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## Bounded and multiperiodic solutions of the system of partial integro-differential equations


The system of partial integro-differential equations with an operator of differentiation with respect to directions of vector field is considered. The considering integro-differential equation does not contain space variables. The matricant is constructed that satisfies the linear matrix equation and some of its properties and estimates are obtained that are related to multiperiodicity in time variables. An integral representation of the multiperiodic solution of this integro-differential system through the resolvent of the resolving kernel is given, recurrent relations are obtained for finding them. Some properties of iterated kernels and resolvent are established, corresponding estimates are found. On the basis of the necessary and sufficient condition of periodicity, multiperiodic solutions of a linear integro-differential equation and additional properties of solutions are found. Sufficient conditions for the existence of a bounded and unique multiperiodic solution on all independent variables of the characteristics system of integro-differential equations with a differentiation operator are established.



*Keywords:* matricant, resolvent, kernel, multiperiodicity, integro-differential, Dirichlet, vector field.


*Introduction*

We consider the partial integro-differential equation

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau}^{\tau+\theta} K(\tau,t,s,\sigma)u(s,\sigma)ds + f(\tau,t), \qquad (1)$$

where $u(\tau,t)$ is an unknown vector-function, $(\tau,t)\in R\times R^m$; $D_c = \frac{\partial}{\partial\tau} + \left\langle c, \frac{\partial}{\partial t}\right\rangle$ is differential operator, $m$ is vector $c$ constant, $\left\langle c, \frac{\partial}{\partial t}\right\rangle$ - scalar product of vectors $c$ and $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_m}\right)$; $\sigma = \sigma_0 + cs = t - c\tau + cs$ is the characteristic of the differentiation operator $D_c$ by the directions of the vector field $\frac{dt}{d\tau} = c$, $s \in R = (-\infty; +\infty)$; $(n\times n)$ are matrices $A(\tau,t)$ and $K(\tau,t,s,\sigma)$, $n$ is vector-function $f(\tau,t)$.

At the present stage of development of the theory of integro-differential equations, of considerable interest is the development of methods for the qualitative study of the solvability of the problems under consideration. Research of the theory integro-differential equations involved many authors. As you know, V. Volterra used integro-differential equations in problems of hereditary elasticity [1], justified the existence of periodic fluctuations in biological associations, created a general theory of functionals [2]. For systems of integro-differential equations, which form the basis of the mathematical theory of oscillatory processes in natural science and engineering, are devoted to the considerable amount of work, we note [3–5]. In [6], the distribution of the results of M. Urabe to systems of integro-differential equations was considered, the representations of the solution of integro-differential equations through the resolvent of the kernel are investigated [7; 158], the role of these equations at describing processes with aftereffects [8] is indicated. Integro-differential equations find applications in problems of the theory heredity [9; 109], hereditary elasticity of the model, metal creep at high temperatures [10]. Note that the rheological processes are described with integro-differential equations [10], [11; 136, 166]. The monograph [12] is devoted to research of almost periodic solutions system of equations with quasiperiodic right-hand sides. In [13, 14] almost periodic solutions of integro-differential transfer type equations were investigated. Questions of the existence and construction of multiperiodic and pseudoperiodic solutions system of integro-differential equations containing a spatial variable are considered in [15]. By various problems of the theory of periodic and almost periodic oscillations were outlined in [16, 17].

The purpose of the work is to establish sufficient conditions to existence and uniqueness of the multiperiodic solution of integro-differential equation (1) with operator of differentiation with respect to direction of vector field.

