DOI 10.31489/2019M1/72-77
UDC 517.95

M.T. Kosmakova, D.M. Akhmanova, S.A. Iskakov, Zh.M. Tuleutaeva, L.Zh. Kasymova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: Svetik_mir69@mail.ru)

Solving one pseudo-Volterra integral equation

In this paper, we study the solvability of a second-kind pseudo-Volterra integral equation. By replacing the right-hand side and the unknown function, the integral equation is reduced to an integral equation, the kernel of which is not «compressible». Using the Laplace transform, the obtained equation is reduced to an ordinary first-order differential equation (linear). Its solution is found. The solution of the homogeneous integral equation corresponding to the original nonhomogeneous integral equation found in explicit form. Special cases of a homogeneous integral equation and its solutions are written for different values of the parameter k. Classes are indicated in which the integral equation has a solution. Singular integral equations were considered in works [1–3]. Their kernels were also «incompressible», but kernels had an another form. In this connection, the weight classes of the solution existence differ from the class of the solution existence for the equation considered in this work.

Keywords: kernel, integral operator, class of essentially bounded functions, Laplace transformation.

Introduction

This paper is devoted to the research of questions of solvability of the following pseudo-Volterra integral equation of the second kind

$$\nu(t)-\frac{a}{2\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t}\,\sqrt{\tau}\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau-\frac{1}{k\,a\sqrt{\pi}}\int_0^t\sqrt{\frac{\tau}{t}}\frac{1}{\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau=f(t), \tag{1}$$

where a, k — are positive constants, $f(t)$ — is the given function.

A similar kind of integral equation arises in solving the boundary value problems of heat conduction with heat generation, which describe the development of the one-dimensional unsteady heat processes with axial symmetry.

1 Reducing the equation (1) to a differential equation in images

We rewrite the equation (1) in the form

$$\frac{\nu(t)}{\sqrt{t}}e^{\frac{t}{4a^2}}-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{t\sqrt{t-\tau}}\cdot\frac{1}{\sqrt{\tau}}e^{\frac{\tau}{4a^2}}\cdot\nu(\tau)d\tau-$$
$$-\frac{1}{k\,a\sqrt{\pi}}\int_0^t\frac{\tau}{t\sqrt{t-\tau}}\left\{\frac{1}{\sqrt{\tau}}e^{\frac{\tau}{4a^2}}\cdot\nu(\tau)\right\}d\tau=\frac{f(t)}{\sqrt{t}}e^{\frac{t}{4a^2}}. \tag{2}$$

After replacements:

$$\frac{1}{\sqrt{t}}e^{\frac{t}{4a^2}}\nu(t)=\nu_1(t),\quad \frac{1}{\sqrt{t}}e^{\frac{t}{4a^2}}f\,(t)=f_1(t) \tag{3}$$

equation (2) takes the form

$$\nu_1(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{t\sqrt{t-\tau}}\cdot\nu_1(\tau)d\tau-\frac{1}{k\,a\sqrt{\pi}}\int_0^t\frac{\tau}{t\sqrt{t-\tau}}\,\nu_1(\tau)\,d\tau=t\,f_1(t);$$

or

$$t\cdot\nu_1(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\cdot\nu_1(\tau)d\tau-\frac{1}{k\,a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\cdot\tau\,\nu_1(\tau)\,d\tau=f_2(t), \tag{4}$$

where $f_2(t)=t\,f_1(t)$.

We note that the integral operator acting in the class of continuous functions $\nu_1(t) \in C(0;\ +\infty)$, of an equation with a kernel:

$$\mathrm{K}(t,\tau) = \frac{a}{2\sqrt{\pi}}\frac{1}{t\sqrt{t-\tau}} + \frac{1}{k\,a\sqrt{\pi}}\frac{\tau}{t\sqrt{t-\tau}}$$

is not bounded.

