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On generic structures preserving elementary equivalence and elementary embeddability

We consider criteria for elementary equivalence and elementary embeddability for generic structures. They use classical characterizations for the general case. The criterion for elementary equivalence is based on the well known Fraïssé–Taimanov–Ehrenfeucht overturning method. The criterion for elementary embeddability uses the known Tarski–Vaught test.

Keywords: generic structure, elementary equivalence, Fraïssé–Taimanov–Ehrenfeucht's method, elementary embeddability, Tarski-Vaught test.

We consider criteria for elementary equivalence and elementary embeddability for generic structures [1-6]. They are based on classical characterizations for the general case. The criterion for elementary equivalence uses the well known Fraïssé–Taimanov–Ehrenfeucht overturning method [7-12]. The criterion for elementary embeddability is based on the Tarski-Vaught test [11, 13].

1 Preliminaries

We consider collections of sentence and formulas in first order logic over a language Σ. Thus, as usual, $\vdash$ means proof from no hypotheses deducing $\vdash \varphi$ for a formula φ of language Σ, which may contain function symbols and constants. If deducing φ, hypotheses in a set Φ of formulas can be used, we write $\Phi \vdash \varphi$. Usually Σ will be fixed in context and not mentioned explicitly.

Below we write $X, Y, Z, \dots$ for finite sets of variables, and denote by $A, B, C, \dots$ finite sets of elements, as well as finite sets in structures, or else the structures with finite universes themselves.

In diagrams, $A, B, C, \dots$ denote finite sets of constant symbols disjoint from the constant symbols in Σ and $\Sigma(A)$ is the vocabulary with the constants from A adjoined. $\Phi(A), \Psi(B), \mathrm{X}(C)$ stand for Σ-*diagrams* (of sets A, B, C), that is, *consistent sets of* $\Sigma(A)$-, $\Sigma(B)$-, $\Sigma(C)$-*sentences*, respectively.

Below we assume that for any considered diagram $\Phi(A)$, if a_1, a_2 are distinct elements in A then $\neg(a_1 \approx a_2) \in \Phi(A)$. This means that if c is a constant symbol in Σ, then there is at most one element $a \in A$ such that $(a \approx c) \in \Phi(A)$.

If $\Phi(A)$ is a diagram and B is a set, we denote by $\Phi(A)|_B$ the set $\{\varphi(\bar{a}) \in \Phi(A) \mid \bar{a} \in B\}$. Similarly, for a language Σ, we denote by $\Phi(A)|_\Sigma$ the restriction of $\Phi(A)$ to the set of formulas in the language Σ.

Definition [1-6]. We denote by $[\Phi(A)]^A_B$ the diagram $\Phi(B)$ obtained by replacing a subset $A' \subseteq A$ by a set $B' \subseteq B$ of constants disjoint from Σ and with $|A'| = |B'|$, where $A \setminus A' = B \setminus B'$. Similarly we call the consistent set of formulas denoted by $[\Phi(A)]^A_X$ the type $\Phi(X)$ if it is the result of a bijective substitution into $\Phi(A)$ of variables of X for the constants in A. In this case, we say that $\Phi(B)$ is a *copy* of $\Phi(A)$ and a *representative* of $\Phi(X)$. We also denote the diagram $\Phi(A)$ by $[\Phi(X)]^X_A$.

Remark. If the vocabulary contains functional symbols then diagrams $\Phi(A)$ containing equalities and inequalities of terms can generate both finite and infinite structures. The same effect is observed for purely predicate vocabularies if it is written in $\Phi(A)$ that the model for $\Phi(A)$ should be infinite. For instance, diagrams containing axioms for finitely axiomatizable theories have this property.

By the definition, for any diagram $\Phi(A)$, each constant symbol in Σ appears in some formula of $\Phi(A)$. Thus, $\Phi(A)$ can be considered as $\Phi(A \cup K)$, where K is the set of constant symbols in Σ.

We now give conditions on a partial ordering of a collection of diagrams which suffice for it to determine a structure. We modify some of the conditions for structures by d to signify they are conditions on diagrams not structures.

