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On the separability of elements and sets in hypergraphs of models of a theory

We consider topological properties of hypergraphs of models of a theory. The separability of elements in these hypergraphs is characterized in terms of algebraic closures. Similarly we specify the separability of sets by the hypergraphs. The separability of finite sets is characterized for special hypergraphs defined by limit models as well as by countable models which are neither almost prime nor limit.

Key words: separability of elements, hypergraph of models, algebraic closure.

In [1–4], the notion of hypergraph of prime models of a theory was introduced and it was clarified that these objects play an important role classifying countable models of complete theories. It was shown in [1, 5] that these hypergraphs can be arbitrary enough.

In the paper, we introduce the notions of hypergraphs $\mathcal{H}$ for elementary submodels of a model of a theory generalizing the notion for prime models. For hypergraphs $\mathcal{H}$, criteria of separability of elements as well as of sets in terms of algebraic closures are proven.

Throughout the paper we assume that T is a complete first order theory without finite models.

1 Hypergraphs of elementary submodels

Definition. Recall that a hypergraph is a pair (X, Y) of sets, where Y is a subset of the set $\mathcal{P}(X)$ being the set of all subsets of X.

Let $\mathcal{M}$ be a model of a theory T. Denote by $H(\mathcal{M})$ the set of all subsets N in the universe M of $\mathcal{M}$ such that these subsets are universes of elementary submodels $\mathcal{N}$ of $\mathcal{M}$. The pair $\mathcal{H}(\mathcal{M}) \rightleftharpoons (M, H(\mathcal{M}))$ is called a *hypergraph of all elementary submodels* of $\mathcal{M}$.

For a cardinality λ we denote by $H_\lambda(\mathcal{M})$ and $\mathcal{H}_\lambda(\mathcal{M})$ the restrictions of $H(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ respectively to the class of elementary submodels $\mathcal{N}$ of $\mathcal{M}$ such that $|N| < \lambda$.

We denote by $\mathcal{H}_p(\mathcal{M})$, $\mathcal{H}_l(\mathcal{M})$, $\mathcal{H}_{\rm npl}(\mathcal{M})$, $\mathcal{H}_h(\mathcal{M})$, $\mathcal{H}_s(\mathcal{M})$ the restrictions of $\mathcal{H}_{\omega_1}(\mathcal{M})$ to the classes of elementary submodels $\mathcal{N}$ of $\mathcal{M}$ which are prime over a finite set, limit, non-prime and non-limit, homogeneous, saturated, respectively. Similarly, $H_p(\mathcal{M})$, $H_l(\mathcal{M})$, $H_{\rm npl}(\mathcal{M})$, $H_h(\mathcal{M})$, $H_s(\mathcal{M})$ denote corresponding restrictions of $H_{\omega_1}(\mathcal{M})$.

By the definition and known properties of special models we have the following properties for ω-saturated models $\mathcal{M}$.

1. The sets $H_p(\mathcal{M})$, $H_l(\mathcal{M})$, and $H_{\rm npl}(\mathcal{M})$ are pairwise disjoint.

2. Each element N in $H_l(\mathcal{M})$, being the universe of a limit model, is represented as a union of countable $\subset$-chain of elements in $H_p(\mathcal{M})$.

3. If $N \in H_{\rm npl}(\mathcal{M})$ then N can not be represented as a union of a $\subseteq$-chain C of elements in $H_p(\mathcal{M})$ such that C consists of universes of prime models over finite sets, forming an elementary chain.

4. If T is countable and $H_p(\mathcal{M}) \neq \varnothing$ then $H_p(\mathcal{M})$, $H_h(\mathcal{M})$, and $H_s(\mathcal{M})$ coincide if and only if T is ω-categorical; otherwise $H_p(\mathcal{M})$ and $H_s(\mathcal{M})$ are disjoint.

