UDC 517.927.6

Z.Yu. Fazullin[1], B.E. Kanguzhin[2], A.A. Seitova[2]

[1]*Bashkir State University, Ufa, Russia;*
[2]*Al-Farabi Kazakh National University, Almaty, Kazakhstan*
(E-mail: fazullinzu@mail.ru)

Stable perturbations of boundary problems for differential equations

In this paper, we expand the class of nondegenerate two-point boundary value problems for the Sturm-Liouville equation, which have a complete system of eigenfunctions and associated functions in special function spaces. Such spaces depend on the length of support of the potential of the Sturm-Liouville equation. The formulated results clarify well-known results of V.A. Marchenko. Two-point boundary value problems for the Sturm-Liouville equation are divided into degenerate and nondegenerate boundary conditions in the sense of V.A. Marchenko. The main result of V.A. Marchenko asserts that systems of eigenfunctions and associated functions of nondegenerate boundary value problems for the Sturm-Liouville equation form a complete system of functions in the space of square-summable functions. In this paper, the result of V.A. Marchenko is clarified in the following direction. There are operators with a complete system of eigenfunctions and associated functions in the space of square-summable functions among the degenerate boundary value problems in the sense of V.A. Marchenko. The presence of the completeness property depends on the length of support of the measure which is antisymmetry to the potential of the Sturm-Liouville equation.

Keywords: finite nonempty set, eigenvalue, three-point boundary value problem, Volterra operator, nondegenerate boundary condition.

1 Introduction

Let $n = 2\mu$ and let us consider a self-adjoint differential expression on the interval $[0,1]$ with real coefficients:

$$l\left(y\right)\equiv\left(p_0y^{(\mu)}\right)^{(\mu)}+\left(p_1y^{(\mu-1)}\right)^{(\mu-1)}+\cdots+\left(p_{\mu-1}y^{(1)}\right)^{(1)}+p_\mu y,$$

where $p_{0,}\cdots,\ p_\mu$ are sufficiently smooth functions.

Let us given a system of linear functionals $\{U_1,\cdots,\ U_n\}$. Let B_p denote the operator generated by the differential expression $l(\cdot)$ and boundary conditions

$$U_j\left(y\right)=0,\ j=1,\cdots,n.$$

In this paper, we study the question: how do coefficients $p_{0,}\cdots,\ p_\mu$ of the operator B_p influence the structure of the spectrum of the operator and the completeness of the system of root functions in $L_2(0,\ 1)$?

As usual, we introduce the fundamental system of solutions $\{y_1(x,\lambda),\cdots,\ y_n(x,\ \lambda)\}$ of a homogeneous differential equation $l\left(y\right)=\lambda\cdots y(x)$ with initial conditions $y_k^{(j-1)}\left(0\right)=\ \delta_{jk}$. Denote by $\Delta_P(\lambda)$ the characteristic determinant of the operator B_p, which is given by the following formula

$$\Delta_P\left(\lambda\right)=det\left\{\|U_j(y_k)\|\right\}.$$

It is well known that $\Delta_P\left(\lambda\right)$ is an entire function of exponential type by a parameter λ. Zeros of the entire function $\Delta_P\left(\lambda\right)$ uniquely characterize the spectrum of the operator B_p. If λ_s is a eigenvalue of the operator B_p with multiplicity m_s, then λ_s is a zero of $\Delta_P\left(\lambda\right)$ with multiplicity m_s. Inverse statement is also true.

Basically, zeros of an entire function $\Delta_P(\lambda)$ can be represented as:

– empty set;

– finite nonempty set;

– a countable set without end points;

– the set coinciding with the complex plane.

The first case is realized when the operator B_p corresponds to the Cauchy problem. Note that, there are examples of operators with an empty spectrum even it is given with boundary conditions

$$U_j\left(y\right)=0,\ j=1,\cdots,n,$$

which is not the Cauchy conditions. Operators with an empty spectrum are often called Volterra operators. Examples of the Volterra operators representing three-point boundary value problems for second-order differential operators can be found in the work of S.A. Dzhumabaev, D.B. Nurakhmetov [1].

However, there are still no examples of differential operators with nonempty finite spectrum. It is proved that if the spectrum of the operator B_p represents a nonempty finite set, then its power is not greater than μ. In the case of $n = 2$. T.Sh. Kalmenov and A. Shaldanbayev [2] proved that there is no boundary value problem with a nonempty finite spectrum.

When the spectrum of the operator B_p is a countable set without finite limit points, then the operator B_p is said to have a discrete spectrum. Operators with discrete spectrum occur frequently.

