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Integro-differentiated singularly perturbed equations with fast oscillating coefficients

In the study of various issues related to dynamic stability, with the properties of media with a periodic structure, in the study of other applied problems, one has to deal with differential equations with rapidly oscillating coefficients. Asymptotic integration of differential systems of equations with such coefficients was carried out by the splitting method and the regularization method. In this paper, a system of integro-differential equations is considered. The main objective of the study is to identify the influence of the integral term on the asymptotics of the solution to the original problem. The case of the absence of resonance is considered, i.e. the case when the integer linear combination of frequencies of the rapidly oscillating coefficient does not coincide with the frequency of the spectrum of the limit operator.

Keywords: singularly perturbation, integro-differential equation, rapidly oscillating coefficient, regularization, asymptotic convergence.

Introduction

Consider the following integro-differential system:

$$\varepsilon \frac{dz}{dt} - A(t)z - \varepsilon g(t) \cos \frac{\beta(t)}{\varepsilon} B(t)z - \int_{t_0}^t K(t,s)z(s,\varepsilon)ds = h(t), \quad z(t_0, \varepsilon) = z^0, \quad t \in [t_0, T], \quad (1)$$

where $z = \{z_1, z_2\}$, $h(t) = \{h_1(t), h_2(t)\}$, $\beta'(t) > 0$, $\omega(t) > 0 (\forall t \in [t_0, T])$, $g(t)$ is a scalar function, $A(t)$ and $B(t)$ are (2×2) -matrices, with $A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix}$, $z^0 = \{z_1^0, z_2^0\}$, $\varepsilon > 0$ is a small parameter. Such a system in the case $\beta(t) = 2\gamma(t)$, $B(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of the absence of an integral term was considered in [1–6].

In the present work, ideas of the regularization method [3–6] are generated on singularly perturbed systems of integro-differential equations with rapidly oscillating coefficients. The study of singularly perturbed integro-differential problems by the regularization method of S.A. Lomov [3, 4] with unstable values of the kernel of an integral operator is reflected in [7–12]. It should also be noted that it is the merit of V.F. Safonov and A.A. Bobodzhyanov in the development of the theory of singularly perturbed integro-differential equations [13–16]. In their studies, various problems for integro-differential systems were considered: with diagonal kernel degenerations, with inverse time, with rapidly changing kernels, with rapidly varying kernels, with partial derivatives, etc. [17–21].

In the system the limiting operator $A(t)$ has a spectrum $\lambda_1(t) = -i\omega(t)$, $\lambda_2(t) = +i\omega(t)$, $\beta'(t)$ is a frequency of rapidly oscillating cosine. In the following, functions $\lambda_3(t) = -i\beta'(t)$, $\lambda_4(t) = +i\beta'(t)$ will be called *the spectrum of a rapidly oscillating coefficient*.

We assume that the following conditions are fulfilled:

1) $\omega(t), \beta(t), g(t) \in C^\infty([t_0, T], \mathbb{C}^1)$, $h(t) \in C^\infty([t_0, T], \mathbb{C}^2)$,

$B(t) \in C^\infty([t_0, T], \mathbb{C}^{2 \times 2})$, $K(t, s) \in C^\infty([t_0, T], \mathbb{C}^{2 \times 2})$,

2) for $\forall t \in [t_0, T]$ and $n_3 \neq n_4$ inequalities

$$\begin{aligned} n_3 \lambda_3(t) + n_4 \lambda_4(t) &\neq \lambda_j(t), \\ \lambda_k(t) + n_3 \lambda_3(t) + n_4 \lambda_4(t) &\neq \lambda_j(t), \quad k \neq j, k, j = 1, 2, \end{aligned}$$

for all multi-indices $n = (n_3, n_4)$ with $|n| \equiv n_3 + n_4 \geq 1$ (n_3 and n_4 are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [3] asymptotic solution of problem (1). Condition 2) is called the *absence of resonance condition*.

1. Regularization of problem (1)

Denote by $\sigma_j = \sigma_j(\varepsilon)$, independent of t magnitudes $\sigma_1 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite system (1) as

$$\begin{aligned} \varepsilon \frac{dz}{dt} - A(t)z - \varepsilon \frac{g(t)}{2} \left(e^{-\frac{i}{\varepsilon} \int_{t_0}^t \beta'(\theta) d\theta} \sigma_1 + e^{+\frac{i}{\varepsilon} \int_{t_0}^t \beta'(\theta) d\theta} \sigma_2 \right) B(t)z - \\ - \int_{t_0}^t K(t, s) z(s, \varepsilon) ds = h(t), \quad z(t_0, \varepsilon) = z^0, \quad t \in [t_0, T]. \end{aligned} \quad (2)$$

We introduce regularizing variables [3, 4]

$$\tau_j = \frac{1}{\varepsilon} \int_{t_0}^t \lambda_j(\theta) d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \quad j = \overline{1, 4} \quad (3)$$

and instead of problem (2), consider the problem

$$\begin{aligned} \varepsilon \frac{\partial \tilde{z}}{\partial t} + \sum_{j=1}^4 \lambda_j(t) \frac{\partial \tilde{z}}{\partial \tau_j} - A(t)\tilde{z} - \varepsilon \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t)\tilde{z} - \\ - \int_{t_0}^t K(t, s) \tilde{z}(s, \frac{\psi(s)}{\varepsilon}, \varepsilon) ds = h(t), \quad \tilde{z}(t, \tau, \varepsilon)|_{t=t_0, \tau=0} = z^0, \quad t \in [t_0, T], \end{aligned} \quad (4)$$

for the function $\tilde{z} = \tilde{z}(t, \tau, \varepsilon)$, where is indicated (by (3)): $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$. It is clear that if $\tilde{z} = \tilde{z}(t, \tau, \varepsilon)$ is a solution to problem (4), then the vector function $z = \tilde{z}(t, \frac{\psi(t)}{\varepsilon}, \varepsilon)$ is an exact solution to problem (2), therefore, problem (4) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral term $J\tilde{z} = \int_{t_0}^t K(t, s) \tilde{z}(s, \frac{\psi(s)}{\varepsilon}, \varepsilon) ds$. To regularize the integral operator, we introduce a class M_ε that is asymptotically invariant with respect to the operator $J\tilde{z}$ [3; 62]. Recall the corresponding concept.

Definition 1. A class M_ε is said to be asymptotically invariant (with $\varepsilon \rightarrow +0$) with respect to an operator P_0 if the following conditions are fulfilled:

- 1) $M_\varepsilon \subset D(P_0)$ with each fixed $\varepsilon > 0$;
- 2) the image $P_0 g(t, \varepsilon)$ of any element $g(t, \varepsilon) \in M_\varepsilon$ decomposes in a power series

$$P_0 g(t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n g_n(t, \varepsilon) (\varepsilon \rightarrow +0, g_n(t, \varepsilon) \in M_\varepsilon, n = 0, 1, \dots),$$

convergent asymptotically for $\varepsilon \rightarrow +0$ (uniformly with $t \in [t_0, T]$).