5. $H_s(\mathcal{M}) \subseteq H_h(\mathcal{M})$; if T is countable then $H_h(\mathcal{M}) \neq \varnothing$.

6. $H_s(\mathcal{M})$ is nonempty if and only if T is small; in such a case $H_p(\mathcal{M}) \cap H_h(\mathcal{M}) \neq \varnothing$ witnessed by prime models over $\varnothing$, moreover, if T is not ω-categorical then $H_s(\mathcal{M}) \subseteq H_l(\mathcal{M})$.

2 Separability of elements by hypergraphs of models

Definition [6]. Let (X,Y) be a hypergraph, x_1,x_2 be distinct elements in X. We say that x_1 is *separated* from x_2, or T_0*-separated*, if there is $y\in Y$ such that $x_1\in y$ and $x_2\notin y$. The elements x_1 and x_2 are *separated*, T_2*-separated*, or *Hausdorff separated*, if there are disjoint $y_1,y_2\in Y$ such that $x_1\in y_1$ and $x_2\in y_2$.

Theorem 2.1. Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements in $\mathcal{M}$. The following conditions are equivalent:

(1) a is separated from b in $H(\mathcal{M})$;

(2) a is separated from b in $H_{\omega_1}(\mathcal{M})$;

(3) $b\notin\mathrm{acl}(a)$.

Proof. Implications (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3) are obvious (clearly that if $b\in\mathrm{acl}(a)$ then b belongs to any model $\mathcal{N}\preccurlyeq\mathcal{M}$ containing a).

For the proof (3) $\Rightarrow$ (2) we need the following

Lemma 2.2. Let $\bar a$ be a tuple whose elements form a set A, B be a finite set such that $\mathrm{acl}(A)\cap B=\varnothing$, and $\varphi(x,\bar a)$ be a consistent formula. Then there is $c\in\varphi(\mathcal{M},\bar a)$ such that $\mathrm{acl}(A\cup\{c\})\cap B=\varnothing$.

Proof of Lemma 2.2. By Compactness and using a consistent formula $\varphi'(x,\bar a)$ with $\varphi'(x,\bar a)\vdash$ $\vdash\varphi(x,\bar a)$ instead of $\varphi(x,\bar a)$, it suffices to show that for any $b\in B$, having a finite set of formulas $\psi_1(x,\bar a,y),\ldots,\psi_n(x,\bar a,y)$ with

$$\psi_i(x,\bar a,y)\vdash\varphi'(x,\bar a)\wedge\forall x\left(\varphi'(x,\bar a)\to\exists^{=k_i}y\psi_i(x,\bar a,y)\right),$$

for some natural k_i, $i=1,\ldots,n$, there is $c\in\varphi'(\mathcal{M},\bar a)$ such that

$$\models\bigwedge_{i=1}^{n}\neg\psi_i(c,\bar a,b).$$

Assume on contrary that for every $c\in\varphi'(\mathcal{M},\bar a)$ there is i such that $\models\psi_i(c,\bar a,b)$. Then the formula $\chi(x,\bar a,y)\rightleftharpoons\bigvee\limits_{i=1}^{n}\psi_i(x,\bar a,y)$ satisfies the following condition: for every $c\in\varphi'(\mathcal{M},\bar a)$, $\models\chi(c,\bar a,b)$, and $\chi(c,\bar a,y)$ has finitely many, at most $m=k_1+\ldots+k_n$, solutions. Now the formula

$$\theta(\bar a,y)\rightleftharpoons\exists x(\chi(x,\bar a,y)\wedge\forall z((\varphi'(z,\bar a)\to(\chi(x,\bar a,y)\wedge\chi(z,\bar a,y)))$$

satisfies b and has at most m solutions. This fact contradicts the condition $b\notin\mathrm{acl}(A)$. $\Box$

Now having $b\notin\mathrm{acl}(a)$ we construct inductively a countable model $\mathcal{N}\preccurlyeq\mathcal{M}$ such that $a\in N$, $b\notin N$, and $N=\bigcup\limits_{n\in\omega}A_n$ for a chain of sets A_n.