Suppose that the conditions of $(\theta,\omega)$-periodicity, continuity by $\tau \in R$ and continuous differentiability by $t \in R^m$:

$$A(\tau+\theta, t+q\omega) = A(\tau,t) \in C^{(0,e)}_{\tau,t}\left(R\times R^m\right), q\in Z^m, \tag{2}$$

$$K(\tau+\theta, t+q\omega, s,\sigma) = K(\tau,t,s,\sigma) \in C^{(0,e,0,e)}_{\tau,t,s,\sigma}\left(R\times R^m\times R\times R^m\right), q\in Z^m, \tag{3}$$

$$K(\tau+\theta, t+q\omega, s+\theta, t+q\omega - c(\tau+\theta)+c(s+\theta)) = K(\tau,t,s,t-c\tau+cs), q\in Z^m,$$

$$f(\tau+\theta, t+q\omega) = f(\tau,t) \in C^{(0,e)}_{\tau,t}\left(R\times R^m\right), q\in Z^m. \tag{4}$$

Here $m$ is vector $e=(1,\dots,1)$, the degree of smoothness by $t=(t_1,\dots,t_m)$; $Z^m$ is set of integer vectors $q=(q_1,\dots,q_m)$ and $q\omega=(q_1\omega_1,\dots,q_m\omega_m)$ is multiple period with multiplicity $q$ period $\omega=(\omega_1,\dots,\omega_m)$ and periods $\omega_0=\theta$, $\omega_1,\dots,\omega_m$ - are rationally incommensurable positive constants. Conditions (2) - (4) are called conditions (P).

*Main results*

We consider the homogeneous integro-differential equation

$$D_c u(\tau,t) = A(\tau,t)u(\tau,t) + \int\limits_{\tau}^{\tau+\theta} K(\tau,t,s,\sigma)u(s,\sigma)ds, \tag{5}$$

corresponding to the nonhomogeneous equation (1).

Taking into account $\sigma=\sigma_0+cs$ by the known technique [13], constructing the matricant of equation $D_c w = A(\tau,t)w$ we define the matrix $W(\tau_0,\tau,t)$ on the basis of integral matrix equation

$$W(\tau_0,\tau,t) = E + \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))W(\tau_0,s,t-c(\tau-s))ds \tag{6}$$

with the unit $n$-matrix $E$. Note, by virtue conditions (P), the matrix $W(s,\tau,t)$ is $(\theta,\theta,\omega)$-periodic by $s,\tau,t$: $W(s+\theta,\tau+\theta,t+q\omega) = W(s,\tau,t) \in C^{(0,0,e)}_{s,\tau,t}\left(R\times R\times R^m\right), q\in Z^m$. We sought the solution of the integral equation (6) in the form of series

$$W(\tau_0,\tau,t) = \sum_{m=0}^{\infty} W_m(\tau_0,\tau,t), \tag{7}$$

members of which we find from the recurrence relations: $W_0(\tau_0,\tau,t)=E$,

$$W_1(\tau_0,\tau,t) = \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))W_0(\tau_0,s,t-c(\tau-s))ds = \int\limits_{\tau_0}^{\tau} A(s,t-c(\tau-s))ds, \dots,$$

$$W_m(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}A(s,t-c(\tau-s))W_{m-1}(\tau_0,s,t-c(\tau-s))ds, m=1,2,\ldots.$$

We show that series (7) converges absolutely and uniformly. Take into account $A(\tau,t)$ matrix boundedness we have $\|W_0(\tau_0,\tau,t)\|=1$,

$$\|W_1(\tau_0,\tau,t)\|\le\int\limits_{\tau_0}^{\tau}\|A(s,t-c(\tau-s))\|ds\le\alpha|\tau-\tau_0|,\ldots,\|W_m(\tau_0,\tau,t)\|\le\alpha^m\frac{|\tau-\tau_0|^m}{m!},$$

where $\|A(\tau,t)\|=\sup\limits_{(\tau,t)\in R\times R^m}|A(\tau,t)|\le\alpha,\alpha=const.$

For the series (7) the majorant is series $1+\alpha M+\frac{\alpha^2M^2}{2!}+\ldots+\frac{\alpha^mM^m}{m!}+\ldots.$
Thus, the matrix series (7) converges absolutely and uniformly at $|\tau-\tau_0|\le M$, $M=const>0$. We assume that the matricant $W(\tau_0,\tau,t)$ for all $\tau\ge\tau_0$ and $t\in R^m$ satisfies condition

$$\|W(\tau_0,\tau,t)\|\le\Gamma e^{-\rho(\tau-\tau_0)},\Gamma\ge1,\rho>0. \quad (8)$$

By a direct verification we can make sure that the matricant $W(\tau_0,\tau,t)$ has the possesses multiperiodicity property with respect by $\tau_0,\tau$ and $t$: $W(\tau_0+\theta,\tau+\theta,t+q\omega)-W(\tau_0,\tau,t)=0,q\in Z^m$.