From this definition it can be seen that the class M_ε depends on the space U , in which the operator P_0 is defined. In our case $P_0 = J$. For the space U we take the space of vector functions $z(t, \tau)$, represented by sums

$$\begin{aligned} z(t, \tau, \sigma) = z_0(t, \sigma) + \sum_{i=1}^4 z_i(t, \sigma) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t, \sigma) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t, \sigma) e^{(e_j+m, \tau)}, \\ m = (0, 0, m_3, m_4), z_i(t, \sigma), z^m(t, \sigma), z^{e_j+m}(t, \sigma) \in C^\infty([t_0, T], \mathbb{C}^2), \\ 1 \leq |m| \equiv m_3 + m_4 \leq N_z, i = \overline{1, 4}, j = 1, 2, \end{aligned} \quad (5)$$

where is denoted: $\lambda(t) \equiv (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $(m, \lambda(t)) \equiv m_3 \lambda_3(t) + m_4 \lambda_4(t)$, $(e_j + m, \lambda(t)) \equiv \lambda_j(t) + m_3 \lambda_3(t) + m_4 \lambda_4(t)$; an asterisk $*$ above the sum sign indicates that the summation for $|m| \geq 1$ it occurs only over multi-indices $m = (0, 0, m_3, m_4)$ with $m_3 \neq m_4$, $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $\sigma = (\sigma_1, \sigma_2)$.

Note that here the degree N_z of the polynomial $z(t, \tau, \sigma)$ relative to the exponentials e^{τ_j} depends on the element z . In addition, the elements of space U depend on bounded in $\varepsilon > 0$ terms of constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$, and which do not affect the development of the algorithm described below, therefore, in the record of element (5) of this space U , we omit the dependence on $\sigma = (\sigma_1, \sigma_2)$ for brevity. We show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant with respect to the operator J . The image of the operator on the element (5) of the space U has the form

$$\begin{aligned}
 Jz(t, \tau) &= \int_{t_0}^t K(t, s) z_0(s) ds + \sum_{i=1}^4 \int_{t_0}^t K(t, s) z_i(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_i(\theta) d\theta} ds + \\
 &\quad + \sum_{2 \leq |m| \leq N_z}^* \int_{t_0}^t K(t, s) z^m(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (m, \lambda(\theta)) d\theta} ds + \\
 &\quad + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* \int_{t_0}^t K(t, s) z^{e_j+m}(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} ds.
 \end{aligned}$$

Integrating in parts, we will have

$$\begin{aligned}
 J_i(t, \varepsilon) &= \int_{t_0}^t K(t, s) z_i(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_i(\theta) d\theta} ds = \varepsilon \int_{t_0}^t \frac{K(t, s) z_i(s)}{\lambda_i(s)} d e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_i(\theta) d\theta} = \\
 &= \varepsilon \frac{K(t, t) z_i(t)}{\lambda_i(t)} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_i(\theta) d\theta} \Big|_{s=t_0}^{s=t} - \varepsilon \int_{t_0}^t \left(\frac{\partial}{\partial s} \frac{K(t, s) z_i(s)}{\lambda_i(s)} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_i(\theta) d\theta} ds = \\
 &= \varepsilon \left[\frac{K(t, t) z_i(t)}{\lambda_i(t)} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_i(\theta) d\theta} - \frac{K(t, t_0) z_i(t_0)}{\lambda_i(t_0)} \right] - \varepsilon \int_{t_0}^t \left(\frac{\partial}{\partial s} \frac{K(t, s) z_i(s)}{\lambda_i(s)} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^s \lambda_i(\theta) d\theta} ds.
 \end{aligned}$$

Continuing this process further, we obtain the decomposition

$$\begin{aligned}
 J_i(t, \varepsilon) &= \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[(I_i^\nu(K(t, s) z_i(s)))_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_i(\theta) d\theta} - (I_i^\nu(K(t, s) z_i(s)))_{s=t_0} \right], \\
 I_i^0 &= \frac{1}{\lambda_i(s)}, I_i^\nu = \frac{1}{\lambda_i(s)} I_i^{\nu-1} \quad (\nu \geq 1, i = \overline{1, 4}).
 \end{aligned}$$

Applying the integration operation in parts to integrals

$$J_m(t, \varepsilon) = \int_{t_0}^t K(t, s) z^m(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (m, \lambda(\theta)) d\theta} ds, \quad J_{e_j+m}(t, \varepsilon) = \int_{t_0}^t K(t, s) z^{e_j+m}(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} ds,$$

we note that for all multi-indices $m = (0, 0, m_3, m_4)$, $m_3 \neq m_4$, inequalities

$$(m, \lambda(t)) \equiv m_3 \lambda_3(t) + m_4 \lambda_4(t) \neq 0 \forall t \in [t_0, T], m_3 + m_4 \geq 2$$

are satisfied. In addition, for the same multi-indices we have

$$(e_j + m, \lambda(t)) \neq 0 \forall t \in [t_0, T], j = 1, 2, m_3 \neq m_4, |m| = m_3 + m_4 \geq 1.$$

Indeed, if $(e_1 + m, \lambda(t)) = 0$ for some $t \in [t_0, T]$ and $m_3 \neq m_4, m_3 + m_4 \geq 1$, then $m_3 \lambda_3(t) + m_4 \lambda_4(t) = -\lambda_1(t) = \lambda_2(t)$, $m_3 + m_4 \geq 1$, which contradicts condition 2). And likewise, if $(e_2 + m, \lambda(t)) = 0$ with some $t \in [t_0, T]$ and $m_3 \neq m_4, m_3 + m_4 \geq 1$, then $m_3 \lambda_3(t) + m_4 \lambda_4(t) = -\lambda_1(t) = \lambda_2(t)$, $m_3 + m_4 \geq 1$, which also contradicts condition 2). Therefore, integration by parts in integrals $J_m(t, \varepsilon)$, $J_{e_j+m}(t, \varepsilon)$ is possible. Performing it, we will have:

$$\begin{aligned}
 J_m(t, \varepsilon) &= \int_{t_0}^t K(t, s) z^m(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (m, \lambda(\theta)) d\theta} ds = \varepsilon \int_{t_0}^t \frac{K(t, s) z^m(s)}{(m, \lambda(s))} d e^{\frac{1}{\varepsilon} \int_{t_0}^s (m, \lambda(\theta)) d\theta} = \\
 &= \varepsilon \left[\frac{K(t, t) z^m(t)}{(m, \lambda(t))} e^{\frac{1}{\varepsilon} \int_{t_0}^t (m, \lambda(\theta)) d\theta} - \frac{K(t, t_0) z^m(t_0)}{(m, \lambda(t_0))} \right] - \varepsilon \int_{t_0}^t \frac{\partial}{\partial s} \frac{K(t, s) z^m(s)}{(m, \lambda(s))} e^{\frac{1}{\varepsilon} \int_{t_0}^s (m, \lambda(\theta)) d\theta} ds = \\
 &= \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[(I_m^\nu(K(t, s) z^m(s)))_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t (m, \lambda(\theta)) d\theta} - (I_m^\nu(K(t, s) z^m(s)))_{s=t_0} \right];
 \end{aligned}$$