At the initial step, we consider the set $A_0=\{a\}$ and enumerate all formulas of the form $\varphi(x,a)$: $\Phi_0=\{\varphi_{0,m}(x,a)\mid m\in\omega\}$. Due the enumeration, we construct at most countable set $A_1=\bigcup\limits_{m\in\omega\cup\{-1\}}A_{1,m}\supset A_0$, with $b\notin\mathrm{acl}(A_1)$, in the following way. We set $A_{1,-1}\rightleftharpoons A_0$. If $A_{1,m-1}$ is already defined and the formula $\varphi_{0,m}(x,a)$ is inconsistent we set $A_{1,m}\rightleftharpoons A_{1,m-1}$; if $\varphi_{0,m}(x,a)$ is consistent, we choose, by Lemma 2.2, an element $c_m\in\varphi_m(\mathcal{M},a)$ such that $b\notin\mathrm{acl}(A_{1,m-1}\cup\{c_m\})$ and put $A_{1,m}\rightleftharpoons A_{1,m-1}\cup\{c_m\}$.

If at most countable set A_n is already constructed we enumerate all formulas of the form $\varphi(x,\bar a)$, $\bar a\in A_n$: $\Phi_n=\{\varphi_{n,m}(x,\bar a_m)\mid m\in\omega\}$. Due the enumeration, we construct at most countable set $A_{n+1}=\bigcup\limits_{m\in\omega\cup\{-1\}}A_{n+1,m}\supset A_n$, with $b\notin\mathrm{acl}(A_{n+1})$, in the following way. We set $A_{n+1,-1}\rightleftharpoons A_n$. If $A_{n+1,m-1}$ is already defined and the formula $\varphi_{n,m}(x,\bar a_m)$ is inconsistent we set $A_{n+1,m}\rightleftharpoons A_{n+1,m-1}$;

if $\varphi_{n,m}(x,\bar{a}_m)$ is consistent, we choose, by Lemma 2.2, an element $c_m \in \varphi_{n,m}(\mathcal{M},\bar{a}_m)$ such that $b \notin \mathrm{acl}(A_{n+1,m-1} \cup \{c_m\})$ and put $A_{n+1,m} \rightleftharpoons A_{n+1,m-1} \cup \{c_m\}$.

By the construction, the set $\bigcup\limits_{n\in\omega} A_n$ forms the required universe N of a countable model $\mathcal{N} \preccurlyeq \mathcal{M}$ such that $a \in N$ and $b \notin N$. $\Box$

As the following example, suggested by E. A. Palyutin, illustrates, it is essential in Theorem 2.1 that $\mathcal{M}$ is ω-saturated.

Example 2.3. Consider a structure $\mathcal{N}$ of the predicate language $\Sigma = \{P_0^{(1)}, P_1^{(1)}, P_2^{(1)}, Q^{(2)}, R_n^{(2)}\}_{n\in\omega}$, where

(1) P_0, P_1, P_2 are infinite disjoint sets with $N = P_0 \cup P_1 \cup P_2$ and such that there are three 1-types over $\varnothing$;

(2) $\delta_Q = P_0$, $\rho_Q = P_1$, for each distinct $a_1, a_2 \in P_0$, the sets $Q(a_1,\mathcal{N})$ and $Q(a_2,\mathcal{N})$ are infinite and disjoint such that all these sets form a partition of P_1; thus $E(x,y) \rightleftharpoons \exists z(Q(z,x) \wedge Q(z,y))$ is an equivalence relation on P_1 with infinitely many classes and all these classes are infinite;