More rarely, there are operators with a spectrum that coincides with the whole complex plane. Similar examples of two-point boundary value problems for higher order differential operators were given in [3].

A detail investigation of the second order differential operators with two-point boundary conditions were studied in the monograph by V.A. Marchenko [4]. The main result of [4] that interests us is that there nondegenerate two-point boundary conditions for a second order differential equation were selected and also it was proved the completeness of the corresponding system of root functions in $L_2(0,1)$.

In this work, it is clarified the result of V.A. Marchenko. We select operators from the class of degenerate boundary conditions for a second order differential equation which have complete system of root functions in a special functional space.

2 Two-point boundary value problems for the Sturm-Liouville equation

Consider an eigenvalue problem

$$-y^{(2)} + q\left(x\right)y = \ \lambda\ y\ ,\ 0 < x < 1$$

with boundary conditions

$$U_j\left(y\right) = \ a_{i1}y\left(0\right) + \ a_{i2}y^{(1)}\left(0\right) + a_{i3}y\left(1\right) + \ a_{i4}y^{(1)}\left(1\right) = 0\ ,\ i = 1,2,$$

where $q(x)$ is a summable function on $(0,1), a_{ij}$ are complex numbers. The boundary conditions are identified using the matrix

$$A = \left\| \begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right\|,\quad rank(A) = 2.$$

We will denote by A_{ij} the minor of the matrix A composed of columns with numbers i and j. We also introduce the function

$$\hat{q}\left(x\right) = q\left(x\right),\ 0 \le x \le \frac{1}{2},\ \ \hat{q}\left(x\right) = q\left(1-x\right),\ \frac{1}{2} < x \le 1.$$

The difference $q\left(x\right) - \hat{q}\left(x\right)$ is denoted by $Q(x)$. It is clear that

$$Q(x) \equiv 0,\ 0 \le x \le \frac{1}{2}.$$

We introduce solutions $\hat{c}\left(x,\lambda\right), \hat{s}\left(x,\lambda\right)$ of a homogeneous equation

$$-y^{(2)} + \hat{q}\left(x\right)y = \lambda y,\ 0 < x < 1$$

with Cauchy conditions

$$\hat{c}\left(\frac{1}{2},\lambda\right) = 1,\ \hat{s}\left(\frac{1}{2},\lambda\right) = 0,\ \hat{c'}\left(\frac{1}{2},\lambda\right) = 0,\ \hat{s'}\left(\frac{1}{2},\lambda\right) = 1.$$

Similarly, we introduce the solutions $c\left(x,\lambda\right),\ s\left(x,\lambda\right)$ of the homogeneous equation

$$-y^{(2)} + q\left(x\right)y = \ \lambda\ y\ ,\ 0 < x < 1.$$

It is easy to verify that for all $0 < x < 1$ the following relations are true

$$\hat{c}\,(x,\lambda) = \hat{c}\,(1-x,\lambda)\,,\, \hat{s}\,(x,\lambda) = -\hat{s}\,(1-x,\lambda)\,.$$

It can also be understood that for all $0 < x < 1$ integral equations are valid

$$c\,(x,\lambda) = \hat{c}\,(x,\lambda) + \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q(t) c\,(t,\lambda)\,dt;$$

$$s\,(x,\lambda) = \hat{s}\,(x,\lambda) + \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q\,(t)\,s\,(t,\lambda)\,dt.$$

The characteristic determinant is given by the formula

$$\Delta_q\,(\lambda) = det\left\{\|U_j\,(c\,(\cdot,\lambda)) \quad U_j\,(s\,(\cdot,\lambda))\,, \quad j=1,\ 2\|\right\}.$$

Let

$$r\,(x,\lambda) = \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q(t) c\,(t,\lambda)\,dt;$$

$$p\,(x,\lambda) = \int_{0,5}^{x} \begin{vmatrix} \hat{c}\,(t,\lambda) & \hat{s}\,(t,\lambda) \\ \hat{c}\,(x,\lambda) & \hat{s}\,(x,\lambda) \end{vmatrix} Q\,(t)\,s\,(t,\lambda)\,dt.$$

Similar to the work of V.A. Marchenko [4], let us formulate a statement in the case of a symmetric potential $\hat{q}\,(x)$.

Lemma 1. We have

$$\Delta_{\hat{q}}\,(\lambda) = A_{12} + A_{34} + A_{14} + A_{32} + 2A_{31}\hat{c}\,(0,\lambda)\,\hat{s}\,(0,\lambda) +$$

$$+2\,(A_{14}+A_{32})\,\hat{s}\,(0,\lambda)\,\hat{c}'\,(0,\lambda) + 2A_{24}\hat{c}'\,(0,\lambda)\,\hat{s}'\,(0,\lambda)\,.$$

We now write the representation of the characteristic determinant in the case of an arbitrary potential.