Definition [1-6]. Let Σ be a vocabulary. We say that $(\mathbf{D}_0;\leqslant)$ (or $\mathbf{D}_0$) is *generic*, or *generative*, if $\mathbf{D}_0$ is a class of Σ-diagrams of finite sets so that $\mathbf{D}_0$ is partially ordered by a binary relation $\leqslant$ such that $\leqslant$ is preserved by bijective substitutions, i. e., if $\Phi(A)\leqslant\Psi(B)$, and $A'\subseteq B'$ such that $[\Phi(A)]^A_{A'}=\Phi(A')$ and $[\Psi(B)]^B_{B'}=\Psi(B')$ are defined, then $[\Phi(A)]^A_{A'}$, $[\Psi(B)]^B_{B'}$ are in $\mathbf{D}_0$ and $[\Phi(A)]^A_{A'}\leqslant[\Psi(B)]^B_{B'}$. [1] Furthermore:

(i) if $\Phi(A)\in\mathbf{D}_0$ then for any quantifier free formula $\varphi(\bar{x})$ and any tuple $\bar{a}\in A$ either $\varphi(\bar{a})\in\Phi(A)$ or $\neg\varphi(\bar{a})\in\Phi(A)$;

(ii) if $\Phi\leqslant\Psi$ then $\Phi\subseteq\Psi$;[2]

(iii) if $\Phi\leqslant \mathrm{X}$, $\Psi\in\mathbf{D}_0$, and $\Phi\subseteq\Psi\subseteq \mathrm{X}$, then $\Phi\leqslant\Psi$;

(iv) some diagram $\Phi_0(\emptyset)$ is the least element of the system $(\mathbf{D}_0;\leqslant)$, and $\mathbf{D}_0\setminus\{\Phi_0(\emptyset)\}$ is nonempty;

(v) (the *d-amalgamation property*) for any diagrams $\Phi(A)$, $\Psi(B)$, $\mathrm{X}(C)\in\mathbf{D}_0$, if there exist injections $f_0\colon A\to B$ and $g_0\colon A\to C$ with $[\Phi(A)]^A_{f_0(A)}\leqslant\Psi(B)$ and $[\Phi(A)]^A_{g_0(A)}\leqslant \mathrm{X}(C)$, then there are a diagram $\Theta(D)\in\mathbf{D}_0$ and injections $f_1\colon B\to D$ and $g_1\colon C\to D$ for which $[\Psi(B)]^B_{f_1(B)}\leqslant\Theta(D)$, $[\mathrm{X}(C)]^C_{g_1(C)}\leqslant\Theta(D)$ and $f_0\circ f_1=g_0\circ g_1$; the diagram $\Theta(D)$ is called the *amalgam* of $\Psi(B)$ and $\mathrm{X}(C)$ over the diagram $\Phi(A)$ and witnessed by the four maps (f_0,g_0,f_1,g_1);

(vi) (the *local realizability property*) if $\Phi(A)\in\mathbf{D}_0$ and $\Phi(A)\vdash\exists x\,\varphi(x)$, then there are a diagram $\Psi(B)\in\mathbf{D}_0$, $\Phi(A)\leqslant\Psi(B)$, and an element $b\in B$ for which $\Psi(B)\vdash\varphi(b)$;

(vii) (the *d-uniqueness property*) for any diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$ if $A\subseteq B$ and the set $\Phi(A)\cup\Psi(B)$ is consistent then $\Phi(A)=\{\varphi(\bar{b})\in\Psi(B)\mid \bar{b}\in A\}$.

A diagram Φ is called a *strong subdiagram* of a diagram Ψ if $\Phi\leqslant\Psi$.

A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a diagram $\Psi(B)$ if there is an injection $f\colon A\to B$ such that $[\Phi(A)]^A_{f(A)}\subseteq\Psi(B)$ ($[\Phi(A)]^A_{f(A)}\leqslant\Psi(B)$). The injection f, in this instance, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in diagram $\Psi(B)$ and is denoted by $f\colon\Phi(A)\to\Psi(B)$. A diagram $\Phi(A)$ is said to be (*strongly*) *embeddable* in a structure $\mathcal{M}$ if $\Phi(A)$ is (strongly) embeddable in some diagram $\Psi(B)$, where $\mathcal{M}\models\Psi(B)$. The corresponding embedding $f\colon\Phi(A)\to\Psi(B)$, in this case, is called a (*strong*) *embedding* of diagram $\Phi(A)$ in structure $\mathcal{M}$ and is denoted by $f\colon\Phi(A)\to\mathcal{M}$.