(3) $\delta_{R_n} = P_1$, $\rho_{R_n} = P_2$ for any $n \in \omega$, the predicates R_n are disjoint, if $a \in P_1$ then there is unique $b \in P_2$ with $\models R_n(a,b)$; if $b \in P_2$ then there are infinitely many elements $a \in P_1$ with $\models R_n(a,b)$, and if $\models R_n(a_1,b) \wedge R_n(a_2,b) \wedge \neg(a_1 \approx a_2)$ then a_1 and a_2 are not E-equivalent;

(4) for any $b \in P_2$ there are infinitely many E-classes E_i with elements $a_n \in E_i$, $n \in \omega$ such that $\{b\} = \bigcap\limits_{n\in\omega} R_n(a_n,\mathcal{N})$, and for any permutation $\sigma \in \omega^\omega$, there is (unique) element b_σ such that $\{b_\sigma\} = \bigcap\limits_{n\in\omega} R_n(a_{\sigma(n)},\mathcal{N})$.

Constructing a generic saturated structure $\mathcal{N}'$ with the properties above and taking its elementary substructure $\mathcal{M}$ such that some E-class $Q(a,\mathcal{M})$ consists of elements a_n with $\{b\} = \bigcap\limits_{n\in\omega} R_n(a_n,\mathcal{M})$ for some b, we have the following: $b \notin \mathrm{acl}(a)$ and $b \in \mathrm{acl}(aa_n)$ for any $a_n \in Q(a,\mathcal{M})$ (witnessed by R_n). At the same time, in $\mathcal{N}'$, by saturation and in view of Lemma 2.2, there is an element $c \in Q(a,\mathcal{N}')$ such that $b \notin \mathrm{acl}(ac)$. $\Box$

Applying Lemma 2.2, by Compactness we get

Lemma 2.4. Let $\mathcal{M}$ be an ω-saturated model of a complete theory T, $\bar{a}, \bar{b} \in M$. If $\mathrm{acl}(\bar{a}) \cap \mathrm{acl}(\bar{b}) = \varnothing$ and $\varphi(x,\bar{a})$ is a consistent formula then there is $c \in \varphi(\mathcal{M},\bar{a})$ such that $\mathrm{acl}(\bar{a}c) \cap \mathrm{acl}(\bar{b}) = \varnothing$.

Remark 2.5. (1) Notice that Lemma 2.4 is not true if $\bar{a}$ and $\bar{b}$ are replaced by arbitrary sets A and B. Indeed, consider a complete bipartite graph with infinite disjoint parts A and B and a binary relation Q consisting of all pairs (a,b), where $a \in A$, $b \in B$. Clearly, for the structure $\mathcal{M} = \langle A \cup B; Q\rangle$ and the formula $Q(a,y)$, $a \in A$, we have $\mathrm{acl}(A) = A$, $\mathrm{acl}(B) = B$, $A \cap B = \varnothing$, and there is no $c \in Q(a,\mathcal{M})$ such that $\mathrm{acl}(A \cup \{c\}) \cap \mathrm{acl}(B) = \varnothing$ since $Q(a,\mathcal{M}) = B$.

(2) At the same time, Lemma 2.4 holds if $\mathcal{M}$ is λ-saturated for $\lambda \geq \max\{|\Sigma(T)|, \omega_1\}$. $\Box$

Theorem 2.6. Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements in $\mathcal{M}$. The following conditions are equivalent:

(1) a and b are separated in $H(\mathcal{M})$;

(2) a and b are separated in $H_{\omega_1}(\mathcal{M})$;

(3) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

Proof. As in the proof of Theorem 2.1 we have to prove (3) $\Rightarrow$ (2). Assuming $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$ we construct inductively disjoint countable models $\mathcal{N}_a, \mathcal{N}_b \preccurlyeq \mathcal{M}$ such that $\mathrm{acl}(a) \subseteq N_a$, $\mathrm{acl}(b) \subseteq N_b$, $N_a = \bigcup\limits_{n\in\omega} A_n$ for a chain of sets A_n, and $N_b = \bigcup\limits_{n\in\omega} B_n$ for a chain of sets B_n.