Now using the replacement

$$u(\tau,t)=W(\tau_0,\tau,t)\upsilon(\tau,t) \quad (9)$$

equation (5) is reduced to the form

$$D_c\upsilon(\tau,t)=\int\limits_{\tau}^{\tau+\theta}Q(\tau_0,\tau,t,s,t-c(\tau-s))\upsilon(s,t-c(\tau-s))ds \quad (10)$$

with the kernel $Q(\tau_0,\tau,t,s,\sigma)=W^{-1}(\tau_0,\tau,t)K(\tau,t,s,\sigma)W(\tau_0,s,\sigma)$, where $\sigma=t-c(\tau-s)$.
We note, based on conditions (P), the kernel $Q(\tau_0,\tau,t,s,t-c(\tau-s))$ has the property:

$$Q(\tau_0,\tau+\theta,t+q\omega,s+\theta,t+q\omega-c(\tau+\theta)+c(s+\theta))=Q(\tau_0,\tau,t,s,t-c(\tau-s)),q\in Z^m.$$

For the matricant $V(\tau_0,\tau,t)$ of equation (10), we have the matrix integral equation

$$V(\tau_0,\tau,t)=E+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))V(\tau_0,s,t-c(\tau-s))dsd\eta. \quad (11)$$

We sought the solution of the integral equation (11) as follows:

$$V(\tau_0,\tau,t)=\sum_{m=0}^{\infty}V_m(\tau_0,\tau,t) \quad (12)$$

with initial approach $V_0(\tau_0,\tau_0,t)=E$. $V_m(\tau_0,\tau,t)$ we find from the relation

$$V_m(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))V_{m-1}(\tau_0,s,t-c(\tau-s))dsd\eta, m=1,2,\ldots. \quad (13)$$

From (13) at $m=1$ follows $V_1(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\times$

$$\times V_0(\tau_0,s,t-c(\tau-s))dsd\eta=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta,$$

where $Q_1(\tau_0,\eta,t,s,t)=Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$.

Applying the Dirichlet permutation rule based on the product of two matrices $Q(\tau_0,\eta_1,t,s_1,t)\left(\int\limits_{\tau_0}^{\eta}\int\limits_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_1d\eta_1\right) \equiv Q_2(\tau_0,\eta,t,s,t)$ by integrating by $\tau=\eta$ at $t:=t-c(\tau-\eta)$ from $\tau_0$ to $\tau$, we obtain the matrix $V_2(\tau_0,\tau,t)$:

$$V_2(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta} Q_2(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta,$$

where $Q_2(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))=$

$$=\int\limits_{\eta}^{\tau}\int\limits_{\eta_1}^{\eta_1+\theta} Q(\tau_0,\eta_1,t-c(\tau-\eta_1),s_1,t-c(\tau-s_1))Q_1(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_1d\eta_1.$$

Further, continuing this process, we obtain

$$V_m(\tau_0,\tau,t)=\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta} Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta, \tag{14}$$

where the iterated kernels are determined from the recurrence relations

$$Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))=\int\limits_{\eta}^{\tau}\int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta} Q(\tau_0,\eta_{m-1},t-c(\tau-\eta_{m-1}),s_{m-1},t-c(\tau-s_{m-1}))\times$$

$$\times Q_{m-1}(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_{m-1}d\eta_{m-1}=$$

$$=\int\limits_{\eta}^{\tau}\int\limits_{\eta}^{\eta+\theta}\dots\int\limits_{\tau_0}^{\eta_m}\int\limits_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))Q(\tau_0,\eta_1,t-c(\tau-\eta_1),s_1,t-c(\tau-s_1))\times\dots\times$$

$$\times Q(\tau_0,\eta_{m-1},t-c(\tau-\eta_{m-1}),s_{m-1},t-c(\tau-s_{m-1}))ds_1d\eta_1ds_2d\eta_2\dots ds_{m-1}d\eta_{m-1}. \tag{15}$$