To equation (4) we apply the Laplace transform, introducing the notation

$$\hat{\nu}_1(p) = \int_0^\infty \nu_1(t)\ \cdot e^{-pt}dt \ \ \Leftrightarrow\ \ \nu_1(t) \div \hat{\nu}_1(p);$$

$$\hat{f}_2(p) = \int_0^\infty f_2(t)\ \cdot e^{-pt}dt \ \ \Leftrightarrow\ \ f_2(t) \div \hat{f}_2(p),$$

that is

$$\hat{\nu}_1(p) = L\left[\nu_1(t)\right];$$

$$\hat{f}_2(p) = L\left[f_2(t)\right].$$

Since

$$\int_0^\infty \frac{1}{\sqrt{t}}\,e^{-pt}dt = \left\| \begin{array}{l} pt = z^2, \\ dt = \frac{2z}{p}\,dz \end{array} \right\| = \frac{2}{\sqrt{p}}\int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{\sqrt{p}};$$

$$t \cdot \nu_1(t) \div -\hat{\nu}_1'(p),$$

integral equation (4) becomes a differential equation in the image space

$$-\hat{\nu}_1'(p) - \frac{a}{2\sqrt{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{p}}\,\hat{\nu}_1(p) - \frac{1}{k\,a\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{\sqrt{p}}\cdot(-\hat{\nu}_1'(p)) = \hat{f}_2(p),$$

which can be rewritten as

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right]\hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = \hat{f}_2(p). \tag{5}$$

2 Solving a homogeneous linear differential equation

We solve a homogeneous equation that corresponds to a linear equation (5)

$$\left[\frac{1}{k\,a\sqrt{p}} - 1\right]\hat{\nu}_1'(p) - \frac{a}{2\sqrt{p}}\ \hat{\nu}_1(p) = 0. \tag{6}$$

The solution to differential equation (6) has the form:

$$\hat{\nu}_1(p) = \frac{Ce^{1/k}}{(ka)^{1/k}}\cdot\frac{e^{-a\sqrt{p}}}{\left(\sqrt{p} - \frac{1}{ka}\right)^{1/k}}, \tag{7}$$

where $\mathcal{C} - const$.

Since from formula No. 149 [4; 272] and from formula No. 9 [4; 259] we have

$$\frac{e^{-as}}{\left(s - \frac{1}{ka}\right)^{1/k}} = L\left[\frac{\tau^{\frac{1}{k}-1}}{\Gamma\left(\frac{1}{k}\right)}e^{\frac{\tau}{ka}}\right],$$

then, taking into account formula No. 29 from [4; 261] and applying the inverse Laplace transformation to (7), we obtain

$$\nu_1(t) = \frac{C}{\Gamma\left(\frac{1}{k}\right)\ (ka)^{1/k}}\int_a^\infty \frac{\tau(\tau-a)^{\frac{1}{k}-1}}{2\sqrt{\pi}\,t^{3/2}}e^{-\frac{\tau^2}{4t}}\cdot e^{\frac{\tau-a}{ka}}d\tau =$$

$$= \frac{Ce^{-\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)\ (ka)^{1/k}}\frac{1}{2\sqrt{\pi}\,t^{3/2}}\int_a^\infty \tau(\tau-a)^{\frac{1}{k}-1}e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau.$$

Introducing the notation

$$I(t,k)=\int_a^{\infty}\tau(\tau-a)^{\frac{1}{k}-1}e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau,$$

we get

$$\nu_1(t)=\frac{Ce^{-\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)\,(ka)^{1/k}}\frac{1}{2\sqrt{\pi}\,t^{3/2}}I(t,k). \tag{8}$$

We calculate the integral

$$I(t,k)=\int_a^{\infty}\tau(\tau-a)^{\frac{1}{k}-1}e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau=$$

$$=\int_a^{\infty}(\tau-a)^{\frac{1}{k}}\;e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau+a\int_a^{\infty}(\tau-a)^{\frac{1}{k}-1}\;e^{-\frac{\tau^2}{4t}+\frac{\tau}{ka}}d\tau.$$

Taking into account the formula 2.3.15 (1) from [5], we obtain

$$I(t,k)=\Gamma\left(\frac{1}{k}+1\right)\left(\frac{1}{2t}\right)^{-\frac{1}{2k}-\frac{1}{2}}\exp\left\{\frac{t}{2k^2a^2}+\frac{1}{2k}-\frac{a^2}{8t}\right\}D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+$$

$$+a\,\Gamma\left(\frac{1}{k}\right)\left(\frac{1}{2t}\right)^{-\frac{1}{2k}}\exp\left\{\frac{t}{2k^2a^2}+\frac{1}{2k}-\frac{a^2}{8t}\right\}\;D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)=$$

$$=\Gamma\left(\frac{1}{k}\right)\exp\left\{\frac{1}{2k}\right\}\;(2t)^{\frac{1}{2k}}\exp\left\{\frac{t}{2k^2a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right].$$