At the initial step, we consider the sets $A_0 = \mathrm{acl}(a)$, $B_0 = \mathrm{acl}(b)$ and enumerate all formulas of the form $\varphi(x,\bar{a})$, $\bar{a} \in A_0$: $\Phi_0 = \{\varphi_{0,m}(x,\bar{a}_m) \mid m \in \omega\}$. Due the enumeration, we construct at most countable set $A_1 = \bigcup\limits_{m\in\omega\cup\{-1\}} A_{1,m} \supset A_0$, with $\mathrm{acl}(A_1) \cap B_0 = \varnothing$, in the following way. We set $A_{1,-1} \rightleftharpoons A_0$. If $A_{1,m-1}$ is already defined and the formula $\varphi_{0,m}(x,\bar{a}_m)$ is inconsistent we set

$A_{1,m} \rightleftharpoons A_{1,m-1}$; if $\varphi_{0,m}(x,\bar{a}_m)$ is consistent, we choose, by Remark (2), an element $c_m \in \varphi_m(\mathcal{M},\bar{a}_m)$ such that $\mathrm{acl}(A_{1,m-1} \cup \{c_m\}) \cap B_0 = \varnothing$ and put $A_{1,m} \rightleftharpoons A_{1,m-1} \cup \{c_m\}$.

If A_1 is defined we extend symmetrically the set B_0 to at most countable set B_1 such that all consistent formulas $\varphi(x,\bar{b})$, $\bar{b} \in B_0$, are realized in B_1 and $\mathrm{acl}(A_1) \cap \mathrm{acl}(B_1) = \varnothing$.

If at most countable sets A_n and B_n are already constructed we enumerate all formulas of the form $\varphi(x,\bar{a})$, $\bar{a} \in A_n$: $\Phi_n = \{\varphi_{n,m}(x,\bar{a}_m) \mid m \in \omega\}$. Due the enumeration, we construct at most countable set $A_{n+1} = \bigcup\limits_{m\in\omega\cup\{-1\}} A_{n+1,m} \supset A_n$, with $b \notin \mathrm{acl}(A_{n+1})$, in the following way. We set $A_{n+1,-1} \rightleftharpoons A_n$. If $A_{n+1,m-1}$ is already defined and the formula $\varphi_{n,m}(x,\bar{a}_m)$ is inconsistent we set $A_{n+1,m} \rightleftharpoons A_{n+1,m-1}$; if $\varphi_{n,m}(x,\bar{a}_m)$ is consistent, we choose, by Remark (2), an element $c_m \in \varphi_{n,m}(\mathcal{M},\bar{a}_m)$ such that $\mathrm{acl}(A_{n+1,m-1} \cup \{c_m\}) \cap \mathrm{acl}(B_n) = \varnothing$ and put $A_{n+1,m} \rightleftharpoons A_{n+1,m-1} \cup \{c_m\}$.

Having A_{n+1} we extend symmetrically the set B_n to at most countable set B_{n+1} such that all consistent formulas $\varphi(x,\bar{b})$, $\bar{b} \in B_n$, are realized in B_{n+1} and $\mathrm{acl}(A_{n+1}) \cap \mathrm{acl}(B_{n+1}) = \varnothing$.

Finding A_n and B_n in an ω_1-saturated models, by finite character of algebraic closure, we can also find these sets inside $\mathcal{M}$.