*Lemma 1.* For iterated kernels the estimates are valid:

$$\|Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|\le Q_0^m\frac{\theta^{m-1}(\tau-\eta)^{m-1}}{(m-1)!}. \tag{16}$$

Proof. On the grounds (15) we have:

$$\|Q_1(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|\le Q_0, Q_0=const,$$

$$\|Q_2(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|\le\int\limits_{\eta}^{\tau}\int\limits_{\eta_1}^{\eta_1+\theta}\|Q(\tau_0,\eta_1,t-c(\tau-\eta_1),s_1,t-c(\tau-s_1))\|\times$$

$$\times\|Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|ds_1d\eta_1\le Q_0^2\theta(\tau-\eta),\dots.$$

Further

$$\|Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|\le\int\limits_{\eta}^{\tau}\int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta}\|Q(\tau_0,\eta_{m-1},t-c(\tau-\eta_{m-1}),s_{m-1},t-c(\tau-s_{m-1}))\times$$

$$\times Q_{m-1}(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|ds_{m-1}d\eta_{m-1}\le Q_0^m\frac{\theta^{m-1}(\tau-\eta)^{m-1}}{(m-1)!}.$$

Taking into account (16) from (14) we get

$$\|V_m(\tau_0,\tau,t)\| \leq Q_0^m \frac{\theta^m(\tau-\tau_0)^m}{m!}, m=1,2,\dots. \tag{17}$$

From (17) it follows that the series (12) converges absolutely and uniformly with respect to $\tau$ and $t$ for $\tau \geq \tau_0$.

Note that all iterated $Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ kernels have the property:

$$Q_m(\tau_0+\theta,\eta,t+q\omega-c((\tau+\theta)-\eta),s,t+q\omega-c((\tau+\theta)-s)) = Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)).$$

On the basis of conditions (P), it can be shown that the matrix $V(\tau_0,\tau,t)$ is $(\theta,\theta,\omega)$-periodic by $\tau_0,\tau,t$: $V(\tau_0+\theta,\tau+\theta,t+q\omega) = V(\tau_0,\tau,t) \in C^{(0,0,e)}_{\tau_0,\tau,t}(R\times R\times R^m), q\in Z^m$.

For the series (12), using (14) we have:

$$V(\tau_0,\tau,t) = E + \int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}\sum_{m=1}^{\infty} Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \tag{18}$$

The series $\sum\limits_{m=1}^{\infty} Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ converges absolutely and uniformly to the continuous function $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$, called the resolvent of kernel.

Resolved kernel satisfies the integral equation

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$

$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta}^{\eta+\theta} Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \tag{19}$$

We note, at $\eta=\tau$, the resolvent becomes the kernel

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q_m(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)).$$

We find the solution of integral equation (19) in the form

$$R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)) = Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$

$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_1}^{\eta_1+\theta} Q(\tau_0,\eta_1,t-c(\tau-\eta_1),s_1,t-c(\tau-s_1))Q_1(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_1d\eta_1+$$

$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_2}^{\eta_2+\theta} Q(\tau_0,\eta_2,t-c(\tau-\eta_2),s_2,t-c(\tau-s_2))Q_2(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_2d\eta_2+\dots+$$

$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta_{m-1}}^{\eta_{m-1}+\theta} Q(\tau_0,\eta_{m-1},t-c(\tau-\eta_{m-1}),s_{m-1},t-c(\tau-s_{m-1}))\times$$

$$\times Q_{m-1}(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds_{m-1}d\eta_{m-1}+\cdots. \tag{20}$$

Further, based on the estimates (16), we obtain $\|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \leq$

$$\leq Q_0\left(1+Q_0\theta(\tau-\eta)+\frac{Q_0^2\theta^2(\tau-\eta)^2}{2!}+\dots+\frac{Q_0^{m-1}\theta^{m-1}(\tau-\eta)^{m-1}}{(m-1)!}\right).$$

Therefore, $\|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \leq Q_0e^{Q_0\theta(\tau-\eta)}$,
where $\|Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\| \leq Q_0$.