Substituting $I(t,k)$ into expression (8), we obtain the general solution of the homogeneous equation that corresponds to the integral equation (4)

$$\nu_1(t)=\frac{Ce^{\frac{1}{k}}}{(ka)^{1/k}}\frac{(2t)^{\frac{1}{2k}}}{2\sqrt{\pi}\,t^{3/2}}\exp\left\{\frac{t}{2k^2a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right],$$

where [6] (see formula 9.241(2))

$$D_{-p}(z)=\frac{e^{-\frac{z^2}{4}}}{\Gamma\left(p\right)}\int_0^{+\infty}e^{-zx-\frac{x^2}{2}}x^{p-1}dx,\quad Rep>0 \tag{9}$$

are Parabolic cylinder functions (Weber functions).

Using the replacement that is inverse to (3), we get

$$\nu(t)=\frac{Ce^{\frac{1}{k}}}{(ka)^{\frac{1}{k}}}\frac{(2t)^{\frac{1}{2k}}}{2\sqrt{\pi}t}\;\exp\left\{\frac{t}{2k^2a^2}-\frac{t}{4a^2}-\frac{a^2}{8t}\right\}\times$$

$$\times\left[\frac{1}{k}\sqrt{2t}\,D_{-\left(\frac{1}{k}+1\right)}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)+a\,D_{-\frac{1}{k}}\left(\frac{ka^2-2t}{ak\sqrt{2t}}\right)\right]. \tag{10}$$

(10) is the general solution of the homogeneous integral equation that corresponds to the initial equation (1).

3 Case $k = 2$

From a practical point of view, the case $k = 2$ is interesting

$$\nu(t) = \frac{C\sqrt{e}}{\sqrt{2a\pi}(2t)^{\frac{3}{4}}} \exp\left\{-\frac{t}{8a^2} - \frac{a^2}{8t}\right\} \times$$

$$\times \left[\frac{\sqrt{2t}}{2} D_{-\frac{3}{2}}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right) + a\, D_{-\frac{1}{2}}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right)\right], \tag{11}$$

where [7] considering formula [9]

$$D_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi\, z}{2}} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right)$$

and $K_\nu(x)$ is the modified Bessel function of the second kind or the Macdonald function.

Since from formula 9.247 (2) [6] we have

$$D_{-\frac{3}{2}}(z) = z\, D_{-\frac{1}{2}}(z) + 2\frac{d}{dz}D_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi\, z^3}{2}} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right) + \frac{d}{dz}\left(\sqrt{2\pi\, z} K_{\frac{1}{4}}\left(\frac{z^2}{4}\right)\right),$$

then, taking into account the formula

$$K'_\nu(x) = -\frac{1}{2}\left(K_{\nu-1}(x) + K_{\nu-1}(x)\right),$$

we conclude that the expression in square brackets in (11) will be a linear combination of functions $K_\nu\left(\frac{z^2}{4}\right)$, where

$$\nu = \left\{\frac{1}{4}; \frac{3}{4}; \frac{5}{4}\right\}, \quad z = \frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}.$$

From asymptotic behavior

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \left(1 + O\left(\frac{1}{x}\right)\right), \quad x \to +\infty$$

and from limit relation

$$\lim_{t\to 0;\, t\to+\infty} z^2 = \lim_{t\to 0;\, t\to+\infty} \left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{t}}{a\sqrt{2}}\right)^2 = +\infty,$$

it follows that function (11) will be bounded when $t \in (0,\ +\infty)$.

Thus, the following theorem is proved.

Theorem 1. The integral equation

$$\nu(t) - \frac{a}{2\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t}\sqrt{\tau}\sqrt{t-\tau}} e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau)d\tau - \frac{1}{2\, a\sqrt{\pi}} \int_0^t \sqrt{\frac{\tau}{t}} \frac{1}{\sqrt{t-\tau}} e^{-\frac{t-\tau}{4a^2}} \cdot \nu(\tau)d\tau = 0$$

in the class of functions $\nu(t) \in L_\infty(0,\ +\infty)$ has a solution defined by the formula (11).