Lemma 2. The following formula holds

$$\Delta_q(\lambda){=}\Delta_{\hat{q}}(\lambda) + \Delta_{\hat{q}}(\lambda)\int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t)Q(z)c(z,\lambda)s(t,\lambda) \begin{vmatrix} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{vmatrix} +$$

$$+\int_{0.5}^{1} Q(t)s(t,\lambda)\{A_{13}\hat{c}(t,\ \lambda)\hat{s}(0,\lambda)\hat{s}(0,\lambda) + (A_{23}{+}A_{41})\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{s'}(0,\lambda){+}A_{42}\hat{c}(t,\ \lambda)\hat{s'}(0,\lambda)\hat{s'}(0,\lambda){+}$$

$$+(A_{23}{+}A_{43})\hat{s}(t,\lambda) + A_{13}\hat{s}(t,\ \lambda)\hat{c}(0,\lambda)\hat{s}(0,\lambda){+}$$

$$+(A_{23}{+}A_{41})\hat{s}(t,\ \lambda)\hat{s}(0,\lambda)\hat{c'}(0,\lambda){+}A_{42}\hat{s}(t,\lambda)\hat{s'}(0,\lambda)\hat{c'}(0,\lambda)\}dt{+}$$

$$+\int_{0.5}^{1} Q(t)c(t,\lambda)\{A_{31}\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{c}(0,\ \lambda) + (A_{32}{+}A_{14})\hat{c}(t,\lambda)\hat{s}(0,\lambda)\hat{c'}(0,\lambda){+}A_{24}\hat{c}(t,\lambda)\hat{c'}(0,\lambda)\hat{s'}(0,\lambda){+}$$

$$+(A_{14}{+}A_{34})\hat{c}(t,\lambda) + A_{31}\hat{s}(t,\lambda)\hat{c}(0,\lambda)\hat{c}(0,\lambda){+}$$

$$+(A_{32}{+}A_{14})\hat{s}(t,\lambda)\hat{c}(0,\lambda)\hat{c'}(0,\lambda){+}A_{24}\hat{s}(t,\lambda)\hat{c'}(0,\lambda)\hat{c'}(0,\lambda)\}dt.$$

It follows from the result of V.A. Marchenko that the matrix A defines nondegenerate boundary conditions if at least one of the following conditions hold:

$A_{24} \neq 0.$

$A_{24} = 0,\ A_{14} + A_{32} \neq 0.$

$A_{24} = 0,\ A_{14} + A_{32} = 0,\ A_{31} \neq 0$

and the system of eigenfunctions and associated functions of the Sturm-Liouville problem represents a complete system in the space $L_2(0,1)$.

Let matrix A define degenerate boundary conditions, i.e.

$$A_{24} = 0,\ A_{14} + A_{32} = 0,\ A_{31} = 0.$$

Then, Lemma 2 implies

$$\Delta_q(\lambda) = (A_{12} + A_{34})$$
$$\left(1 + \int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t) Q(z) c(z,\lambda) s(t,\lambda) \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{array} \right| \right) +$$
$$+ \int_{0.5}^{1} Q(t) s(t,\lambda) \{(A_{14} - A_{34}) \hat{s}(t, \lambda)\} dt +$$
$$+ \int_{0.5}^{1} Q(t) c(t,\lambda) \{(A_{14} + A_{34}) \hat{c}(t, \lambda)\} dt.$$

Since $rankA = 2$, it follows that $A_{12} \neq 0$.

Hence, for $A_{12} \neq 0$, it follows from the last relation that

$$\frac{\Delta_q(\lambda)}{A_{12}} = \left(1 - \theta^2\right) \times$$
$$\times \left(1 + \int_{0,5}^{1} dt \int_{0.5}^{1} dz Q(t) Q(z) c(z,\lambda) s(t,\lambda) \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}(z,\lambda) & \hat{s}(z,\lambda) \end{array} \right| \right) -$$
$$-\theta(1+\theta) \int_{0.5}^{1} Q(t) s(t,\lambda) \hat{s}(t, \lambda) dt -$$
$$-\theta(1-\theta) \int_{0.5}^{1} Q(t) c(t,\lambda) \hat{c}(t, \lambda) dt$$

for some θ.