By the construction, the sets $\bigcup\limits_{n\in\omega} A_n$ and $\bigcup\limits_{n\in\omega} B_n$ form the required universes N_a and N_b, respectively, of disjoint countable models $\mathcal{N}_a, \mathcal{N}_b \preccurlyeq \mathcal{M}$ such that $a \in N_a$ and $b \in N_b$. $\square$

Assuming that a theory T has a prime models over a tuple $\bar{a}$ (is small, i. e., has countably many types), we have that for any consistent formula $\varphi(\bar{x},\bar{a})$ there is an $\bar{a}$-principal formula $\psi(\bar{x},\bar{a})$ such that $\psi(\bar{x},\bar{a}) \vdash \varphi(\bar{x},\bar{a})$. In such a case, elements c in Lemmas 2.2 and 2.4 can be chosen so that $\mathrm{tp}(c/\bar{a})$ is principal. Applying this choice for the proof of Theorems 2.1 and 2.6 we obtain models $\mathcal{N}$, $\mathcal{N}_a$, which are prime over a, and $\mathcal{N}_b$ which is prime over b.

Thus we get the following

Corollary 2.7. Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements in $\mathcal{M}$, and there is a prime model over a. The following conditions are equivalent:

(1) a is separated from b in $H(\mathcal{M})$;

(2) a is separated from b in $H_{\omega_1}(\mathcal{M})$;

(3) a is separated from b in $H_p(\mathcal{M})$;

(4) $b \notin \mathrm{acl}(a)$.

Corollary 2.8. Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements in $\mathcal{M}$, and there are prime models over a and b respectively. The following conditions are equivalent:

(1) a and b are separated in $H(\mathcal{M})$;

(2) a and b are separated in $H_{\omega_1}(\mathcal{M})$;

(3) a and b are separated in $H_p(\mathcal{M})$;

(4) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

3 Separability of sets by hypergraphs of models

Definition. Let (X,Y) be a hypergraph, X_1, X_2 be disjoint subsets of X. We say that X_1 is separated from X_2, or T_0-separated , if there is $y \in Y$ such that $X_1 \subseteq y$ and $X_2 \cap y = \varnothing$. The sets X_1 and X_2 are separated, T_2-separated , or Hausdorff separated , if there are disjoint $y_1, y_2 \in Y$ such that $X_1 \subseteq y_1$ and $x_2 \subseteq y_2$.

Using arguments for the proofs of Theorems 2.1 and 2.6 we get the following generalizations.

Theorem 3.1. Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|,\omega\}$, A and B be sets in $\mathcal{M}$ of cardinalities $< \lambda$. The following conditions are equivalent:

(1) A is separated from B in $H(\mathcal{M})$;

(2) A is separated from B in $H_\lambda(\mathcal{M})$;

(3) $\mathrm{acl}(A) \cap B = \varnothing$.

Theorem 3.2. Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega_1\}$, A and B be sets in $\mathcal{M}$ of cardinalities $< \lambda$. The following conditions are equivalent:

(1) A and B are separated in $H(\mathcal{M})$;

(2) A and B are separated in $H_\lambda(\mathcal{M})$;

(3) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

Similarly Corollaries 2.7 and 2.8 we get

Corollary 3.3. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite sets in $\mathcal{M}$, and there is a prime model over A. The following conditions are equivalent:

(1) A is separated from B in $H(\mathcal{M})$;

(2) A is separated from B in $H_{\omega_1}(\mathcal{M})$;

(3) A is separated from B in $H_p(\mathcal{M})$;

(4) $\mathrm{acl}(A) \cap B = \varnothing$.

Corollary 3.4. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite sets in $\mathcal{M}$, and there are prime models over A and B respectively. The following conditions are equivalent:

(1) A and B are separated in $H(\mathcal{M})$;

(2) A and B are separated in $H_{\omega_1}(\mathcal{M})$;

(3) A and B are separated in $H_p(\mathcal{M})$;

(4) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

Note that by Property (4) in Section 1, if T is ω-categorical, we can add for Corollaries 3.3 and 3.4 equivalent conditions that A is separated from B (A and B are separated) in $H_h(\mathcal{M})$ as well as in $H_s(\mathcal{M})$.

