω. For each Λ there is the standard homomorphism of F_Λ onto G_Λ. Then for each subset Ξ of Λ free group G_Ξ is a subgroup of G_Λ, and for every $\Psi, \Xi \subseteq \Psi \subseteq \Lambda$, free group G_Ξ is a subgroup of G_Ψ.

If α is an automorphism of G_Λ then $\{\alpha(\bar{x}_\lambda) : \lambda \in \Lambda\}$ is also a basis of G_Λ and every basis of G_Λ has this form.

Any automorphism ϕ of F_Λ induces an automorphism $\bar{\phi}$ of G_Λ. Thus every basis of F_Λ induces a basis of G_Λ. The converse however is not always true; in general, there are automorphisms of G_Λ which are not induced by automorphisms of F_Λ. See [2, 3] for relevant results.

An automorphism of G_Λ which is induced by an automorphism of F_Λ is called *tame.* If $\{g_\xi : \xi \in \Xi\}$ are distinct elements of G_Λ such that $\{g_\xi : \xi \in \Xi\}$ is contained in a basis of G_Λ then $\{g_\xi : \xi \in \Xi\}$ is called a *primitive system* of G_Λ.

Any primitive system $\{f_\xi : \xi \in \Xi\}$ of F_Λ induces a primitive system of G_Λ that is called *tame.* But, in general, not every primitive system of G_Λ is induced by a primitive system of F_Λ. We observe different tameness properties in the following Section 2.

An other topic of this paper is small index property. A countable first-order structure M is said to have the small index property if every subgroup of the automorphism group $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$ contains the pointwise stabilizer $\mathrm{C}(U)$ of a finite subset U of the domain of M. In Section 3, we give results about small index property for relatively free groups of countably infinite rank.

Further in the paper, $\mathcal{A}$ denotes the variety of all abelian groups, $\mathcal{N}_k$ means the variety of all nilpotent groups of class $\leq k$, and $\mathcal{A}^2$ stands for the variety of all metabelian groups (for any varieties $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{CD}$ denotes the variety of all groups with a normal subgroup in $\mathcal{C}$ and factor group in $\mathcal{D}$, thus $\mathcal{A}^2 = \mathcal{AA}$). We also denote by $A_\infty, N_{k,\infty}$ and M_ω the free groups of the countably infinite rank in the varieties $\mathcal{A}, \mathcal{N}_k$ and $\mathcal{A}^2$, respectively. For any group H and each positive integer k we denote by $\gamma_k(H)$ the kth member of the low central series in H. In particular, $\gamma_1(H) = H$ and $\gamma_2(H) = H'$, the derived subgroup of H.

Tameness

In this section, we present known results about tame automorphisms of relatively free groups of infinite rank. It is well known that every automorphism of A_∞ can be lifted to an automorphism of F_∞, thus is tame. The following results also belong to this direction.

Theorem 1 (Bryant and Macedonska [4]). Let F_∞ be a free group of infinite rank and let V be a characteristic subgroup of F_∞ such that F_∞/V is nilpotent. Then $G_\infty = F_\infty/V$ is relatively free group of infinite rank in a nilpotent variety and every automorphism of G_∞ is induced by an automorphism of F_∞, thus is tame.

If F_∞ and V are as in the statement of the theorem then V contains $\gamma_k(F_\infty)$ for some positive integer k. Since V is characteristic in F, it follows by a result of Cohen [5] that V is fully characteristic in F_∞. Thus V defines nilpotent variety $\mathcal{N}$ of groups, then $V = \mathcal{N}(F_\infty)$ is the corresponding verbal subgroup of F_∞. Hence F_∞/V is a relatively free group in $\mathcal{N}$.

To prove Theorem 1 the authors defined a property called the *finitary lifting property* (see details below) and obtained the following two results.

Proposition 1. Every nilpotent variety of groups has the finitary lifting property.

Proposition 2. If $\mathcal{C}$ is any variety of groups with the finitary lifting property and F_∞ is a free group of infinite rank then every automorphism of $F_\infty/\mathcal{C}(F_\infty)$ is induced by an automorphism of F_∞.

