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Convex fragmens of strongly minimal Jonsson sets

This article introduced and discussed the concepts of minimal Jonsson sets and respectively strongly minimal Jonsson sets. On this basis, we introduce the concept of independence of special subsets of existentially closed submodel of semantic model. The concept of independence leads to the concept of basis and then we have the Jonsson analogue of the theorem on uncountable categoricity.

Key words: Jonsson sets, Jonsson strongly minimal sets, fragments of Jonsson sets.

This article is devoted to the study of the concept Jonsson sets and its application. Jonsson sets concept defined in [1] and further results were obtained, which were presented in [2–4].

The concept of strongly minimality, as for the sets and for the theories played a decisive role in obtaining results which describe the uncountable-categorical theories [5].

We will consider all theories with additional property, it is a property of convexity.

So all theory will be convex.

Recall that theory T will be convex, if for any model $\mathfrak{A}$ of theory T and any collection $\{\mathfrak{B}_i : i \in I\}$ of substructures of $\mathfrak{A}$ which are models of T, the intersection $\cap\, \mathfrak{B}_i$ is a model of T.

The Jonsson theory are a natural subclass of broad class of theories, as a class of inductive theories. As is known, the main examples of the theories of algebras are examples of inductive theories, and they tend to represent an example of incomplete theories.

In modern model theory an technical apparatus developed mainly for complete theories, so today appliances study of incomplete theories are noticeably poorer than for complete theories.

On the one hand the Jonsson conditions area natural algebraic requirements that arise in the study of a wide class of algebras.

On the other hand natural examples of Jonsson theories are many, it is, for example, the theory of boolean algebras, abelian groups, fields of fixed characteristics, polygons (S-Acts, where S is monoid), and etc.

All of these examples are important in an algebra and in the various areas of mathematics. As can be seen, the list of the following scope of application of the technique developed for studying Jonsson theories can be quite broad.

Thus, all of the above suggests that the study of model-theoretic properties of Jonsson theories is atopicaltask.

Studying the inductive theories [6], it follows that Jonsson theory, as a subclass of inductive theories are such a part where there are certain methods of investigation incomplete theories, namely the method of transfer of properties of first-order theory of Jonsson center on the its Jonsson theory.

On this method and on research in the study of Jonsson theories and unrelated to the material in this article, we refer the reader to the following: [7–10].

As noted above, the basic technique associated with more subtle methods of studying the behavior of model elements, is the prerogative of the art study complete theories.

Therefore, even just trying to find a generalization of standard concepts from the arsenal of complete theories, we can come to a tautology or a concept that is not technically justified.

Therefore even been proposed Jonsson set.

Recall the basic definitions of [1], which are associated with these sets.

Suppose we are given an arbitrary language L.

The theory T is called Jonsson, if:

1) the theory T has infinite models;
2) the theory T of inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the property of amalgam (AP).

Jonsson theory T be a perfect theory, if its semantic model saturated.

Let T be perfect Jonsson theory complete for existential sentences in the language L and its semantic model is C.

We say that a set X be Σ-definable if it is definable by some existential formula.

a) The set X is said Jonsson in the theory T if it satisfies the following properties.

X is the Σ-definable subset of C;

dcl (X) is a support of some existentially closed submodel C.

b) The set X is said to be algebraically Jonsson in the theory T if it satisfies the following properties:

X is Σ-definable subset of C;

acl (X) is the support of some existentially closed submodel C.

From the definition of Jonsson sets can be seen that they work very simply in the sense of Morley rank [1]. It turns out that the elements of the set-theoretic difference(wells) of the closure and a Jonsson set have rank 0, i.e, they are algebraic. So, this is a case where we can work with the elements even in the case of incomplete.

The second point the utility of such a definition Jonsson set is that we closinga given set immediately obtain some existentially closed model. This in turn enables usfirst to determine Jonsson fragment from the set, and in principle and in an arbitrary theory.

At this point quite well studied are perfect Jonsson theory. For them, was proved a criterion of perfectness [7], which provide to carry out many model-theoretic facts about Jonsson theory and its center. There are complete descriptions as the center of such theories and models of their classes.