Applying the resolvent (20) to (18) we have

$$V(\tau_0,\tau,t)=E+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta. \tag{21}$$

It is directly possible to show that $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ has continuous partial derivatives with respect to $\tau$ and $t$.

We note, the following properties of the resolvent:
1) We show that $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ satisfies equation (10). Indeed, applying the operator $D_c$ to both sides of equation (19) we find:

$$D_cR(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))=D_cQ(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))+$$

$$+\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds+$$

$$+\int\limits_{\eta}^{\tau}\int\limits_{\eta}^{\eta+\theta}D_c[Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))]dsd\eta=$$

$$=\int\limits_{\eta}^{\eta+\theta}Q(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))ds.$$

Thus, the resolvent satisfies the integro-differential equation. Note that by the way we used some of the properties of operator $D_c$, which was to be proven.
2) Resolvent $R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))$ is the multiperiodic function:

$$R(\tau_0+\theta,\eta,t+q\omega-c((\tau+\theta)-\eta),s,t+q\omega-c((\tau+\theta)-s))=R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s)),q\in Z^m.$$

*Lemma 2.* For the function $V(\tau_0,\tau,t)$, the following estimate takes place:

$$\|V(\tau_0,\tau,t)\|\le e^{Q_0\theta(\tau-\tau_0)}. \tag{22}$$

Proof. Using (13), (16), (21) we get

$$\|V(\tau_0,\tau,t)\|\le 1+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}\|R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))\|dsd\eta\le$$

$$\le 1+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}Q_0e^{Q_0\theta(\tau-\eta)}dsd\eta\le e^{Q_0\theta(\tau-\tau_0)}.$$

That which was to be shown.

Further, taking into account (9), (21) we have

$$U(\tau_0,\tau,t)=W(\tau_0,\tau,t)\left(E+\int\limits_{\tau_0}^{\tau}\int\limits_{\eta}^{\eta+\theta}R(\tau_0,\eta,t-c(\tau-\eta),s,t-c(\tau-s))dsd\eta\right). \tag{23}$$

Using (8) and (22) we estimate (23)

$$\|U(\tau_0,\tau,t)\|\le\|W(\tau_0,\tau,t)\|\|V(\tau_0,\tau,t)\|\le\Gamma e^{\chi(\tau-\tau_0)},\chi=Q_0\theta-\rho<0. \tag{$*$}$$

Find a function $u(\tau,t)$, that satisfies for all $\tau>\tau_0$ and $t\in R^m$ to integro-differential equation (1) and initial condition

$$u(\tau_0,t)=\varphi(t)\in C_t^e(R^m). \tag{24}$$

Solution to the Cauchy problem (1), (24) is sought in the form

$$u(\tau,t)=U(\tau_0,\tau,t)\varphi(t-c(\tau-\tau_0))+\int\limits_{\tau_0}^{\tau}U(s,\tau,t)f(s,t-c(\tau-s))ds. \tag{25}$$

In (25) assuming that the vector function is $\varphi(t)$ any from $C_t^e(R^m)$, using the necessary and sufficient condition for the multiperiodicity of Umbetzhanov-Sartabanov

$$u(\tau_0+\theta,t)=u(\tau_0,t)\in C_{\tau,t}^{(0,e)}(R\times R^m) \tag{26}$$

we search among the solutions (25) for the multiperiodic solution of system (1).