$$I_m^0 = \frac{1}{(m, \lambda(s))}, I_m^\nu = \frac{1}{(m, \lambda(s))} \frac{\partial}{\partial s} I_m^{\nu-1} (\nu \geq 1, |m| \geq 2);$$

$$\begin{aligned}
 J_{e_j+m}(t, \varepsilon) &= \int_{t_0}^t K(t, s) z^{e_j+m}(s) e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} ds = \\
 &= \varepsilon \int_{t_0}^t \frac{K(t, s) z^{e_j+m}(s)}{(e_j + m, \lambda(s))} d e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} =
 \end{aligned}$$

$$\begin{aligned}
 &= -\varepsilon \left[\frac{K(t, t) z^{e_j+m}(t)}{(e_j+m, \lambda(t))} e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} - \frac{K(t, t_0) z^{e_j+m}(t_0)}{(e_j+m, \lambda(t_0))} \right] - \\
 &\quad - \varepsilon \int_{t_0}^t \frac{\partial}{\partial s} \frac{K(t, s) z^{e_j+m}(s)}{(e_j+m, \lambda(s))} e^{\frac{1}{\varepsilon} \int_{t_0}^s (e_j+m, \lambda(\theta)) d\theta} ds = \\
 &= \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} [(I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t (e_j+m, \lambda(\theta)) d\theta} - \\
 &\quad - (I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t_0}], \\
 I_{j,m}^0 &= \frac{1}{(e_j+m, \lambda(s))}, \quad I_{j,m}^\nu = \frac{1}{(e_j+m, \lambda(s))} \frac{\partial}{\partial s} I_{j,m}^{\nu-1} (\nu \geq 1, |m| \geq 1, j = 1, 2),
 \end{aligned}$$

Therefore, the image of the operator J on the element (5) of the space U is represented as a series

$$\begin{aligned}
 Jz(t, \tau) &= \int_{t_0}^t K(t, s) z_0(s) ds + \sum_{i=1}^4 \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} [(I_i^\nu (K(t, s) z_i(s)))_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_i(\theta) d\theta} - \\
 &\quad - (I_i^\nu (K(t, s) z_i(s)))_{s=t_0}] + \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} [(I_m^\nu (K(t, s) z^m(s)))_{s=t} e^{\frac{1}{\varepsilon} \int_{t_0}^t (m, \lambda(\theta)) d\theta} - \\
 &\quad - (I_m^\nu (K(t, s) z^m(s)))_{s=t_0}] + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} [(I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t} \times \\
 &\quad \times e^{\frac{1}{\varepsilon} \int_{t_0}^t (e_j+m, \lambda(\theta)) d\theta} - (I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t_0}]_{\tau=\psi(t)/\varepsilon}.
 \end{aligned}$$

It is easy to show [22; 291–294] that this series converges asymptotically for $\varepsilon \rightarrow +0$ (uniformly in $t \in [t_0, T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon \rightarrow +0$) with respect to the operator J .

We introduce operators $R_\nu : U \rightarrow U$, acting on each element $z(t, \tau) \in U$ of the form (5) according to the law:

$$R_0 z(t, \tau) = \int_{t_0}^t K(t, s) z_0(s) ds, \quad (6_0)$$

$$\begin{aligned}
 R_1 z(t, \tau) &= \sum_{i=1}^4 [(I_i^0 (K(t, s) z_i(s)))_{s=t} e^{\tau_i} - (I_i^0 (K(t, s) z_i(s)))_{s=t_0}] + \\
 &\quad + \sum_{1 \leq |m| \leq N_z}^* (I_m^0 (K(t, s) z^m(s)))_{s=t} e^{(m, \tau)} - (I_m^0 (K(t, s) z^m(s)))_{s=t_0} + \\
 &\quad + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* [(I_{j,m}^0 (K(t, s) z^{e_j+m}(s)))_{s=t} e^{(e_j+m, \tau)} - (I_{j,m}^0 (K(t, s) z^{e_j+m}(s)))_{s=t_0}]; \quad (6_1)
 \end{aligned}$$

$$\begin{aligned}
 R_{\nu+1} z(t, \tau) &= \sum_{i=1}^4 (-1)^\nu \varepsilon^{\nu+1} [(I_i^\nu (K(t, s) z_i(s)))_{s=t} e^{\tau_i} - (I_i^\nu (K(t, s) z_i(s)))_{s=t_0}] + \\
 &\quad + \sum_{\nu=0}^* (-1)^\nu \varepsilon^{\nu+1} [(I_m^\nu (K(t, s) z^m(s)))_{s=t} e^{(m, \tau)} - (I_m^\nu (K(t, s) z^m(s)))_{s=t_0}] + \\
 &\quad + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* (-1)^\nu [(I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t} e^{(e_j+m, \tau)} - (I_{j,m}^\nu (K(t, s) z^{e_j+m}(s)))_{s=t_0}], \nu \geq 1.
 \end{aligned} \tag{6_{\nu+1}}$$

Now let $\tilde{z}(t, \tau, \varepsilon)$ be an arbitrary continuous function on $(t, \tau) \in [t_0, T] \times \{\tau : \operatorname{Re} \tau_j \leq 0, j = \overline{1, 4}\}$ with asymptotic expansion

$$\tilde{z}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z_k(t, \tau), \quad z_k(t, \tau) \in U, \quad (7)$$

converging as $\varepsilon \rightarrow +0$ (uniformly in $(t, \tau) \in [t_0, T] \times \{\tau : \operatorname{Re}\tau_j \leq 0, j = \overline{1, 4}\}$). Then the image $J\tilde{z}(t, \tau, \varepsilon)$ of this function is decomposed into an asymptotic series

$$J\tilde{z}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Jz_k(t, \tau) = \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^r R_{r-s} z_s(t, \tau) |_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of an operator J on series of the form (7):

$$\tilde{J}\tilde{z}(t, \tau, \varepsilon) \equiv \tilde{J} \left(\sum_{k=0}^{\infty} \varepsilon^k z_k(t, \tau) \right) \triangleq \sum_{r=0}^{\infty} \varepsilon^r \sum_{s=0}^r R_{r-s} z_s(t, \tau). \quad (8)$$

Although the operator (8) is formally defined, its utility is obvious, since in practice it is usual to construct the N -th approximation of the asymptotic solution of the problem (2), in which impose only N -th partial sums of the series (7), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$\varepsilon \frac{\partial \tilde{z}}{\partial t} + \sum_{j=1}^4 \lambda_j(t) \frac{\partial \tilde{z}}{\partial \tau_j} - A(t)\tilde{z} - \varepsilon \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B\tilde{z} - \tilde{J}\tilde{z} = h(t),$$

$$\tilde{z}(t, \tau, \varepsilon)|_{t=t_0, \tau=0} = z^0, \quad t \in [t_0, T]. \quad (9)$$