If in the caseof study of complete theories we mainly deal with two objects, it is the theory itself and its models, in the case of study of Jonsson theory we consider as models the class of existential closed models of the theory, as well as some additional condition is the completness of the theory in logical sense. At least, this theory must be existentially complete.

We give a definition of Jonsson fragment:

We say that all ∀∃-consequences of an arbitrary theory create Jonsson fragment of this theory, if the deductive closure of these ∀∃-consequences will beJonsson theory.

Due to the fact that this is not always true, it would be interesting to be able to allocate in arbitrary theory this part that will Jonsson theory. Such a task the place to be if only because of the fact that morleyzation of a theory it provides us, moreover, the resulting theory is perfect [6].

Another way is to use such a fact that any countable model of inductive theory necessarily isomorphically embeds in some existentially closed model of this theory [6].

Next, consider all ∀∃-consequences which are true in this model. Then in the case of Jonsson theory is well known fact that ∀∃-consequences whichtrue in this existentially closed model form aJonsson theory.

To study the behavior of the elements of wells in the case of Jonsson sets, we can always consider the ∀∃-consequences which true in the above closures ofJonsson set. In view of the above, in this case, consideredset of sentences would be Jonsson theory.

Obtained in this case Jonsson theory will be called the Jonsson fragment of corresponding Jonsson set. It is clear that we can carry out research on the relationship Jonsson fragments from the original theory, which is a new formulation of the problem of study Jonsson theory.

The main objective of this article is the following problem:

In the frame of these newly introduced definitions, consider and try to describe strongly minimal Jonsson sets.

This in turn will entail a number of new formulations of problems, such as refinement of Lachlan-BaldwinTheorem in the framework of the newly introduced subjects.

Recall that Jonsson theory T has a semantic model C in enough large cardinality. If this model is saturated, this theorycalled perfect Jonsson theory.

Semantic model of perfect Jonsson theory uniquely determined by their power.

Further, since we have to deal with perfect Jonsson theory, it is convenient to work within a large semantic existentially closed model containing all other existentially closed models of considered perfect Jonsson theory. We call this model of universal existential domain (UED).

It can also be characterized by the following conditions.

1. Each model of this theory is isomorphically embeddable in C.

2. Every isomorphism between two its submodels which are models of considered theory extends to an automorphism model C.

We will not consider all subsets of *C*, but only a Jonsson subset.

For any Σ — definable subsets of a semantic model we have that the following result is yields.

Lemma 1. Σ — definablesubset of the semantic model is definable over a set of parameters from *A* if and only if it invariant under all automorphisms of model *C*, leaving in place each element of *A*.

It follows that the definable closure dcl (*A*) of Jonsson set *A*, i.e. the set of all elements definable over *A* is the set of elements that are invariant under all automorphisms of *A*.

From Lemma 1 it follows that the element b is algebraic over A if and only if it has only a finite number of conjugate elementsover *A*.

We define the Morleyrank for existentially definable subsets of the semantic model.

We want to assign to each Σ — definable subset *M* of the semantic model ordinal number (or maybe 1 or ∞) — its Morley rank, denoted by MR. First, we define the ratio $MR(M)\geq\alpha$ by recursion on the ordinal α.

Let *T* be perfect Jonsson theory and *C* its *UED*.

Definition 1. $MR(M)\geq 0$, if and only if *M* is not empty;

$MR(M)\geq\lambda$, if and only if $MR(M)\geq\alpha$ for all $\alpha<\lambda$ (λ is a limit ordinal);

$MR(M)\geq(\alpha+1)$, if and only if there exists in *M* an infinite family of (M_i) disjoint Σ — definable subsets such that $MR(M_i)\geq\alpha$ at all *i*.

Then the Morley rank of *M* is $MR(M)=\sup\{\alpha / MR(M)\geq\alpha\}$

Moreover, we assume that $MR(\varnothing)=-1$ and $MR(D)=\infty$, if $MR(M))\geq\alpha$ for all α (in the latter case we say that *M* has no rank).