Using the necessary and sufficient condition for periodicity (26) to solve (25), we have

$$\varphi(t)=U(\tau_0,\tau_0+\theta,t)\varphi(t-c((\tau_0+\theta)-s))+$$

$$+\int\limits_{\tau_0}^{\tau_0+\theta}U(s,\tau_0+\theta,t)f(s,t-c((\tau_0+\theta)-s))ds. \tag{27}$$

Introducing the designation $\psi(t)=\int\limits_{\tau_0}^{\tau_0+\theta}U(s,\tau_0+\theta,t)f(s,t-c((\tau_0+\theta)-s))ds$, making a shift by period $\theta$ and taking into account the $\theta$-periodicity of the matricant $U(\tau_0,\tau,t-c(\tau-s))$ and vector-functions $f(\tau,t-c(\tau-s))$ we find $\psi(t)=\int\limits_{\tau_0-\theta}^{\tau_0}U(s,\tau_0,t)f(s,t-c(\tau_0-s))$. Putting $\varphi_0(t)=\psi(t)$, solving equation (27) by the method of successive approximations, we have:

$$\varphi_m(t)=U(\tau_0,\tau_0+\theta,t-c((\tau_0+\theta)-s))\varphi_{m-1}(t-c(\tau_0-s))+\psi(t), m=1,2,\ldots. \tag{28}$$

Along the way, using the convolution type formula for $m=1$ from (28) we get

$$\varphi_1(t)=U(\tau_0,\tau_0+\theta,t)\varphi_0(t-c(\tau_0-s))+\psi(t)=U(\tau_0,\tau_0+\theta,t)\int\limits_{\tau_0-\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau_0-s))ds+$$

$$+\int\limits_{\tau_0-\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau_0-s))ds=\int\limits_{\tau_0-2\theta}^{\tau_0-\theta}U(s,\tau,t)f(s,t-c(\tau_0-s))ds+$$

$$+\int\limits_{\tau_0-\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau_0-s))ds=\int\limits_{\tau_0-2\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau_0-s))ds.$$

Further, using the method of complete mathematical induction, assuming at $k=m$, we have $\varphi_m(t)=\int\limits_{\tau_0-m\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau_0-s))ds$. In the last integral, replacing the integration variable. Using the $\theta$-periodicity of vector-function $f(\tau,t-c(\tau-s))$ with respect to the variable $\tau$ we set $\varphi_k(t)=\int\limits_{\tau_0-k\theta}^{\tau_0}U(s,\tau,t)f(s,t-$ $-c(\tau_0-s))ds$, $k=0,1,2,\ldots$. Thus we have

$$\varphi^*(t)=\sum_{k=0}^{\infty}\int\limits_{\tau_0-k\theta}^{\tau_0}U(s,\tau,t)f(s,t-c(\tau-s))ds=$$

$$= \int_{-\infty}^{\tau_0} U(s,\tau,t)f(s,t-c(\tau-s))ds. \qquad (29)$$

Substituting (29) into (25), by the way using the group property, we obtain:

$$u(\tau,t) = U(\tau_0,\tau,t)\int_{-\infty}^{\tau_0} U(s,\tau,t)f(s,t-c(\tau-s))ds+$$

$$+\int_{\tau_0}^{\tau} U(s,\tau,t)f(s,t-c(\tau-s))ds = \int_{-\infty}^{\tau} U(s,\tau,t))f(s,t-c(\tau-s))ds.$$

In this way,

$$u^*(\tau,t) = \int_{-\infty}^{\tau} U(s,\tau,t)f(s,t-c(\tau-s))ds. \qquad (30)$$

The convergence of the improper integral in the right-hand side of (30) is ensured by the boundedness of the vector function $f(\tau,t-c(\tau-s))$.

We note some properties of the vector-function $u^*(\tau,t)$:

1) function $u^*(\tau,t)$ satisfies the integro-differential equation (1) and at $\tau\to\tau_0+0$ it turns into $\varphi^*(t)$;

2) it is the multiperiodic function by $\tau$ and $t$ with a period vector $(\theta,\omega)$.