Let F_∞ be a free group of infinite rank and let $\{x_\lambda : \lambda \in \Lambda\}$ be a basis of F_∞. An automorphism ϕ of F_∞ will be called *finitary* if there is a finite subset U of this basis such that $\phi(x) = x$ for each free generator $x \notin U$. Let $\mathcal{C}$ be a variety of groups and write $V = \mathcal{C}(F_\infty)$. Suppose that Γ and Δ are subsets of Λ such that $\Gamma \cap \Delta = \emptyset$, Δ is finite, and $\Lambda \setminus (\Gamma \cup \Delta)$ is infinite. Let α be automorphism of F_∞/V such that $\alpha(x_\lambda V) = x_\lambda V$ for all $\lambda \in \Gamma$. We say that the triple (Γ, Δ, α) can be lifted if there exists a finitary automorphism ξ of F_∞ such that $\xi(x_\lambda) = x_\lambda$ for all λ in Γ and $\xi(x_\lambda V) = \alpha(x_\lambda V)$ for all $\lambda \in \Delta$. Such a finitary automorphism ξ is called a *lifting* of (Γ, Δ, α). We say that $\mathcal{C}$ has the *finitary lifting property* if, for every F_∞ of infinite rank, every triple (Γ, Δ, α) can be lifted.

The theorem generalises some previously known results. The case where $V = \gamma_2(F_\infty)$ is a result of Swan (see [6]). A closely related result had been obtained a few years earlier by Burns and Farouqi [7] who proved that if $A_\omega(p)$ is a free abelian of exponent p group of countably infinite rank and p is a prime number then every automorphism of $A_\omega(p)$ is induced by an automorphism of A_ω. In [8], Gawron and Macedonska proved the discussed property in the cases $V = \gamma_i(F_\omega)$ for $i = 3, 4$.

For each positive integer m, we denote by $\mathcal{A}(m)$ the variety of all abelian groups of exponent dividing m. Also we denote by $\mathcal{A}(0)$ the variety of all abelian groups $\mathcal{A}$.

Theorem 2 (Bryant and Groves [9]). Let m and n be non-negative integers. Every automorphism a free group of infinite rank in the metabelian product variety $\mathcal{A}(m)\mathcal{A}(n)$ is tame. In particular, every automorphism of M_∞ is tame.

Theorem 3 (Bryant and Gupta [10]). Let $\mathcal{C}$ be a variety such that $\mathcal{A}^2 \subseteq \mathcal{C} \subseteq \mathcal{N}_k\mathcal{A}$ for some k, and G_∞ be a free group of infinite rank in $\mathcal{C}$. Then every automorphism of G_∞ is tame.

The following result generalizes Theorems 1, 2 and 3.

Theorem 4 (Bryant and Roman'kov [11]). Let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$ for some k. Let G_∞ be a free group of infinite rank in $\mathcal{C}$. Then every automorphism of G_∞ is tame.

The main ingredient in the proof of this theorem are the following result that has its own interest.

Theorem 5 (Bryant and Roman'kov [11]). Let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$, where $k \geq 1$. Let n be a positive integer and write $l = 2^n(n+1) + 2k$. Then every primitive system of $F_n/\mathcal{C}(F_n)$ is induced by some primitive system of F_l.

Above we presented some positive results about tameness of the automorphisms of relatively free groups of infinite rank. However, there are negative results for other varieties.

Theorem 6 (Bryant and Groves [12]). Let $\mathcal{K} = \mathrm{var}(K)$ be the variety generated by a non-abelian finite simple group K, and G_∞ is the free group of the countable infinite rank in $\mathcal{K}$. Then there is an automorphism of G_∞ which is not induced by an automorphism of F_∞.

Small index property

Hodges, Hodkinson, Lascar and Shelah established in [13] that ω-categorical and ω-stable structures, and so called random graph have the small index property. In [14], Bryant and Evans use the methods of the paper [13] to show that the free group of countably infinite rank and certain relatively free groups of countably infinite rank have the small index property.

Theorem 7 has some immediate consequences through the results of [14] and [15].

Theorem 8 (Bryant and Roman'kov [11]). Let F_ω be a free group of countably infinite rank and let $\mathcal{C}$ be a subvariety of $\mathcal{N}_k\mathcal{A}$, where $k \geq 1$. Then $F_\omega/\mathcal{C}(F_\omega$ has the basis cofinality property and the small index property. The automorphism group $\mathrm{Aut}(F_\omega/\mathcal{C}(F_\omega)$ is not the union of a countable chain of proper subgroups. Also, $\mathrm{Aut}(F_\omega/\mathcal{C}(F_\omega)$ has no proper normal subgrou of index less than $2^{\aleph_0}$ and it is a perfect group.

Recall that a group is called *perfect* if it equals its derived subgroup.

Other properties

Completeness. A group G is said to be complete if G is centreless and every automorphism of G is inner. By the Burnside's criterion for a centerless group G its the automorphism group $\mathrm{Aut}(G)$ is complete if and only if the subgroup $\mathrm{Inn}(G)$ of all inner automorphisms of G is a characteristic subgroup of the group $\mathrm{Aut}(G)$ (that is, preserved under the action of all automorphisms of the group $\mathrm{Aut}(G)$).