Let $\mathbf{D}_0$ be a class of diagrams, $\mathbf{P}_0$ be a class of structures of some language, and $\mathcal{M}$ be a structure in $\mathbf{P}_0$. The class $\mathbf{D}_0$ is *cofinal* in the structure $\mathcal{M}$ if for each finite set $A\subseteq M$, there are a finite set B, $A\subseteq B\subseteq M$, and a diagram $\Phi(B)\in\mathbf{D}_0$ such that $\mathcal{M}\models\Phi(B)$. The class $\mathbf{D}_0$ is *cofinal* in $\mathbf{P}_0$ if $\mathbf{D}_0$ is cofinal in every structure of $\mathbf{P}_0$. We denote by $\mathbf{K}(\mathbf{D}_0)$ the class of all structures $\mathcal{M}$ with the condition that $\mathbf{D}_0$ is cofinal in $\mathcal{M}$, and by $\mathbf{P}$ a subclass of $\mathbf{K}(\mathbf{D}_0)$ such that each diagram $\Phi\in\mathbf{D}_0$ is true in some structure in $\mathbf{P}$.

Now we extend the relation $\leqslant$ from the generative class $(\mathbf{D}_0;\leqslant)$ to a class of subsets of structures in the class $\mathbf{K}(\mathbf{D}_0)$.

Let $\mathcal{M}$ be a structure in $\mathbf{K}(\mathbf{D}_0)$, A and B be finite sets in $\mathcal{M}$ with $A\subseteq B$. We call A a *strong subset* of the set B (in the structure $\mathcal{M}$), and write $A\leqslant B$, if there exist diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$, for which $\Phi(A)\leqslant\Psi(B)$ and $\mathcal{M}\models\Psi(B)$.

A finite set A is called a *strong subset* of a set $M_0\subseteq M$ (in the structure $\mathcal{M}$), where $A\subseteq M_0$, if $A\leqslant B$ for any finite set B such that $A\subseteq B\subseteq M_0$ and $\Phi(A)\subseteq\Psi(B)$ for some diagrams $\Phi(A),\Psi(B)\in\mathbf{D}_0$ with $\mathcal{M}\models\Psi(B)$. If A is a strong subset of M_0 then, as above, we write $A\leqslant M_0$. If $A\leqslant M$ in $\mathcal{M}$ then we refer to A as a *self-sufficient set* (in $\mathcal{M}$).

Notice that, by the d-uniqueness property, the diagrams $\Phi(A)$ and $\Psi(B)$ specified in the definition of strong subsets are defined uniquely. A diagram $\Phi(A)\in\mathbf{D}_0$, corresponding to a self-sufficient set A in $\mathcal{M}$, is said to be a *self-sufficient diagram* (in $\mathcal{M}$).

Definition [1-6]. class $(\mathbf{D}_0;\leqslant)$ possesses the *joint embedding property* (JEP) if for any diagrams $\Phi(A)$, $\Psi(B)\in\mathbf{D}_0$, there is a diagram $\mathrm{X}(C)\in\mathbf{D}_0$ such that $\Phi(A)$ and $\Psi(B)$ are strongly embeddable in $\mathrm{X}(C)$.

Clearly, every generative class has JEP since JEP means the d-amalgamation property over the empty set.

Definition [1-6]. A structure $\mathcal{M}\in\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0;\leqslant)$, or is *finitely generated over* Σ, if any finite set $A\subseteq M$ is contained in some finite self-sufficient set in $\mathcal{M}$, i. e., there is a finite set B with $A\subseteq B\subseteq M$ and $\Psi(B)\in\mathbf{D}_0$ such that $\mathcal{M}\models\Psi(B)$ and $\Psi(B)\leqslant\mathrm{X}(C)$ for any $\mathrm{X}(C)\in\mathbf{D}_0$ with $\mathcal{M}\models\mathrm{X}(C)$ and $\Psi(B)\subseteq\mathrm{X}(C)$. A class $\mathbf{P}$ has *finite closures* with respect to the class $(\mathbf{D}_0;\leqslant)$, or is *finitely generated over* Σ, if each structure in $\mathbf{P}$ has finite closures (with respect to $(\mathbf{D}_0;\leqslant)$).

[1]Note that $\mathbf{D}_0$ is closed under bijective substitutions since $\leqslant$ is preserved by bijective substitutions and $\leqslant$ is reflexive.