Indeed, consider

$$u^*(\tau+\theta,t+q\omega) = \int_{-\infty}^{\tau+\theta} U(s,\tau+\theta,t+q\omega)f(s,t+q\omega-c((\tau+\theta)-s))ds =$$

$$= \int_{-\infty}^{\tau} U(s+\theta,\tau+\theta,t+q\omega)f(s+\theta,t+q\omega-c((\tau+\theta)-(s+\theta)))ds =$$

$$= \int_{-\infty}^{\tau} U(s,\tau,t)f(s,t-c(\tau-s))ds = u^*(\tau,t).$$

That which was to be demonstrated.

3) for functions $u^*(\tau,t)$ the inequality is fair

$$\|u^*(\tau,t)\| \le \Gamma f_0 \left|\chi\right|^{-1}, \qquad (31)$$

where $\|f(\tau,t-c(\tau-s))\| = \sup\limits_{(\tau,t)\in R\times R^m} |f(\tau,t-c(\tau-s))| \le f_0, f_0 = const.$

4) the solution $u^*(\tau,t)$ is unique.

Result obtained is formulated as the theorem.

*Theorem.* If conditions (P) and (*) are satisfied, then there is a unique multiperiodic solution $u^*(\tau,t)$ to system (1), defined as (30) and satisfying condition (31).

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# Дербес туындылы интегро-дифференциалдық теңдеулер жүйесінің шенелген және көппериодты шешімдері

Векторлық өріс бағыттары бойынша дифференциалдау операторлы дербес туындылы интегро-дифференциалдық теңдеулер жүйесі зерттелді. Қарастырылатын интегро-дифференциалдық теңдеуге кеңістік айнымалылары енбейді. Сызықты матрицалық теңдеуге қанағаттандыратын матрицант тұрғызылды және уақыт айнымалылары бойынша көппериодтылықпен байланысты оның кейбір қасиеттері мен бағалаулары алынды. Қарастырылатын интегро-дифференциалдық жүйенің көппериодты шешімінің шешуші ядроның резольвентасы арқылы интегралдық түрі келтірілген, оларды табу үшін рекурренттік қатынастар алынған. Итерацияланған ядролар мен резольвентаның кейбір қасиеттері орнатылған, сәйкес бағалаулар тағайындалған. Периодтылықтың қажетті және жеткілікті шарты негізінде сызықты интегро-дифференциалдық теңдеудің көппериодты шешімдері табылған, сонымен қатар шешімнің қосымша қасиеттері анықталған. Дифференциалдау операторлы интегро-дифференциалдық теңдеулер жүйесінің сипаттамаларында шенелген, барлық тәуелсіз айнымалылары бойынша көппериодты шешімінің бар және жалғыз болуының жеткілікті шарттары орнатылған.

*Кілт сөздер:* матрицант, резольвента, ядро, көппериодтылық, интегро-дифференциалды, Дирихле, векторлық өріс.

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# Ограниченные и многопериодические решения системы интегро-дифференциальных уравнений в частных производных

Исследована система интегро-дифференциальных уравнений в частных производных с оператором дифференцирования по направлениям векторного поля. Рассматриваемое интегро-дифференциальное уравнение не содержит пространственных переменных. Построен матрицант, удовлетворяющий линейному матричному уравнению, и получены некоторые его свойства и оценки, связанные с многопериодичностью по временным переменным. Приведено интегральное представление многопериодического решения данной интегро-дифференциальной системы через резольвенту разрешающего ядра, получены рекуррентные соотношения для их нахождения. Установлены некоторые свойства итерированных ядер и резольвенты, найдены соответствующие оценки. На основе необходимого и достаточного условия периодичности найдены многопериодические решения линейного интегро-дифференциального уравнения, а также выявлены дополнительные свойства решений. Установлены достаточные условия существования ограниченного и единственного многопериодического по всем независимым переменным решения на характеристиках системы интегро-дифференциальных уравнений с оператором дифференцирования.

*Ключевые слова:* матрицант, резольвента, ядро, многопериодичность, интегро-дифференциальное, Дирихле, векторное поле.

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