Theorem 9 (Tolstykh [16]. The automorphism group $\mathrm{Aut}(F_\infty)$ of any free group of infinite rank is complete.

This statement was derived from the following assertions:

• The family of all inner automorphisms of F_∞ determined by powers of primitive elements of F_∞ is first-order definable in $\mathrm{Aut}(F_\infty)$, hence $\mathrm{Inn}(F_\infty)$ is a characteristic subgroup of $\mathrm{Aut}(F_\infty)$.

• The subgroup $\mathrm{Inn}(F_\infty)$ is then first-order definable in $\mathrm{Aut}(F_\infty)$.

Theorem 10 (Tolstykh [17, 18]). For any $k \geq 2$, the automorphism group $\mathrm{Aut}(N_{\infty,k})$ of any free nilpotent group $N_{\infty,k}$ of infinite rank is complete.

Note, that $\mathrm{Inn}(\mathrm{Aut}(A_\infty)) = \mathrm{Aut}(\mathrm{Aut}(A_\infty))$ ([19], [20]). Anyway $\mathrm{Aut}(A_\infty)$ is not complete because it contains a non-central the inverting authomorphism.

In [17], this statement was proved for the case $k = 2$, and in [18], for the general case.

Theorem 11 (Tolstykh [21]). Let F_∞ be an infinitely generated free group, $R \leq F'_\infty$ a fully characteristic subgroup of F_∞ such that the quotient group F_∞/R is residually torsion-free nilpotent. Then the group $\mathrm{Aut}(F_\infty/R)$ is complete.

Corollary. Let G_∞ be a free abelian-by-nilpotent (in particular metabelian or free solvable of class ≥ 3) relatively free group of infinite rank. Then the group $\mathrm{Aut}(G_\infty)$ is complete.

Generalized small index property. Let F be a relatively free algebra of infinite rank κ. We say that F has the *generalized small index property* if any subgroup of $\mathrm{Aut}(F)$ of index at most κ contains the pointwise stabilizer $\mathrm{C}(U)$ of a subset U of the domain of F of cardinality less than κ.

Theorem 12 (Tolstykh [22] Every infinitely generated free nilpotent (in particular free abelian) group N_∞ has the generalized small index property.

Bergman property. A group G is said to have the *Bergman property* (the property of *uniformity of finite width*) if given any generating X with $X = X^{-1}$ of G, we have that $G = X^l$ for some natural l, that is, every element of G is a product of at most l elements of X. The property is named after Bergman, who found in [23] that it is satisfied by all infinite symmetric groups. The first example of an infinite group with the Bergman property was apparently found by Shelah in the 1980s.

Theorem 13 (Tolstykh [24]). The automorphism group $\mathrm{Aut}(F_\omega)$ of the free group F_ω of countably infinite rank has the Bergman property.

Theorem 14 (Tolstykh [24]). For any positive integer k, the automorphism group $\mathrm{Aut}(N_{\infty,k})$ of any free nilpotent group $N_{\infty,k}$ of infinite rank has the Bergman property.

Some other discussion on the automorphism groups of free relatively free groups can be found in survey [25].

Acknowledgments

Supported by Russian Foundation for Basic Research, project 16-01-00577.

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В.А. Романьков

Шексіз рангті еркін группаларға қатысты автоморфизмдер группалары жайында: шолу

Мақаланың мақсаты шексіз рангті еркін группаларға қатысты автоморфизмдер тобын зерттеудің кейбір мәселелерін шолу болып табылады. Автоморфизмнің «нұсқаулы» болуы, қарабайырлық, кіші индекс, Бергман қасиеті және тағы басқадай қасиеттері жан-жақты талқыланады.

Кілт сөздер: группалардың көпбейнелілігі, салыстырмалы еркін топ, санамалы шексіз ранг, автоморфизмдер группасы, «нұсқаулы» автоморфизм, кіші индекс қасиеті, кофиналдылық, Бергман қасиеті.

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О группах автоморфизмов относительно свободных групп бесконечного ранга: обзор

Целью статьи является обзор некоторых вопросов исследований групп автоморфизмов относительно свободных групп бесконечного ранга. Обсуждаются такие свойства, как быть «ручным» автоморфизмом, примитивность, свойства малого индекса, Бергмана и т.п.

Ключевые слова: многообразие групп, относительно свободная группа, счетный бесконечный ранг, группа автоморфизмов, «ручной» автоморфизм, свойство малого индекса, кофинальность, свойство Бергмана.

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