Note that Σ — definable subset *M* has rank 1 if it is empty; rank 0 if it is finite; rank 1 if it is infinite, but does not contain an infinite family of disjoint infinite Σ — definable classes.

Lemma 2. We have the relation $MR(M_1\cup M_2)=\max(MR(M_1),MR(M_2))$.

Definition 2. Morley degree $MD(M)$ of Jonsson subset *M* of the semantic model that has Morley rank α, *d* is the maximum length of its decomposition $M=M_1\cup\ldots\cup M_d$ existentially definable disjoint subsets of rank α.

In the case of rank 0 a degree of existentially definable subset *M* is a number of its elements. If existentially definable subset has no rank, is not defined and its degree of Morley.

Let us consider Jonsson minimal sets. Further, under the structure of the model refers to the signature or the language $\mathcal{L}$of Jonsson theory under consideration.

Let $\mathcal{M}$ a structure, and let $D\subseteq M^n$ infinite Σ — definable subset. We say that *D* is minimal in $\mathcal{M}$, if for any Σ — definable $Y\subseteq D$ or *Y* is finite, or $D\setminus Y$ finite.If $\varphi(\overline{\upsilon},\overline{a})$ is the formula that determines the *D*, then we can also say that $\varphi(\overline{\upsilon},\overline{a})$ is minimal.

We say that *D* and φ be Jonsson strongly minimal, if φ is minimal in any existentially closed extension $\mathcal{N}$of$\mathcal{M}$.

We say that a theory *T* Jonsson strongly minimal if$\forall\mathcal{M}\in E_T$, *M* is Jonsson strongly minimal.

The following properties of the algebraic closure true for any algebraically Jonssonset *D*.

i) $acl(acl(A))=acl(A)\supseteq A$;

ii) If $A\subseteq B$, then $acl(A)\subseteq acl(B)$;

iii) If $a\in acl(A)$, then $a\in acl(A_0)$ for some finite $A_0\subseteq A$.

More subtle property holds if *D* Jonsson strongly minimal.

Lemma on a replacement. Suppose that *D* is a subset of the semantic model of the theory and it Jonsson strongly minimal, $A\subseteq D$ and $a,b\in D$. If $\in acl(A\cup\{b\})\setminus acl(A)$, then $b\in acl(A\cup\{a\})$.

Remark. Jonsson strongly minimal set is existentially definable subset of the semantic model of the theory of rank 1 and degree 1 in the sense of Morley.

Definition 3. 1. Jonsson theory *T* Jonsson totally transcendental, if each existentially definable subset of its semantic model has Morley rank.

2. A theory *T* is ω-stable Jonsson, if the number of existential types is countable over every countable subset *A* semantic model.

Theorem 1. Jonsson theory *T* Jonsson totally transcendental, if and only if it Jonsson ω-stable.

Lemma 3. Let a and b be arbitrary elements of the semantic model. If the element b is algebraic over *A* and *a*, where *A* is existentially definable subset of the semantic model, the $MR(b/A) \leq MR(a/A)$.

Corollary 1. Let *M* — some ω-saturated existentially closed submodel of semantic model, and some φ is a $L(M)$ formula of rank α and Morley degree *d*. Then φ can be decomposed to $L(M)$ formulas $\varphi_1,...,\varphi_m$ rank α and degree 1.

In any Jonsson strongly minimal set, we can define the concept of independence, which generalizes the linear independence in vector spaces and algebraic independence of algebraically closed fields.

We fix $\mathcal{M} \models T$ and *D* is Jonsson strongly minimal set in the $\mathcal{M}$ — existential closed submodel ofsemantic model of *T*, where *T* is Jonsson theory.

Definition 2. We say that $A \subseteq D$ independent if $a \notin acl\left(A \setminus \{a\}\right)$ for all $a \in A$. If $C \subset D$, we say that *A* independent over *C*, if $a \notin acl\left(C \cup \left(A \setminus \{a\}\right)\right)$ for all $a \in A$.

Definition 4. We say that *A* is a basis for $Y \subseteq D$, if $A \subseteq Y$ independent and $acl(A) = acl(Y)$.