2. Iterative problems and their solvability in space solution of the first iterative problem

Substituting the series (7) into (9) and equating the coefficients with the same degrees, we obtain the following iterative problems:

$$Lz_0(t, \tau) \equiv \sum_{j=1}^4 \lambda_j(t) \frac{\partial z_0}{\partial \tau_j} - A(t)z_0 - R_0 z_0 = h(t), \quad z_0(t_0, 0) = z^0; \quad (10_0)$$

$$Lz_1(t, \tau) = -\frac{\partial z_0}{\partial t} + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) z_0 + R_1 z_0, \quad z_1(t_0, 0) = 0; \quad (10_1)$$

$$Lz_2(t, \tau) = -\frac{\partial z_1}{\partial t} + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) z_1 + R_1 z_1 + R_2 z_0, \quad z_0(t_0, 0) = 0; \quad (10_2)$$

...

$$Lz_k(t, \tau) = -\frac{\partial z_{k-1}}{\partial t} + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) z_{k-1} + R_k z_0 + \dots + R_1 z_{k-1}, \quad z_k(t_0, 0) = 0, \quad k \geq 1. \quad (10_k)$$

Each of the iterative problems (10_k) can be written as

$$Lz(t, \tau) \equiv \sum_{j=1}^4 \lambda_j(t) \frac{\partial z}{\partial \tau_j} - A(t)z - R_0 z = H(t, \tau), \quad z(t_0, 0) = z^*, \quad (10)$$

where

$$H(t, \tau) = H_0(t) + \sum_{i=1}^4 H_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* H^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_H}^* H^{e_j+m}(t) e^{(e_j+m, \tau)}$$

is the known vector function of space U , z^* is the known constant vector of the complex space \mathcal{C}^2 , and the operator R_0 has the form (see (6₀))

$$R_0 z \equiv R_0 \left(z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{0 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)} \right) \triangleq \int_{t_0}^t K(t, s) z_0(s) ds.$$

In the future, we will need $\lambda_j(t)$ -eigenvectors of the matrix $A(t)$:

$$\varphi_1(t) = \begin{pmatrix} 1 \\ -i\omega(t) \end{pmatrix}, \varphi_2(t) = \begin{pmatrix} 1 \\ +i\omega(t) \end{pmatrix},$$

and also $\bar{\lambda}_j(t)$ -eigenvectors of the matrix $A^*(t)$:

$$\chi_1(t) = \begin{pmatrix} 1 \\ -\frac{i}{\omega(t)} \end{pmatrix}, \chi_2(t) = \begin{pmatrix} 1 \\ +\frac{i}{\omega(t)} \end{pmatrix}.$$

These vectors form a biorthogonal system, i.e.

$$(\varphi_k(t), \chi_j(t)) = \begin{cases} 2, k = j, \\ 0, k \neq j \end{cases} \quad (k, j = 1, 2).$$

We introduce scalar (for each $t \in [t_0, T]$) product in space U :

$$\begin{aligned} \langle z, w \rangle &\equiv \\ &\equiv \langle z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}, \\ w_0(t) + \sum_{i=1}^4 w_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_w}^* w^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_w}^* w^{e_j+m}(t) e^{(e_j+m, \tau)} \rangle &\triangleq \\ &\triangleq \langle z_0(t), w_0(t) \rangle + \sum_{i=1}^4 \langle z_i(t), w_i(t) \rangle + \\ &+ \sum_{2 \leq |m| \leq \min(N_z, N_w)}^* \langle z^m(t), w^m(t) \rangle + \sum_{j=1}^2 \sum_{1 \leq |m| \leq \min(N_z, N_w)}^* \langle z^{e_j+m}(t), w^{e_j+m}(t) \rangle, \end{aligned}$$

where we denote by $(*, *)$ the usual scalar product in the complex space \mathbb{C}^2 . Let us prove the following statement.

Theorem 1. Let conditions 1) and 2) be fulfilled and the right-hand side $H(t, \tau) = H_0(t) + \sum_{i=1}^4 H_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* H^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_H}^* H^{e_j+m}(t) e^{(e_j+m, \tau)}$ of system (10) belongs to the space U . Then the system (10) is solvable in U , if and only if

$$\langle H(t, \tau), \chi_k(t) e^{\tau_k} \rangle \equiv 0, \quad k = 1, 2, \quad \forall t \in [t_0, T]. \quad (11)$$

Proof. We will determine the solution of system (10) as an element (5) of the space U :

$$\begin{aligned} z(t, \tau) &= z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \\ &+ \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_H}^* z^{e_j+m}(t) e^{(e_j+m, \tau)} \equiv \\ &\equiv z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{k=1}^2 \sum_{2 \leq |m^k| \leq N_H}^* z^{m^k}(t) e^{(m^k, \tau)}, \end{aligned} \quad (12)$$

where for convenience are introduced multi-indices

$$m^1 = e_1 + m \equiv (1, 0, m_3, m_4), m^2 = e_2 + m \equiv (0, 1, m_3, m_4), |m^k| = 1 + m_3 + m_4 \geq 2,$$

m_3 and m_4 are non-negative integer numbers. Substituting (12) into system (10), we will have

$$\begin{aligned} &\sum_{i=1}^4 [\lambda_i(t) I - A(t)] z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* [(m, \lambda(t)) I - A(t)] z^m(t) e^{(m, \tau)} + \\ &+ \sum_{k=1}^2 \sum_{2 \leq |m^k| \leq N_H}^* [(m^k, \lambda(t)) I - A(t)] z^{m^k}(t) e^{(m^k, \tau)} - A(t) z_0(t) - \int_{t_0}^t K(t, s) z_0(s) ds = \\ &= H_0(t) + \sum_{i=1}^4 H_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* H^m(t) e^{(m, \tau)} + \sum_{k=1}^2 \sum_{2 \leq |m^k| \leq N_H}^* H^{m^k}(t) e^{(m^k, \tau)}. \end{aligned}$$

Equating here the free terms and coefficients separately for identical exponents, we obtain the following systems of equations:

$$-A(t) z_0(t) - \int_{t_0}^t K(t, s) z_0(s) ds = H_0(t), \quad (13)$$

$$[\lambda_i(t) I - A(t)] z_i(t) = H_i(t), \quad i = \overline{1, 4}; \quad (13_i)$$

$$[(m, \lambda(t)) I - A(t)] z^m(t) = H^m(t), \quad m_3 \neq m_4, \quad 2 \leq |m| \leq N_H; \quad (13_m)$$

$$[(m^k, \lambda(t)) I - A(t)] z^{m^k}(t) = H^{m^k}(t), \quad m_3 \neq m_4, \quad 2 \leq |m^k| \leq N_H, \quad k = 1, 2. \quad (14)$$

Since the matrix $A(t)$ is reversible, the system (13) can be written as

$$z_0(t) = \int_{t_0}^t (-A^{-1}(t) K(t, s)) z_0(s) ds - A^{-1}(t) H_0(t). \quad (13_0)$$