[2]Note that $\Phi(A)\leqslant\Psi(B)$ implies $A\subseteq B$, since if $a\in A$ then $(a\approx a)\in\Phi(A)$, so $\Phi(A)\leqslant\Psi(B)$ implies $\Phi(A)\subseteq\Psi(B)$ and we have $(a\approx a)\in\Psi(B)$, whence $a\in B$.

Clearly, an at most countable structure $\mathcal{M}$ has finite closures with respect to $(\mathbf{D}_0;\leqslant)$ if and only if $M=\bigcup\limits_{i\in\omega}A_i$ for some self-sufficient sets A_i with $A_i\leqslant A_{i+1}$, $i\in\omega$.

Note that the finite closure property is defined modulo Σ and does not correlate with the cardinalities of algebraic closures. For instance, if Σ contains infinitely many constant symbols then $\mathrm{acl}(A)$ is always infinite whereas a finite set A can or can not be extended to a self-sufficient set.

Besides, for the finite closures of sets A we consider finite self-sufficient extensions B in a given structure $\mathcal{M}$ with respect to $(\mathbf{D}_0;\leqslant)$ only and B can be both a universe of a substructure of $\mathcal{M}$ or not. Moreover, it is permitted that corresponding diagrams $\Psi(B)$ can have only finite, finite and infinite, or only infinite models.

Thus, for instance, a finitely axiomatizable theory without finite models and with a generative class $(\mathbf{D}_0;\subseteq)$, containing diagrams for all finite sets and with axioms in diagrams, has identical finite closures whereas each diagram in $\mathbf{D}_0$ has only infinite models.

Definition [1-6]. A structure $\mathcal{M}\in\mathbf{K}(\mathbf{D}_0)$ is $(\mathbf{D}_0;\leqslant)$-*generic*, or a *generic limit for the class* $(\mathbf{D}_0;\leqslant)$ and denoted by $\mathrm{glim}(\mathbf{D}_0;\leqslant)$, if it satisfies the following conditions:

(a) $\mathcal{M}$ has finite closures with respect to $\mathbf{D}_0$;

(b) if $A\subseteq M$ is a finite set, $\Phi(A),\Psi(B)\in\mathbf{D}_0$, $\mathcal{M}\models\Phi(A)$ and $\Phi(A)\leqslant\Psi(B)$, then there exists a set $B'\leqslant M$ such that $A\subseteq B'$ and $\mathcal{M}\models\Psi(B')$.

Clearly, uncountable $(\mathbf{D}_0;\leqslant)$-generic structures can be non-isomorphic. Indeed, for instance, all infinite structures in the empty language are generic for a given generative class although these structures are non-isomorphic for distinct cardinalities. But, as the following theorem shows, they are isomorphic for at most countable cases.

Theorem 1.1 [1-6]. *For any generative class* $(\mathbf{D}_0;\leqslant)$ *with at most countably many diagrams whose copies form* $\mathbf{D}_0$*, there exists at most countable* $(\mathbf{D}_0;\leqslant)$*-generic structure, unique up to isomorphism.*

Theorem 1.2 [1-6]. *Every* ω*-homogeneous structure* $\mathcal{M}$ *is* $(\mathbf{D}_0;\leqslant)$*-generic for some generative class* $(\mathbf{D}_0;\leqslant)$*.*

Thus any first-order theory has a generic model and therefore can be represented by it.

2 Elementary equivalence and elementary embeddability

Recall that structures $\mathcal{M}_1$ and $\mathcal{M}_2$ in a language Σ are *elementarily equivalent* (denoted by $\mathcal{M}_1\equiv\mathcal{M}_2$) if for any sentence φ in the language Σ, $\mathcal{M}_1\models\varphi$ if and only if $\mathcal{M}_2\models\varphi$.

Definition [11]. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be structures in a language Σ. An injective map $f\colon X\to M_2$, where $X\subseteq M_1$, is a *partial isomorphism of* $\mathcal{M}_1$ *into* $\mathcal{M}_2$ if for every elements $a_1,\ldots,a_n\in X$ the following conditions hold:

1) for any functional symbol $F^{(n)}\in\Sigma$ and correspondent operations $F_{\mathcal{M}_1}$ and $F_{\mathcal{M}_2}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$f(F_{\mathcal{M}_1}(a_1,\ldots,a_n))=F_{\mathcal{M}_2}(f(a_1),\ldots,f(a_n));$$

2) for any predicate symbol $P^{(n)}\in\Sigma$ and correspondent predicates $P_{\mathcal{M}_1}$ and $P_{\mathcal{M}_2}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$(a_1,\ldots,a_n)\in P_{\mathcal{M}_1}\Leftrightarrow(f(a_1),\ldots,f(a_n))\in P_{\mathcal{M}_2}.$$

A partial isomorphism $f\colon X\to M_2$ is called *finite* if the set X is finite.