4 Case $k = 1$

When $k = 1$ from representation (10) we get

$$\nu(t) = \frac{Ce}{a\sqrt{2\pi\, t}} \exp\left\{\frac{t}{4a^2} - \frac{a^2}{8t}\right\} \left[\sqrt{2t}\, D_{-2}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{2t}}{a}\right) + a\, D_{-1}\left(\frac{a}{\sqrt{2t}} - \frac{\sqrt{2t}}{a}\right)\right]. \tag{12}$$

From formulas 9.254 (1) and 9.254 (2) [6] we have for (12):

$$\nu(t) = \frac{Ce}{a\sqrt{2\pi\, t}} \exp\left\{\frac{t}{4a^2} - \frac{a^2}{8t}\right\} \left[\sqrt{\frac{\pi}{2}}\sqrt{2t} \exp\left\{\frac{a^2}{8t} + \frac{t}{2a^2} - \frac{1}{2}\right\} \times\right.$$

$$\times\left\{\sqrt{\frac{2}{\pi}}\,\exp\left\{1-\frac{a^2}{4t}-\frac{t}{a^2}\right\}-\left(\frac{a}{\sqrt{2t}}-\frac{\sqrt{2t}}{a}\right)erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right\}+$$

$$+a\sqrt{\frac{\pi}{2}}\exp\left\{\frac{a^2}{8t}+\frac{t}{2a^2}-\frac{1}{2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right]=$$

$$=\frac{Ce^{\frac{3}{2}}}{a\sqrt{\pi}}\left[\frac{1}{\sqrt{t}}\exp\left\{-\frac{t}{4a^2}-\frac{a^2}{4t}\right\}+\frac{\sqrt{\pi}}{ae}\exp\left\{\frac{3t}{4a^2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right].$$

So, when $k=1$, representation (10) has the form:

$$\nu(t)=\frac{Ce^{\frac{3}{2}}}{a\sqrt{\pi}}\left[\frac{1}{\sqrt{t}}\exp\left\{-\frac{t}{4a^2}-\frac{a^2}{4t}\right\}+\frac{\sqrt{\pi}}{ae}\exp\left\{\frac{3t}{4a^2}\right\}erfc\left(\frac{a}{2\sqrt{t}}-\frac{\sqrt{t}}{a}\right)\right]. \tag{13}$$

Thus, the following theorem is valid.

Theorem 2. The integral equation

$$\nu(t)-\frac{a}{2\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t}\,\sqrt{\tau}\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau-\frac{1}{a\sqrt{\pi}}\int_0^t\sqrt{\frac{\tau}{t}}\frac{1}{\sqrt{t-\tau}}\,e^{-\frac{t-\tau}{4a^2}}\cdot\nu(\tau)d\tau=0$$

in the class of functions $\exp\left\{-\frac{t}{a^2}\right\}\nu(t)\in L_\infty(0,\,+\infty)$ has a solution defined by the formula (13).

References

1 Amangaliyeva M.M. About Dirichlet boundary value problem for the heat equation in the infinite angular domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Boundary Value Problems. – 2014. — 213. – P. 1–21. DOI: 10.1186/s13661-014-0213-4.

2 Amangaliyeva M.M. On one homogeneous problem for the heat equation in an infinite angular domain / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Siberian Mathematical Journal. — 2015. — Vol. 56, No. 6. — P. 982–995.

3 Amangaliyeva M.M. On a Volterra equation of the second kind with 'incompressible' kernel / M.M. Amangaliyeva, M.T. Jenaliyev, M.T. Kosmakova, M.I. Ramazanov // Advances in Difference Equations. — 2015. — 71. — P.1–14. DOI: 10.1186/s13662-015-0418-6.

4 Дёч Г. Руководство к практическому применению преобразования Лапласа и Z-преобразования / Г. Дёч. — М.: Наука, 1971. — 288 с.

5 Прудников А.П. Интегралы и ряды. Элементарные функции / А.П. Прудников, Ю.А. Брычков, О.И. Маричев. — М.: Наука, 1981. — 791 с.

6 Gradshteyn I.S. Table of Integrals, Series, and Products / I.S. Gradshteyn, I.M. Ryzhik. — Seventh Ed., AP, New York, 2007. — 171 p.

7 Бейтмен Г. Высшие трансцендентные функции / Г. Бейтмен, А. Эрдейи. — Т. 2. — М.: Наука, 1974. — 296 с.