4 Separability of finite sets by hypergraphs of limit models

We shall use the following

Theorem 4.1 [4, Theorem 6.4.0.6]. Let $\mathbf{q} = (q_n)_{n\in\omega}$ be a $\leq_{\mathrm{RK}}$-sequence of types of a small theory T. The following conditions are equivalent:

(1) there exists a limit model over $\mathbf{q}$;

(2) there are infinitely many $n \in \omega$ such that $(I_{q_n,q_{n+1}})^{-1} \neq I_{q_{n+1},q_n}$ for some (any) model $\mathcal{M} \models T$ realizing all types in $\mathbf{q}$;

(3) there are infinitely many $n \in \omega$ such that for some (any) model $\mathcal{M} \models T$ realizing all types in $\mathbf{q}$, and for some realizations $\bar{a}_n$ and $\bar{a}_{n+1}$ (in $\mathcal{M}$) of q_n and q_{n+1} respectively, $(\bar{a}_{n+1}, \bar{a}_n) \in I_{q_{n+1},q_n}$ and $(\bar{a}_n, \bar{a}_{n+1}) \notin \mathrm{SI}_{q_n,q_{n+1}}$ hold; in particular, $(\mathrm{SI}_{q_n,q_{n+1}})^{-1} \neq \mathrm{SI}_{q_{n+1},q_n}$.

Theorem 4.2 [4, Theorem 7.5.0.4]. *A countable model $\mathcal{M}$ of a countable theory T is prime over a finite set or limit if and only if each tuple $\bar{a} \in M$ can be extended to a tuple $\bar{b} \in M$ such that every consistent formula $\varphi(\bar{x}, \bar{b})$ belongs to an isolated type over $\bar{b}$.*

Let A be a finite set in an ω-saturated model $\mathcal{M}$ of T, $\bar{a}$ be a tuple of all elements in A, $q = \mathrm{tp}(\bar{a})$. Applying the proof of Theorem 4.1 and Theorem 4.2 we obtain

Theorem 4.3. The tuple $\bar{a}$ belongs to a limit model $\mathcal{N} \preccurlyeq \mathcal{M}$ if and only if there is a $\leq_{\mathrm{RK}}$-sequence $\mathbf{q} = (q_n)_{n\in\omega}$ of types in $S(T)$ and a sequence $(\bar{a}_n)_{n\in\omega}$, such that $\bar{a} \subseteq \bar{a}_0$, every consistent formula $\varphi(\bar{x}, \bar{a}_n)$ belongs to an isolated type over $\bar{a}_n$ and there are infinitely many $n \in \omega$ such that $(\bar{a}_{n+1}, \bar{a}_n) \in I_{q_{n+1},q_n}$ and $(\bar{a}_n, \bar{a}_{n+1}) \notin \mathrm{SI}_{q_n,q_{n+1}}$ hold. In this case, a model $\bigcup\limits_{n\in\omega} \mathcal{M}(\bar{a}_n)$, being the union of an elementary chain $(\mathcal{M}(\bar{a}_n))_{n\in\omega}$ of prime models $\mathcal{M}(\bar{a}_n)$ over $\bar{a}_n$, is limit over $\mathbf{q}$.

Remark 4.4. Note that as in Lemma 2.2 assuming that $\mathrm{acl}(\bar{a}) \cap B = \varnothing$ and $\varphi(x, \bar{a})$ is a consistent formula belonging to a complete type $t(x, \bar{a})$, by Compactness there is a solution c of $\varphi(x, \bar{a})$ realizing $t(x, \bar{a})$ such that $\mathrm{acl}(\bar{a}c) \cap B = \varnothing$.