The set of finite partial isomorphisms of $\mathcal{M}_1$ into $\mathcal{M}_2$ is denoted by $P(\mathcal{M}_1,\mathcal{M}_2)$.

The following well-known theorem uses the Fraïssé–Taimanov–Ehrenfeucht overturning method [7-10]. It is broadly used, in particular, in [12].

Theorem 2.1 [11]. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be structures in a language* Σ*. The following conditions are equivalent:*

(1) *the structures* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *are elementarily equivalent;*

(2) *for any* $n\in\omega$ *and any finite language* $\Sigma_0\subseteq\Sigma$ *there are nonempty sets* $Z_1(\Sigma_0,n),\ldots,Z_n(\Sigma_0,n)$ *of finite partial isomorphisms of* $\mathcal{M}_1|_{\Sigma_0}$ *into* $\mathcal{M}_2|_{\Sigma_0}$ *such that for any* $f\in Z_i(\Sigma_0,n)$, $1\leq i<n$*, and for any* $a\in M_1$, $b\in M_2$ *there are* $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$*, for which* $a\in\delta_{g_1}$, $b\in\rho_{g_2}$ *and* $f\subseteq g_1\cap g_2$*.*

Notice that considering $(\mathbf{D}_i;\leqslant_i)$-generic structures $\mathcal{M}_i$ in a language Σ, $i=1,2$, we take elements for extensions $g_1,g_2\in Z_{i+1}(\Sigma_0,n)$ in diagrams $\Phi(A)$ and $\Psi(B)$ in generative classes satisfying $\mathcal{M}_1\models\Phi(A)$ and $\mathcal{M}_2\models\Psi(B)$. Moreover, since the sets A and B are finite, we can replace addition of elements a and b by addition of self-sufficient sets A and B. Finite partial isomorphisms $f\colon X\to M_2$ with $X=A$ or $\rho_f=B$ are called *coordinated* with given generative classes, *coordinated generic*, or simply *generic*.

The set of generic finite partial isomorphisms of $\mathcal{M}_1$ into $\mathcal{M}_2$ is denoted by $\mathrm{PG}(\mathcal{M}_1,\mathcal{M}_2)$.

We have $\mathrm{PG}(\mathcal{M}_1, \mathcal{M}_2) \subseteq P(\mathcal{M}_1, \mathcal{M}_2)$ and each partial isomorphism in $P(\mathcal{M}_1, \mathcal{M}_2)$ is extensible till a partial isomorphism in $\mathrm{PG}(\mathcal{M}_1, \mathcal{M}_2)$. Thus, for generic structures in Theorem 2.1 it suffices to consider generic finite partial isomorphisms in $\mathrm{PG}(\mathcal{M}_1, \mathcal{M}_2)$, with their restrictions, and a modification of that theorem holds allowing syntactically, in terms of generative classes, characterize the elementary equivalence for generic structures. Below we consider that generic modification, whose proof can be easily obtained from the proof of [11, Theorem 5.1.1].

Theorem 2.2 Let $\mathcal{M}_i$ be $(\mathbf{D}_i; \leqslant_i)$-generic structures in a language Σ, $i = 1, 2$. The following conditions are equivalent:

(1) *the structures $\mathcal{M}_1$ and $\mathcal{M}_2$ are elementarily equivalent;*

(2) *for any $n \in \omega$ and any finite language $\Sigma_0 \subseteq \Sigma$ there are nonempty sets $Z_1(\Sigma_0, n), \ldots, Z_n(\Sigma_0, n)$ of restrictions of generic finite partial isomorphisms of $\mathcal{M}_1|_{\Sigma_0}$ into $\mathcal{M}_2|_{\Sigma_0}$ such that the following condition holds:*

$(*)$ *for any $f \in Z_i(\Sigma_0, n)$, $1 \le i < n$, and for any $a \in M_1$, $b \in M_2$ there are $g_1, g_2 \in Z_{i+1}(\Sigma_0, n)$, for which $a \in \delta_{g_1}$, $b \in \rho_{g_2}$ and $f \subseteq g_1 \cap g_2$.*