In the case of degenerate boundary conditions, it is necessary to investigate completeness of the system of root functions in $L_2(0,1)$. As in [4], we introduce the solutions of a homogeneous equation

$$-y^{(2)} + q(x) y = \lambda\, y, \ 0 < x < 1$$

by formulas

$$\omega_1(x,\lambda) = \left| \begin{array}{cc} c(x,\lambda) & s(x,\lambda) \\ U_2(c(\cdot,\lambda)) & U_2(s(\cdot,\lambda)) \end{array} \right|;$$
$$\omega_2(x,\lambda) = \left| \begin{array}{cc} U_1(c(\cdot,\lambda)) & U_1(s(\cdot,\lambda)) \\ c(x,\lambda) & s(x,\lambda) \end{array} \right|.$$

The following representations are also useful

$$\omega_1(x,\lambda) = \left| \begin{array}{cc} c(x,\lambda) & s(x,\lambda) \\ U_2(\hat{c}(\cdot,\lambda)) & U_2(\hat{s}(\cdot,\lambda)) \end{array} \right| + \left| \begin{array}{cc} c(x,\lambda) & s(x,\lambda) \\ U_2(r(\cdot,\lambda)) & U_2(p(\cdot,\lambda)) \end{array} \right| =$$
$$= \left\{ a_{21} \left| \begin{array}{cc} c(x,\lambda) & s(x,\lambda) \\ (1+\theta)\hat{c}(0,\lambda) & (1-\theta)\hat{s}(0,\lambda) \end{array} \right| + a_{22} \left| \begin{array}{cc} c(x,\lambda) & s(x,\lambda) \\ (1+\theta)\hat{c}'(0, \lambda) & (1-\theta)\hat{s}'(0, \lambda) \end{array} \right| \right\} +$$
$$+\theta \int_{0,5}^{1} \left| \begin{array}{cc} c(t,\lambda) & s(t,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{array} \right| Q(t) \left\{ a_{21} \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ -\hat{c}(0,\lambda) & \hat{s}(0,\lambda) \end{array} \right| - a_{22} \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}'(0, \lambda) & -\hat{s}'(0, \lambda) \end{array} \right| \right\} dt;$$
$$\omega_2(x, \lambda) = \left\{ a_{11} \left| \begin{array}{cc} (1+\theta)\hat{c}(0,\lambda) & (1-\theta)\hat{s}(0,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{array} \right| + a_{12} \left| \begin{array}{cc} (1+\theta)\hat{c}'(0, \lambda) & (1-\theta)\hat{s}'(0, \lambda) \\ c(x,\lambda) & s(x,\lambda) \end{array} \right| \right\} -$$
$$-\theta \int_{0,5}^{1} \left| \begin{array}{cc} c(t,\lambda) & s(t,\lambda) \\ c(x,\lambda) & s(x,\lambda) \end{array} \right| Q(t) \left\{ a_{11} \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ -\hat{c}(0,\lambda) & \hat{s}(0,\lambda) \end{array} \right| - a_{12} \left| \begin{array}{cc} \hat{c}(t,\lambda) & \hat{s}(t,\lambda) \\ \hat{c}'(0, \lambda) & -\hat{s}'(0, \lambda) \end{array} \right| \right\} dt.$$

Main result: Let

$$r = max_{x \in suppQ} > \frac{1}{2} \text{ and } A_{24} = 0,\ A_{14} + A_{32} = 0,\ A_{31} = 0.$$

The system of eigenfunctions and associated functions of the two-point boundary value problem for the Sturm-Liouville equation is complete in the space $L_2(\frac{1}{2} - r, \frac{1}{2} + r)$. The proof of the main result is carried out by the method of V.A. Marchenko [4] with the involvement of some modifications.

References

1 Джумабаев С.А. О вольтерровых трехточечных задачах для оператора Штурма-Лиувилля, связанных с симметрией потенциала / С.А. Джумабаев, Д.Б. Нурахметов // Матем. заметки. — 2018. — 104:4. — С. 632–636.

2 Кальменов Т.Ш. О структуре спектра краевой задачи Штурма-Лиувилля на конечном отрезке времени / Т.Ш. Кальменов, А.Ш. Шалданбаев // Известия АН РК. Сер. физ.-мат. — 2000. — № 3. — С. 29–34.

3 Садовничий В.А. О связи между спектром дифференциального оператора с симметричными коэффициентами и краевыми условиями / В.А. Садовничий, Б.Е. Кангужин // ДАН СССР. — 1982. — Т. 267. — № 2. — С. 310–313.