Obviously, that any maximal independent subset of *Y* is the basis for *Y*.

Let $I(E_T, \aleph_0)$denotes the number of countable existentially closed models of Jonssontheory *T*.

Using the technique of proofs for complete theories and concepts relevant to the changing techniques for Jonsson sets, we can prove Jonsson analogues of the results to appropriate spectrum of countable models [6].

Corollary 1. If *T* is Jonssonstrongly minimal Jonsson theory, complete for existential sentences, then *T* is *k* — categorical for $\kappa \geq \aleph_1$and $I(E_T, \aleph_0) \leq \aleph_0$.

Corollary 2. If *T* Jonsson theory complete for the existential sentence uncountably categoricaland there is Jonsson strongly minimal $\mathcal{L}$-formula, then either $T\aleph_0$-categorical or $I(E_T, \aleph_0) = \aleph_0$.

Theorem 2. If *T* Jonsson theory complete for the existential sentence is uncountably categorical, but not $\aleph_0$-categorical, then $I(E_T, \aleph_0) = \aleph_0$.

Definition 4. Jonsson stability (*J*-stability).Let *T* is a Jonsson theory, $S^J(X)$ is the set of all existential complete *n*-type over *X*, in accordance with the *T*, for any finite *n*.We shall say that Jonsson theory *T* be $J-\lambda$ stable if for any *T*-existentially closed model and for any its subset *X*

$|X| \leq \lambda \Rightarrow |S^J(X)| \leq \lambda.$

Theorem 3. If *T* Jonssonsuperstable, but not $\aleph_0$-categorical, then $I(E_T, \aleph_0) \geq \aleph_0$.

Let us consider the stability for fragments of Jonsson sets.

Let *X* Jonssonset and *M* is existentially closed model, where *dcl* (*X*) = *M*.

Consider the fragment of Jonsson set *X* as the theory $Th_{\forall\exists}(M) = T_M$.

Lemma 2. T_M will be Jonsson theory.

Theorem 1. Let T_M, as described above. If $\lambda \geq \omega$, then the following conditions are equivalent:

T_M is $J-\lambda$ — stable;

T^* is λ — stable, where T^* is the centerof *T*.

Theorem 2. Then the following conditions are equivalent:

(1) $T_M^* - \omega$ -categorical; (2) $T_M - \omega$ -categorical.

Definition 9. Let $A, B \in E_T$ and $A \subset B$. Then *B* is algebraically simpleextension *A* in E_T, if for any model $C \in E_T$ so that if *A* isomorphically embedded in *C*, then *B* is isomorphically embedded in *C*.

Let *X* be algebraically Jonsson set, *acl* (*X*) = *M*, the formula that determines the set *X* is strongly minimal existential formula.

Theorem 3. Then the following conditions are equivalent:

(1) $T_M^* - \omega_1$ — categorical;

(2) Any countable model from E_{T_M} has algebraically simple extension in E_{T_M}.

All undefined in this article definitions, as well as more detailed information about Jónsson theories can be found in [7].

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А.Р.Ешкеев

Қатты минималды йонсондық жиындардың дөңес фрагменттері

Мақалада йонсондық минималды жиындардың және қатты минималды йонсондық жиындардың ұғымдары енгізілген және қарастырылған. Осы негізде семантикалық модельдің экзистенционалды тұйық ішкі модельдің арнайы ішкі жиындары үшін тәуелсіздік ұғымы енгізілді. Бұл ұғым арқылы базис ұғымына келуге болады және әрі қарай саналымсыз категориялық туралы теореманың йонсондық баламасына ие боламыз.

А.Р.Ешкеев

Выпуклые фрагменты сильно минимальных йонсоновских множеств

В статье введены и рассмотрены понятия минимальных йонсоновских множеств и соответственно сильно минимальных йонсоновских множеств. На этой основе введено понятие независимости специальных подмножеств экзистенциально замкнутой подмодели семантической модели. Понятие независимости приводит к понятию базиса, и далее мы имеем йонсоновский аналог теоремы о несчетной категоричности.

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