Remark 2.3. Following Theorem 2.2 and adding for any $f \in Z_i(\Sigma_0, n)$ and for any $a \in M_1$, $b \in M_2$ all elements in some self-sufficient sets $A \supset \delta_f \cup \{a\}$ and $B \supset \rho_f \cup \{b\}$ we can consider sequences $Z_1(\Sigma_0, n), \ldots, Z_n(\Sigma_0, n)$ of nonempty families of generic finite partial isomorphisms with the property of sequential extensions by $g_1, g_2 \in Z_{i+1}(\Sigma_0, n)$ with $a \in \delta_{g_1}$, $b \in \rho_{g_2}$ and $f \subseteq g_1 \cap g_2$.

Proposition 2.4. If $\mathcal{M}_i$ are elementarily equivalent $(\mathbf{D}_i; \leqslant_i)$-generic structures, $i = 1, 2$, then the classes $(\mathbf{D}_i; \leqslant_i)$ can be extended, with some extensions of their diagrams, till a common generative class $(\mathbf{D}_0; \leqslant)$.

Proof. Since $\mathcal{M}_1 \equiv \mathcal{M}_2$, complete diagrams $\Phi^*(A)$ for finite sets in $\mathcal{M}_1$ and in $\mathcal{M}_2$ can be collected for a homogeneous model $\mathcal{M}$ of the theory $\mathrm{Th}(\mathcal{M}_1) = \mathrm{Th}(\mathcal{M}_2)$ realizing the complete types $\Phi^*(X)$. The complete diagrams for $\mathcal{M}$ form the required generative class $(\mathbf{D}_0; \subseteq)$. □

Proposition 2.4 immediately implies

Corollary 2.5. Any elementarily equivalent generic structures are isomorphic to some restrictions of a common generic structure.

Since any countable structure has a countable homogeneous elementary extension and homogeneous structures are generic, Corollary 2.5 has the following modification:

Corollary 2.6. Any elementarily equivalent countable structures are isomorphic to some restrictions of a common (countable) generic structure.

Recall that a substructure $\mathcal{M}_1 = \langle M_1; \Sigma \rangle$ of $\mathcal{M}_2 = \langle M_2; \Sigma \rangle$ is called an *elementary substructure* (denoted by $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$), if for any formula $\varphi(x_1, \ldots, x_n)$ in the language Σ and for any elements $a_1, \ldots, a_n \in M_1$ the condition $\mathcal{M}_1 \models \varphi(a_1, \ldots, a_n)$ is equivalent to $\mathcal{M}_2 \models \varphi(a_1, \ldots, a_n)$. Here the structure $\mathcal{M}_2$ is an *elementary extension* of $\mathcal{M}_1$. If $M_1 \neq M_2$, we write $\mathcal{M}_1 \prec \mathcal{M}_2$ instead of $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$. If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and the condition $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$ ($\mathcal{M}_1 \prec \mathcal{M}_2$) does not hold, we write $\mathcal{M}_1 \not\preccurlyeq \mathcal{M}_2$ (respectively $\mathcal{M}_1 \not\prec \mathcal{M}_2$).

The following well-known *Tarski–Vaught test* [11, 13] is used for the checking that a substructure is an elementary one.

Theorem 2.7. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be structures in a language Σ, $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The following conditions are equivalent:

(1) $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$;

(2) *for any formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language Σ and for any elements $a_1, \ldots, a_n \in M_1$, if $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M_1$ such that $\mathcal{M}_2 \models \varphi(a_0, a_1, \ldots, a_n)$.*

In the following theorem, we obviously modify Theorem 2.7 for generic cases.

Theorem 2.8. Let $\mathcal{M}_1$ be a $(\mathbf{D}_1; \leqslant_1)$-generic structures in a language Σ, $\mathcal{M}_1 \subseteq \mathcal{M}_2$. The following conditions are equivalent:

(1) $\mathcal{M}_1 \preccurlyeq \mathcal{M}_2$;

(2) *for any formula $\varphi(x_0, x_1, \ldots, x_n)$ in the language Σ and for any elements $a_1, \ldots, a_n$ forming a self-sufficient set $A \leqslant_1 M_1$, if $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ then there is an element $a_0 \in M_1$ in a self-sufficient set $B \leqslant_1 M_1$ such that $A \leqslant_1 B$ and $\mathcal{M}_2 \models \varphi(a_0, a_1, \ldots, a_n)$.*

Remark 2.9. If in Theorem 2.8 the diagrams $\Phi(A), \Psi(B) \in \mathbf{D}_1$, for the sets A and B, force the complete types $\mathrm{tp}(A)$, $\mathrm{tp}(B)$, respectively, we take formulas $\exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ and $\varphi(a_0, a_1, \ldots, a_n)$ which are forced by $\Phi(A)$ and $\Psi(B)$, respectively.