Due to the smoothness of the kernel $-A^{-1}(t) K(t, s)$ and heterogeneity $-A^{-1}(t) H_0(t)$, this Volterra integral system has a unique solution $z_0(t) \in C^\infty([t_0, T], \mathbb{C}^2)$. The systems (13₃) and (13₄) also have unique solutions

$$z_i(t) = [\lambda_i(t) I - A(t)]^{-1} H_i(t) \in C^\infty([t_0, T], \mathbb{C}^2), \quad i = 3, 4,$$

since $\lambda_3(t), \lambda_4(t)$ do not belong to the spectrum of the matrix $A(t)$. Systems (13₁) and (13₂) are solvable in space $C^\infty([t_0, T], \mathbb{C}^2)$ if and only if there are identities

$$(H_i(t), \chi_i(t)) \equiv 0 \quad \forall t \in [t_0, T], \quad i = 1, 2.$$

It is not difficult to see that these identities coincide with identities (11). Further, since $(m, \lambda(t)) \equiv m_3 \lambda_3(t) + m_4 \lambda_4(t) \neq \lambda_j(t), j = 1, 2, |m| = m_3 + m_4 \geq 2, m_3 \neq m_4$ (see condition 2) the absence of resonance), the system (13_m) has a unique solution

$$z^m(t) = [(m, \lambda(t)) I - A(t)]^{-1} H^m(t), \quad 2 \leq |m| \leq N_H \in C^\infty([t_0, T], \mathbb{C}^2).$$

We now consider systems (14). Let us show that when $|m^k| \geq 2$ the functions $(m^k, \lambda(t))$ are not eigenvalues of the matrix $A(t)$. Indeed, let $(m^1, \lambda(t)) = \lambda_1(t), |m^1| \geq 2$. Then

$$\lambda_1(t) + m_3 \lambda_3(t) + m_4 \lambda_4(t) = \lambda_2(t), \quad m_3 + m_4 \geq 1,$$

which contradicts condition 2) the absence of resonance. And likewise, equality $(m^2, \lambda(t)) = \lambda_1(t), |m^2| \geq 2, m_3 + m_4 \geq 1$ cannot be fulfilled.

Therefore, when $|m^k| \geq 2$ the matrix $(m^k, \lambda(t)) I - A(t)$ is reversible, we get a unique solution of system (14) for $|m^k| \geq 2$ in the class $C^\infty([t_0, T], \mathbb{C}^2)$:

$$z^{m^k}(t) = [(m^k, \lambda(t)) I - A(t)]^{-1} H^{m^k}(t), \quad 2 \leq |m^k| \leq N_H, \quad k = 1, 2.$$

Thus, condition (11) is necessary and sufficient for the solvability of system (10) in the space U . The theorem is proved.

Remark 1. If identity (11) holds, then under conditions 1) and 2), system (10) has the following solution in the space U :

$$\begin{aligned} z(t, \tau) = & z_0(t) + \sum_{k=1}^2 \alpha_k(t) \varphi_k(t) e^{\tau k} + \frac{(H_1(t), \chi_2(t))}{\lambda_1(t) - \lambda_2(t)} \varphi_2(t) e^{\tau_1} + \\ & + \frac{(H_2(t), \chi_1(t))}{\lambda_2(t) - \lambda_1(t)} \varphi_1(t) e^{\tau_2} + \sum_{i=3}^4 [\lambda_i(t) I - A(t)]^{-1} H_i(t) e^{\tau_i} + \\ & + \sum_{2 \leq |m| \leq N_H}^* [(m, \lambda(t)) I - A(t)]^{-1} H^m(t) e^{(m, \tau)} + \\ & + \sum_{k=1}^2 \sum_{1 \leq |m| \leq N_H}^* [(e_k + m, \lambda(t)) I - A(t)]^{-1} H^{e_k+m}(t) e^{(e_k+m, \tau)}, \end{aligned} \quad (15)$$

where $\alpha_k(t) \in C^\infty([t_0, T], \mathbb{C}^1)$ are arbitrary functions, $k = 1, 2, z_0(t)$ is the solution of an integral system (13₀), $m \equiv (0, 0, m_3, m_4), m_3 \neq m_4, |m| = m_3 + m_4 \geq 1$.

3. The unique solvability of the general iterative problem in the space U . Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of system (10) in space U . Along with problem (10), we consider the system

$$Lw(t, \tau) = -\frac{\partial z}{\partial t} + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) z + Q(t, \tau), \quad (16)$$

where $z = z(t, \tau)$ is the solution (15) of the system (10), $Q(t, \tau)$ is the well-known function of the space U . The right part of this system:

$$\begin{aligned} G(t, \tau) \equiv & -\frac{\partial z}{\partial t} + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) z + Q(t, \tau) = -\frac{\partial}{\partial t} [z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \\ & + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}] + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) \times \\ & \times [z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}] + Q(t, \tau), \end{aligned}$$

may not belong to space U , if $z = z(t, \tau) \in U$. Indeed, taking into account the form (15) of the function $z = z(t, \tau) \in U$, we will have

$$\begin{aligned} Z(t, \tau) \equiv & G(t, \tau) + \frac{\partial z}{\partial t} = \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) [z_0(t) + \\ & + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}] = \\ & = \frac{g(t)}{2} B(t) z_0(t) (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) + \sum_{i=3}^4 \frac{g(t)}{2} B(t) z_i(t) (e^{\tau_i+\tau_3} \sigma_1 + e^{\tau_i+\tau_4} \sigma_2) + \\ & + \sum_{k=1}^2 \frac{g(t)}{2} B(t) z_k(t) (e^{\tau_k+\tau_3} \sigma_1 + e^{\tau_k+\tau_4} \sigma_2) + \\ & + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) [\sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}] + \\ & + Q(t, \tau). \end{aligned}$$

Here are terms with exponents

$$\begin{aligned} e^{\tau_4+\tau_3} &= e^{(m, \tau)}|_{m=(0,0,1,1)}, \\ e^{\tau_3+(m, \tau)} & \text{ (if } m_3+1=m_4), e^{\tau_4+(m, \tau)} \text{ (if } m_4+1=m_3), \\ e^{\tau_3+(e_1+m, \tau)} & \text{ (if } m_3+1=m_4), e^{\tau_4+(e_2+m, \tau)} \text{ (if } m_4+1=m_3) \end{aligned} \quad (*)$$

do not belong to space U , since in multi-index $m = (0, 0, m_3, m_4)$ of the space U must be $m_3 \neq m_4, m_3+m_4 \geq 1$. Then, according to the well-known theory [3; 234], we embed these terms in the space U according to the following rule (see (*)):

$$\begin{aligned} \widehat{e^{\tau_4+\tau_3}} &= e^0 = 1, \widehat{e^{\tau_3+(m, \tau)}} = \\ &= e^0 = 1 (m_3+1=m_4, m_3 \neq m_4), \widehat{e^{\tau_4+(m, \tau)}} = e^0 = 1 (m_4+1=m_3, m_3 \neq m_4), \\ \widehat{e^{\tau_3+(e_1+m, \tau)}} &= e^{\tau_1} (m_3+1=m_4, m_3 \neq m_4), \widehat{e^{\tau_4+(e_2+m, \tau)}} = e^{\tau_2} (m_4+1=m_3, m_3 \neq m_4). \end{aligned} \quad (**)$$