4 Марченко В.А. Операторы Штурма-Лиувилля и их приложения / В.А. Марченко. — Киев: Наук. думка, 1977. — С. 330.

З.Ю. Фазуллин, Б.Е. Кангужин, А.А. Сеитова

Дифференциалдық теңдеулер үшін шекаралық есептердің орнықты ауытқулары

Мақалада Штурм-Лиувилл теңдеуіне ақаулы емес екінүктелі шекаралық есептердің кластары кеңейтілді. Кеңейтілген класқа тиісті есептердің меншекті және қосалқы функциялар жүйелері арнайы функционалдық кеңістіктерде толық болады. Авторлармен құрастырылған арнайы кеңістіктер Штурм-Лиувилл теңдеуінің потенциалының тасушысының ұзындығына тәуелді. Мақаладағы нәтижелер В.А. Марченконың белгілі жетістіктерін кеңейтеді. Штурм-Лиувилл теңдеуіне сәйкес екінүктелі шекаралық шарттар, В.А. Марченко бойынша, ақаулы және ақаулы емес шеттік шарттарға бөлінеді. В.А. Марченконың негізгі тұжырымы бойынша, Штурм-Лиувилл теңдеуіне сәйкес ақаулы емес шекаралық есептердің меншікті және қосалқы функциялар жүйесі арнайы функционалдық кеңістікте толық жүйе құрайды. В.А. Марченко бойынша, ақаулы шекаралық есептердің арасында меншікті және қосалқы функциялар жүйелері арнайы кеңістіктерде толық болуы мүмкін екендігі көрсетілген.

Кілт сөздер: ақырлы бос емес жиынтық, меншікті мән, үшнүктелі шекаралық есептер, Вольтерра операторлары, ерекше емес шекаралық шарттар.

З.Ю. Фазуллин, Б.Е. Кангужин, А.А. Сеитова

Устойчивые возмущения граничных задач для дифференциальных уравнений

В статье расширен класс невырожденных двухточечных граничных задач для уравнения Штурма-Лиувилля, имеющих полную систему собственных и присоединенных функций в специальных функциональных пространствах. Указанные специальные пространства зависят от длины носителя потенциала уравнения Штурма-Лиувилля. Сформулированные результаты уточняют известные результаты В.А. Марченко. Двухточечные краевые задачи для уравнения Штурма-Лиувилля делятся на вырожденные и невырожденные, по В.А. Марченко, граничные условия. Основной результат В.А. Марченко утверждает, что системы собственных и присоединенных функций невырожденных граничных задач для уравнения Штурма-Лиувилля в пространстве квадратично суммируемых функций образуют полную систему функций. Авторами статьи уточнен результат В.А. Марченко в следующем направлении. Среди вырожденных граничных задач, по В.А. Марченко, имеются задачи с полной системой собственных и присоединенных функций в пространстве квадратично суммируемых функций. Наличие свойства полноты зависит от длины носителя меры антисимметрии носителя потенциала уравнения Штурма-Лиувилля.

Ключевые слова: конечное непустое множество, собственное значение, трехточечные краевые задачи, вольтерровы операторы, невырожденные граничные условия.

References

1 Dzhumabaev, S.A., & Nurakhmetov, D.B. (2018). O volterrovykh trekhtochechnykh zadachakh dlia operatora Shturma-Liuvillia, sviazannykh s simmetriei potentsiala [On Volterra three-point problems for the Sturm–Liouville operator related to potential symmetry]. *Matematicheskie zametki – Mathematical notes, 104:4,* 632–636 [in Russian].

2 Kalmenov, T.Sh., & Shaldanbaev, A.A. (2000). O strukture spektra kraevoi zadachi Shturma-Liuvillia na konechnom otrezke vremeni [On the structure of the spectrum of the Sturm-Liouville boundary value problem on a finite time interval]. *Izvestiia AN RK. Seriia fiziko-matematicheskaia – Proceedings of the Academy of Sciences of the Republic of Kazakhstan. Physics and Mathematics Series, 3,* 29–34 [in Russian].

3 Sadovnichy, V.A., & Kanguzhin, B.E. (1982). O sviazi mezhdu spektrom differentsialnoho operatora s simmetrichnymi koeffitsientami i kraevymi usloviiami [On the relationship between the spectrum of a differential operator with symmetric coefficients and boundary conditions]. *DAN SSSR – DAN USSR, Vol. 267, 2,* 310–313 [in Russian].

4 Marchenko, V.A. (1977). Operatory Shturma-Liuvillia i ix prilozheniia [The Sturm-Liouville Operators and Their Applications]. Kiev: Naukova Dumka [in Russian].