Recall that an *elementary embedding* of a structure $\mathcal{M}_1$ into a structure $\mathcal{M}_2$ of the same language Σ is a map $f \colon M_1 \to M_2$ such that for every Σ-formula $\varphi(x_1, \ldots, x_n)$ and all elements $a_1, \ldots, a_n$ of M_1, $\mathcal{M}_1 \models \varphi(a_1, \ldots, a_n)$ if and only if $\mathcal{M}_2 \models \varphi(f(a_1), \ldots, f(a_n))$. In such a case, f is really an embedding denoted by $f \colon \mathcal{M}_1 \to \mathcal{M}_2$ and for $\mathcal{M}_1$ and $\mathcal{M}_2$ we say that $\mathcal{M}_1$ is *elementarily embeddable* into $\mathcal{M}_2$.

Similarly to Theorems 2.7 and 2.8, the following theorems characterize the elementary embeddability in general case and for generic structures, respectively.

Theorem 2.10. Let $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be structures in a language* Σ, $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ *be an embedding. The following conditions are equivalent:*

(1) *the embedding* f *is elementary*;

(2) *for any formula* $\varphi(x_0, x_1, \ldots, x_n)$ *in the language* Σ *and for any elements* $a_1, \ldots, a_n \in M_1$, *if* $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, f(a_1), \ldots, f(a_n))$ *then there is an element* $a_0 \in M_1$ *such that* $\mathcal{M}_2 \models \varphi(f(a_0), f(a_1), \ldots, f(a_n))$.

Theorem 2.11. Let $\mathcal{M}_1$ *be a* $(\mathbf{D}_1; \leqslant_1)$*-generic structure in a language* Σ*, and* $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ *be an embedding. The following conditions are equivalent:*

(1) *the embedding* f *is elementary*;

(2) *for any formula* $\varphi(x_0, x_1, \ldots, x_n)$ *in the language* Σ *and for any elements* $a_1, \ldots, a_n$ *forming a self-sufficient set* $A \leqslant_1 M_1$*, if* $\mathcal{M}_2 \models \exists x_0\, \varphi(x_0, f(a_1), \ldots, f(a_n))$ *then there is an element* $a_0 \in M_1$ *in a self-sufficient set* $B \leqslant_1 M_1$ *such that* $A \leqslant_1 B$ *and* $\mathcal{M}_2 \models \varphi(f(a_0), f(a_1), \ldots, f(a_n))$.

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С.В. Судоплатов

Элементарлық енгізілуді және элементарлық эквиваленттілікті сақтайтын генерикалық құрылымдар туралы

Мақалада генерикалық құрылымдар үшін элементарлық эквиваленттілік және элементарлық енгізілу критерийлері қарастырылды. Олар үшін жалпы жағдайдағы классикалық сипаттама қолданылды. Элементарлы эквивалентті критерийі Фраиссе-Тайманов-Эренфойхтың жақсы танымал «ауыстыру» әдісінде негізделген. Элементарлық енгізілу критерийінде Тарский-Воottың танымал тесті пайдаланылды.

Кілт сөздер: генерикалық құрылымдар, элементарлық эквиваленттілік, Фраиссе-Тайманов-Эренфойхтың әдісі, элементарлық енгізілу, Тарский-Воот тесті.

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О генерических структурах, сохраняющих элементарную эквивалентность и элементарную вложимость

В статье рассмотрены критерии элементарной эквивалентности и элементарной вложимости для генерических структур, которые используют классические характеризации для общего случая. Критерий элементарной эквивалентности базирован на хорошо известном методе «перекидывания» Фраиссе-Тайманова-Эренфойхта, а критерий элементарной вложимости — на известном тесте Тарского-Воота.

Ключевые слова: генерические структуры, элементарная вложимость, метод Фраиссе-Тайманова-Эренфойхта, тест Тарского-Воота.

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