In $Z(t, \tau)$ need of embedding only the terms

$$\begin{aligned} M(t, \tau) \equiv & \sum_{i=3}^4 \frac{g(t)}{2} B(t) z_i(t) (e^{\tau_i+\tau_3} \sigma_1 + e^{\tau_i+\tau_4} \sigma_2) + \sum_{k=1}^2 \frac{g(t)}{2} B(t) z_k(t) (e^{\tau_k+\tau_3} \sigma_1 + e^{\tau_k+\tau_4} \sigma_2), \\ S(t, \tau) \equiv & \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) [\sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}]. \end{aligned}$$

We describe this embedding in more detail, taking into account formulas (**):

$$\begin{aligned}
 M(t, \tau) &\equiv \sum_{k=1}^2 \frac{g(t)}{2} B(t) z_k(t) (e^{\tau_k + \tau_3} \sigma_1 + e^{\tau_k + \tau_4} \sigma_2) + \\
 &+ \sum_{i=3}^4 \frac{g(t)}{2} B(t) z_i(t) (e^{\tau_i + \tau_3} \sigma_1 + e^{\tau_i + \tau_4} \sigma_2) = \\
 &= \frac{g(t)}{2} B(t) [z_1(t) e^{\tau_1 + \tau_3} \sigma_1 + z_1(t) e^{\tau_1 + \tau_4} \sigma_2 + z_2(t) e^{\tau_2 + \tau_3} \sigma_1 + z_2(t) e^{\tau_2 + \tau_4} \sigma_2 + \\
 &+ z_3(t) e^{2\tau_3} \sigma_1 + z_3(t) e^{\tau_3 + \tau_4} \sigma_2 + z_4(t) e^{\tau_4 + \tau_3} \sigma_1 + z_4(t) e^{2\tau_4} \sigma_2] \Rightarrow \\
 \Rightarrow \widehat{M}(t, \tau) &= \frac{g(t)}{2} B(t) [z_1(t) e^{\tau_1 + \tau_3} \sigma_1 + z_1(t) e^{\tau_1 + \tau_4} \sigma_2 + z_2(t) e^{\tau_2 + \tau_3} \sigma_1 + \\
 &+ z_2(t) e^{\tau_2 + \tau_4} \sigma_2 + z_3(t) e^{2\tau_3} \sigma_1 + z_3(t) \sigma_2 + z_4(t) \sigma_1 + z_4(t) e^{2\tau_4} \sigma_2
 \end{aligned}$$

(note that in $\widehat{M}(t, \tau)$ there are no members containing e^{τ_1}, e^{τ_2} measurement exponents $|m| = 1$);

$$\begin{aligned}
 S(t, \tau) &\equiv \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) \left[\sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \right. \\
 &+ \left. \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j + m}(t) e^{(e_j + m, \tau)} \right] = \\
 &= \frac{g(t)}{2} B(t) \left[\sum_{2 \leq |m| \leq N_z}^* z^m(t) (e^{\tau_3 + (m, \tau)} \sigma_1 + e^{\tau_4 + (m, \tau)} \sigma_2) + \right. \\
 &+ \sum_{1 \leq |m| \leq N_z}^* z^{e_1 + m}(t) (e^{(e_1 + m, \tau) + \tau_3} \sigma_1 + e^{(e_1 + m, \tau) + \tau_4} \sigma_2) + \\
 &+ \left. \sum_{1 \leq |m| \leq N_z}^* z^{e_2 + m}(t) (e^{(e_2 + m, \tau) + \tau_3} \sigma_1 + e^{(e_2 + m, \tau) + \tau_4} \sigma_2) \right] \Rightarrow \widehat{S}(t, \tau) = \\
 &= \frac{g(t)}{2} B(t) \sum_{\substack{2 \leq |m| \leq N_z, \\ m_3 + 1 = m_4}} z^m(t) \sigma_1 + \sum_{\substack{2 \leq |m| \leq N_z, \\ m_4 + 1 = m_3}} z^m(t) \sigma_2 + \sum_{\substack{2 \leq |m| \leq N_z, \\ m_3 + 1 \neq m_4, m_4 + 1 \neq m_3}} z^m(t) e^{(m, \tau)} + \\
 &+ \left(\sum_{\substack{1 \leq |m| \leq N_z, \\ m_3 + 1 = m_4}} z^{e_1 + m}(t) \sigma_1 + \sum_{\substack{1 \leq |m| \leq N_z, \\ m_4 + 1 = m_3}} z^{e_1 + m}(t) \sigma_2 \right) e^{\tau_1} + \\
 &+ \left(\sum_{\substack{1 \leq |m| \leq N_z, \\ m_3 + 1 = m_4}} z^{e_2 + m}(t) \sigma_1 + \sum_{\substack{1 \leq |m| \leq N_z, \\ m_4 + 1 = m_3}} z^{e_2 + m}(t) \sigma_2 \right) e^{\tau_2} + \\
 &+ \sum_{j=1}^2 \sum_{\substack{1 \leq |m| \leq N_z, \\ m_3 + 1 \neq m_4, m_4 + 1 \neq m_3}}^* z^{e_j + m}(t) e^{(e_j + m, \tau)}.
 \end{aligned}$$

After embedding, the right-hand side of system (20) will look like

$$\begin{aligned} \widehat{G}(t, \tau) = & -\frac{\partial}{\partial t} [z_0(t) + \sum_{i=1}^4 z_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* z^m(t) e^{(m, \tau)} + \\ & + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_z}^* z^{e_j+m}(t) e^{(e_j+m, \tau)}] + \widehat{M}(t, \tau) + \widehat{S}(t, \tau) + Q(t, \tau), \end{aligned}$$

moreover, in $\widehat{S}(t, \tau)$ the coefficients at e^{τ_1}, e^{τ_2} do not depend on $z_k(t), k = 1, 2$. As indicated in [3], the embedding $G(t, \tau) \rightarrow \widehat{G}(t, \tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $\widehat{Z}(t, \tau)|_{\tau=\psi(t)/\varepsilon} \equiv Z(t, \tau)|_{\tau=\psi(t)/\varepsilon}$.