М.Т. Космакова, Д.М. Ахманова, С.А. Искаков, Ж.М. Тулеутаева, Л.Ж. Касымова

Псевдо-Вольтерраның интегралдық теңдеуінің шешілуі

Мақалада псевдо-Вольтерраның екінші текті интегралдық теңдеуінің шешілу сұрақтары зерттелді. Интегралдық теңдеу оң жақтағы және ізделінді функцияны ауыстыру арқылы ядросы «сығылмалы» болмайтын интегралдық теңдеуге келтірілді. Алынған теңдеу Лаплас түрлендіруі арқылы қарапайым бірінші ретті (сызықтық) дифференциалдық теңдеуге келтірілді. Бұл теңдеудің шешуі табылды. Бастапқы біртекті емес интегралдық теңдеуге сәйкес келетін біртекті интегралдық теңдеудің шешуі айқын түрде табылды. Біртекті интегралдық теңдеудің дербес жағдайлары және оның k параметрінің әртүрлі мәндеріндегі шешулері жазылды. Шешулері болатын интегралдық теңдеулердің кластары көрсетілген. Сингулярлық интегралдық теңдеулер [1–3] жұмыстарда қарастырылған. Сонымен қатар олардын ядролары «сығылмайтын» болды, бірақ түрі өзгеше. Осыған байланысты шешудің бар болуының салмақтық кластарының аталған жұмыстағы зерттеліп отырған теңдеулердің шешулері бар болуының кластарынан айырмашылығы бар.

Кілт сөздер: ядро, интегралдық оператор, шектелген функциялар кластары, Лаплас түрлендіруі.

М.Т. Космакова, Д.М. Ахманова, С.А. Искаков, Ж.М. Тулеутаева, Л.Ж. Касымова

Решение одного псевдо-Вольтеррового интегрального уравнения

В статье исследованы вопросы разрешимости псевдо-Вольтеррового интегрального уравнения второго рода. С помощью замен правой части и искомой функции интегральное уравнение сведено к интегральному уравнению, ядро которого не является «сжимаемым». С помощью преобразования Лапласа полученное уравнение сведено к обыкновенному дифференциальному уравнению первого порядка (линейному). Найдено его решение. Решение однородного интегрального уравнения, соответствующего исходному неоднородному интегральному уравнению, найдено в явном виде. Выписаны частные случаи однородного интегрального уравнения и его решения при различных значениях параметра k. Указаны классы, в которых интегральное уравнение имеет решение. Сингулярные интегральные уравнения были рассмотрены в работах [1–3]. Их ядра также были «несжимаемы», но имели другой вид. В связи с этим весовые классы существования решения отличаются от класса существования решения уравнения, исследуемого в данной работе.

Ключевые слова: ядро, интегральный оператор, класс существенно ограниченных функций, преобразование Лапласа.

References

1 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2014). About Dirichlet boundary value problem for the heat equation in the infinite angular domain. *Boundary Value Problems, 213,* 1–21. DOI: 10.1186/s13661-014-0213-4.

2 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On one homogeneous problem for the heat equation in an infinite angular domain. *Siberian Mathematical Journal, Vol. 56, 6,* 982–995.

3 Amangaliyeva, M.M., Jenaliyev, M.T., Kosmakova, M.T., & Ramazanov, M.I. (2015). On a Volterra equation of the second kind with 'incompressible' kernel. *Advances in Difference Equations, 71,* 1–14. DOI: 10.1186/s13662-015-0418-6.

4 Dech, G. (1971). *Rukovodstvo k prakticheskomu primeneniiu preobrazovaniia Laplasa i Z-preobrazovaniia [Guide to the practical application of Laplace transform and Z-transform].* Moscow: Nauka [in Russian].

5 Prudnikov, A.P., Brychkov, Iu.A., & Marichev, O.I. (1981). *Integraly i riady. Elementarnye funktsii [Integrals and series. Elementary functions].* Moscow: Nauka [in Russian].

6 Gradshteyn, I.S., & Ryzhik, I.M. (2007). *Table of Integrals, Series, and Products.* (Seventh Ed.), AP, New York.

7 Beitmen, G., & Erdeii, A. (1974). *Vysshie transtsendentnye funktsii [Higher transcendental functions].* (Vol. 2). Moscow: Nauka [in Russian].