Theorem 2. Let conditions 1) and 2) be fulfilled and the right-hand side $H(t, \tau) = H_0(t) + \sum_{i=1}^4 H_i(t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_z}^* H^m(t) e^{(m, \tau)} + \sum_{j=1}^2 \sum_{1 \leq |m| \leq N_H}^* H^{e_j+m}(t) e^{(e_j+m, \tau)} \in U$ of system (10) satisfy condition (11). Then problem (10) under additional conditions

$$\langle \widehat{G}(t, \tau), \chi_k(t) e^{\tau_k} \rangle \equiv 0 \quad \forall t \in [t_0, T], k = 1, 2, \quad (17)$$

where

$$Q(t, \tau) = Q_0(t) + \sum_{k=1}^4 Q_k(t) e^{\tau_k} + \sum_{2 \leq |m| \leq N_z}^* Q^m(t) e^{(m, \tau)} + \sum_{k=1}^2 \sum_{1 \leq |m| \leq N_Q}^* Q^{e_k+m}(t) e^{(e_k+m, \tau)}$$

is the known vector function of space U , is uniquely solvable in U .

Proof. Since the right-hand side of system (10) satisfies condition (11), this system has a solution in space U in the form (15), where $\alpha_k(t) \in C^\infty([t_0, T], \mathbb{C}^1)$ are arbitrary functions so far, $k = 1, 2$. Submit (15) to the initial condition $z(t_0, 0) = z^*$. We get $\sum_{k=1}^2 \alpha_k(t_0) \varphi_k(t_0) = z_*$, where denoted

$$\begin{aligned} z_* = & z^* + A^{-1}(t_0) H_0(t_0) - \sum_{i=3}^4 [\lambda_i(t_0) I - A(t_0)]^{-1} H_i(t_0) - \\ & - \frac{(H^{e_1}(t_0), \chi_2(t_0))}{\lambda_1(t_0) - \lambda_2(t_0)} \varphi_2(t_0) - \frac{(H^{e_2}(t_0), \chi_1(t_0))}{\lambda_2(t_0) - \lambda_1(t_0)} \varphi_1(t_0) - \\ & - \sum_{2 \leq |m| \leq N_z}^* z^m(t_0) - \sum_{k=1}^2 \sum_{1 \leq |m^k| \leq N_H}^* [(m^k, \lambda(t_0)) I - A(t_0)]^{-1} H^{m^k}(t_0). \end{aligned}$$

Multiplying the equality $\sum_{k=1}^2 \alpha_k(t_0) \varphi_k(t_0) = z_*$ scalarly by $\chi_j(t_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\{\chi_j(t)\}$, we find the values $\alpha_k(t_0) = \frac{1}{2} (z_*, \chi_k(t_0))$, $k = 1, 2$. Now we submit the solution (15) to the condition of orthogonality (17). Considering that under these conditions, scalar multiplication performed by vector functions $\chi_k(t) e^{\tau_k}$, containing only exponents $e^{\tau_k}, k = 1, 2$, it is necessary to keep in the expression $\widehat{G}(t, \tau)$ only terms with exponents e^{τ_1} and e^{τ_2} . Then condition (17) takes the form

$$\begin{aligned} & \langle -\frac{\partial}{\partial t} \left(\sum_{k=1}^2 \alpha_k(t) \varphi_k(t) e^{\tau_k} + \frac{(H_1(t), \chi_2(t))}{\lambda_1(t) - \lambda_2(t)} \varphi_2(t) e^{\tau_1} + \frac{(H_2(t), \chi_1(t))}{\lambda_2(t) - \lambda_1(t)} \varphi_1(t) e^{\tau_2} \right) + \\ & + \left(\sum_{\substack{1 \leq |m| \leq N_z, \\ m_3 + 1 = m_4}} z^{e_1+m}(t) \sigma_1 + \sum_{\substack{1 \leq |m| \leq N_z, \\ m_4 + 1 = m_3}} z^{e_1+m}(t) \sigma_2 \right) e^{\tau_1} + \\ & + \left(\sum_{\substack{1 \leq |m| \leq N_z, \\ m_3 + 1 = m_4}} z^{e_2+m}(t) \sigma_1 + \sum_{\substack{1 \leq |m| \leq N_z, \\ m_4 + 1 = m_3}} z^{e_2+m}(t) \sigma_2 \right) e^{\tau_2} + \\ & + Q_1(t) e^{\tau_1} + Q_2(t) e^{\tau_2}, \chi_k(t) e^{\tau_k} \rangle \equiv 0 \quad \forall t \in [t_0, T], k = 1, 2. \end{aligned}$$

Performing here scalar multiplication, we obtain linear ordinary differential equations with respect to the functions $\alpha_k(t)$, involved in the solution (15) of system (10). Attaching to them the initial conditions

$\alpha_k(t_0) = \frac{1}{2}(z_*, \chi_k(t_0))$, $k = 1, 2$, computed earlier, we find uniquely the functions $\alpha_k(t) \in C^\infty([t_0, T], \mathbb{C}^1)$, $k = 1, 2$, and, therefore, we construct solution (15) in the space in a unique way. The theorem is proved.

Applying Theorems 1 and 2 to iterative problems (10_k) (in this case, the right-hand sides $H^{(k)}(t, \tau)$ of these problems are embedded in the space U , i.e. $H^{(k)}(t, \tau)$ we replace with $\hat{H}^{(k)}(t, \tau) \in U$), we find uniquely their solutions in space U and construct series (7). Just as in [3], we prove the following statement.

Theorem 3. Suppose that conditions (1) – (2) are satisfied for system (2). Then, when $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ is sufficiently small), system (2) has a unique solution $z(t, \varepsilon) \in C^1([0, T], \mathbb{C}^2)$; in this case, the estimate

$$\|z(t, \varepsilon) - z_{\varepsilon N}(t)\|_{C[0, T]} \leq c_N \varepsilon^{N+1},$$

holds true, where $z_{\varepsilon N}(t)$ is the restriction (for $\tau = \frac{\psi(t)}{\varepsilon}$) of the N -partial sum of series (9) (with coefficients $z_k(t, \tau) \in U$, satisfying the iteration problems (10_k)), and the constant $c_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

4. Construction of the solution of the first iteration problem in space U .

Using Theorem 1, we will try to find a solution to the first iteration problem (10₀). Since the right side $h(t)$ of the system (10₀) satisfies condition (11), this system has (according to (15)) a solution in space U in the form

$$z_0(t, \tau) = z_0^{(0)}(t) + \sum_{k=1}^2 \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau k}, \tag{18}$$

where $z_0^{(0)}(t)$ is the solution of the integrated system

$$z_0^{(0)}(t) = \int_{t_0}^t (-A^{-1}(t) K(t, s) z_0^{(0)}(s) ds - A^{-1}(t) h(t), \tag{19}$$

$\alpha_k^{(0)}(t) \in C^\infty([t_0, T], \mathbb{C}^1)$ are arbitrary functions. Subjecting (18) to the initial condition $z_0(t_0, 0) = z^0$, we will have

$$z_0^{(0)}(t_0) + \sum_{k=1}^2 \alpha_k^{(0)}(t_0) \varphi_k(t_0) = z^0 \Leftrightarrow \sum_{k=1}^2 \alpha_k^{(0)}(t_0) \varphi_k(t_0) = z^0 + A^{-1}(t_0) h(t_0).$$

Multiplying this equality scalarly by $\chi_j(t_0)$ and taking into account the biorthogonality of the systems $\{\varphi_k(t)\}$ and $\{\chi_j(t)\}$, we find the values $\alpha_k^{(0)}(t_0) = \frac{1}{2}(z^0 + A^{-1}(t_0) h(t_0), \chi_k(t_0))$, $k = 1, 2$. To fully compute the functions $\alpha_k^{(0)}(t)$, we proceed to the next iteration problem (10₁). Substituting into it the solution (16) of the system (10₀), we arrive at the following system:

$$\begin{aligned} L z_1(t, \tau) = & -\frac{d}{dt} z_0^{(0)}(t) - \sum_{k=1}^2 \frac{d}{dt} (\alpha_k^{(0)}(t) \varphi_k(t)) e^{\tau k} + \\ & + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) \left(z_0^{(0)}(t) + \sum_{k=1}^2 \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau k} \right) + \\ & + \sum_{j=1}^2 \left[\frac{(K(t, t) \alpha_j^{(0)}(t) \varphi_j(t))}{\lambda_j(t)} e^{\tau_j} - \frac{(K(t, t_0) \alpha_j^{(0)}(t_0) \varphi_j(t_0))}{\lambda_j(t_0)} \right] \end{aligned} \tag{20}$$

(here we used the expression (6₁) for $R_1 z(t, \tau)$ and took into account that for $z(t, \tau) = z_0(t, \tau)$ only the terms with e^{τ_1} and e^{τ_2} remain in the sum (6₁)). It is not difficult to see that the right side

$$\begin{aligned} H(t, \tau) = & -\frac{d}{dt} z_0^{(0)}(t) - \sum_{k=1}^2 \frac{d}{dt} (\alpha_k^{(0)}(t) \varphi_k(t)) e^{\tau k} + \\ & + \frac{g(t)}{2} (e^{\tau_3} \sigma_1 + e^{\tau_4} \sigma_2) B(t) \left(z_0^{(0)}(t) + \sum_{k=1}^2 \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau k} \right) + \\ & + \sum_{j=1}^2 \left[\frac{(K(t, t) \alpha_j^{(0)}(t) \varphi_j(t))}{\lambda_j(t)} e^{\tau_j} - \frac{(K(t, t_0) \alpha_j^{(0)}(t_0) \varphi_j(t_0))}{\lambda_j(t_0)} \right] \end{aligned}$$

of system (20) belongs to space U . System (20) is solvable in this space U if and only if conditions (11) are satisfied, which in our case take the form

$$\begin{aligned} & \left(-\frac{d}{dt}(\alpha_k^{(0)}(t)\varphi_k(t)) + \frac{(K(t,t)\alpha_k^{(0)}(t)\varphi_k(t))}{\lambda_k(t)}, \chi_k(t) \right) = 0 \Leftrightarrow \\ & \Leftrightarrow 2\frac{d\alpha_k^{(0)}(t)}{dt} = \left(\frac{(K(t,t)\varphi_k(t))}{\lambda_k(t)} - \dot{\varphi}_k(t), \chi_k(t) \right) \alpha_k^{(0)}(t), k = 1, 2. \end{aligned}$$

Attaching to this system the initial conditions $\alpha_k^{(0)}(t_0) = \frac{1}{2}(z^0 + A^{-1}(t_0)h(t_0), \chi_k(t_0))$, we find uniquely functions

$$\alpha_k^{(0)}(t) = \alpha_k^{(0)}(t_0) \exp \left\{ \frac{1}{2} \int_{t_0}^t \left(\frac{(K(s,s)\varphi_k(s))}{\lambda_k(s)} - \dot{\varphi}_k(s), \chi_k(s) \right) ds \right\}, k = 1, 2,$$

therefore, we uniquely calculate the solution (18) of the problem (10₀) in the space U . Moreover, the main term of the asymptotic of the solution to problem (2) has the form

$$\begin{aligned} z_{\varepsilon 0}(t) &= z_0^{(0)}(t) + \\ &+ \sum_{k=1}^2 \alpha_k^{(0)}(t_0) \exp \left\{ \frac{1}{2} \int_{t_0}^t \left(\frac{(K(s,s)\varphi_k(s))}{\lambda_k(s)} - \dot{\varphi}_k(s), \chi_k(s) \right) ds \right\} \varphi_k(t) e^{\frac{1}{\varepsilon} \int_{t_0}^t \lambda_k(\theta) d\theta}, \end{aligned}$$

where $\alpha_k^{(0)}(t_0) = \frac{1}{2}(z^0 + A^{-1}(t_0)h(t_0), \chi_k(t_0))$, $k = 1, 2$, $z_0^{(0)}(t)$ is the solution of the integra system (19).

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Б.Т. Калимбетов, В.Ф. Сафонов

Жылдам осцилляцияланатын коэффициентті сингуляр ауытқыған интегро-дифференциалдық теңдеулер

Динамикалық орнықтылықпен, периодты құрылымға ие ортамен байланысты және басқа да қолданбалы мәселелерді зерттеулер жылдам осцилляцияланатын коэффициентті дифференциалдық теңдеулермен айналысуды қажет етеді. Мұндай коэффициентті дифференциалдық теңдеулер жүйелерінің асимптотикалық интегралдау бөлшектеу және регуляризация әдістермен жүргізілген. Осы жұмыста интегралды-дифференциалдық теңдеулер жүйесі қарастырылған. Зерттеудің негізгі мақсаты — интегралдық мүшенің алғашқы есептің шешімінің асимптотикасына әсерін зерттеу. Есепте резонанс болмаған жағдай, яғни жылдам осцилляцияланатын коэффициенттің бүтін сызықтық комбинациясының жиіліктері шекті оператордың спектрінің жиілігімен сәйкес келмейтін жағдай, зерттелген.

Кілт сөздер: сингуляр ауытқу, интегро-дифференциалдық теңдеу, жылдам осцилляцияланатын коэффициент, регуляризация, асимптотикалық жинақтылық.

Б.Т. Калимбетов, В.Ф. Сафонов

Интегро-дифференциальные сингулярно возмущенные уравнения с быстро осциллирующими коэффициентами

При исследовании различных вопросов, связанных с динамической устойчивостью, со свойствами сред с периодической структурой, при исследовании других прикладных задач приходится иметь дело с дифференциальными уравнениями с быстро осциллирующими коэффициентами. Асимптотическое интегрирование дифференциальных систем уравнений с такими коэффициентами проводилось методами расщепления и регуляризации. В настоящей работе рассмотрена система интегрально-дифференциальных уравнений. Основная цель исследования состоит в выявлении влияния интегрального члена на асимптотику решения исходной задачи. Изучен случай отсутствия резонанса, т.е. случай, когда целочисленная линейная комбинация частот быстро осциллирующего коэффициента не совпадает с частотой спектра предельного оператора.

Ключевые слова: сингулярное возмущение, интегрально-дифференциальное уравнение, быстро осциллирующий коэффициент, регуляризация, асимптотическая сходимость.

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