Having finite sets A and B such that $\mathrm{acl}(A) \cap B = \varnothing$, tuples $\bar{a}$ and $\bar{b}$ consisting of all elements of A and B, respectively, as well a limit model over a sequence $\mathbf{q} = (q_n)_{n\in\omega}$ and with a tuple $\bar{a}_0 \supseteq \bar{a}$ realizing

q_0, we can choose solutions c of consistent formulas $\varphi(x, \bar{a})$ so that, preserving $\mathrm{acl}(\bar{a}'c) \cap B = \varnothing$ for $\bar{a}' \supset \bar{a}$ as in Lemma 2.2 and Remark 4.4, all types q_n are step-by-step realized by some tuples $\bar{a}_n$ as in Theorem 4.3 and an elementary chain $(\mathcal{M}(\bar{a}_n))_{n\in\omega}$ of prime models $\mathcal{M}(\bar{a}_n)$ is formed, where $\bigcup\limits_{n\in\omega} \mathcal{M}(\bar{a}_n)$ is a limit model disjoint with B.

Thus we get

Theorem 4.5. Let $\mathcal{M}$ be an ω-saturated model of a theory T, A and B be finite sets in $\mathcal{M}$, and there is a limit model over over a sequence $\mathbf{q} = (q_n)_{n\in\omega}$, where the set A is extensible to a tuple realizing the type q_0. The following conditions are equivalent:

(1) A is separated from B in $H(\mathcal{M})$;

(2) A is separated from B in $H_{\omega_1}(\mathcal{M})$;

(3) A is separated from B in $H_l(\mathcal{M})$ and it is witnessed by a limit model over $\mathbf{q}$;

(4) $\mathrm{acl}(A) \cap B = \varnothing$.

Since any countable model of a small theory T is either prime over a finite set or limit and there are limit models for small theories which are not ω-categorical, by Theorem 4.5 and Corollary 3.3, we have

Corollary 4.6. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, which is not ω-categorical, A and B be finite sets in $\mathcal{M}$. The following conditions are equivalent:

(1) A is separated from B in $H(\mathcal{M})$;

(2) A is separated from B in $H_{\omega_1}(\mathcal{M})$;

(3) A is separated from B in $H_p(\mathcal{M})$;

(4) A is separated from B in $H_l(\mathcal{M})$;

(5) $\mathrm{acl}(A) \cap B = \varnothing$.

Similarly, having finite sets A and B in an ω-saturated model $\mathcal{M}$, with $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$, such that there is a limit model over over a sequence $\mathbf{q} = (q_n)_{n\in\omega}$, where the set A is extensible to a tuple realizing the type q_0, and there is a limit model over over a sequence $\mathbf{q}' = (q'_n)_{n\in\omega}$, where the set B is extensible to a tuple realizing the type q'_0, then using the proof of Lemma 2.4 we construct disjoint limit models $\mathcal{M}_A = \bigcup\limits_{n\in\omega} \mathcal{M}(\bar{a}_n)$ and $\mathcal{M}_B = \bigcup\limits_{n\in\omega} \mathcal{M}(\bar{b}_n)$ over $\mathbf{q}$ and $\mathbf{q}'$ and containing A and B respectively, where $(\mathcal{M}(\bar{a}_n))_{n\in\omega}$ and $(\mathcal{M}(\bar{b}_n))_{n\in\omega}$ are elementary chains, $\models q_n(\bar{a}_n)$, $\models q'_n(\bar{b}_n)$, $n \in \omega$. Hence, the following assertions hold.

Theorem 4.7. Let $\mathcal{M}$ be an ω-saturated model of a theory T, A and B be finite sets in $\mathcal{M}$, and there are limit model over over sequences $\mathbf{q} = (q_n)_{n\in\omega}$ and $\mathbf{q}' = (q'_n)_{n\in\omega}$, respectively, where the set A is extensible to a tuple realizing the type q_0 and B is extensible to a tuple realizing the type q'_0. The following conditions are equivalent:

(1) A and B are separated in $H(\mathcal{M})$;

(2) A and B are separated in $H_{\omega_1}(\mathcal{M})$;

(3) A and B are separated in $H_l(\mathcal{M})$ and it is witnessed by limit models over $\mathbf{q}$ and $\mathbf{q}'$ respectively;

(4) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

Theorem 4.7 and Corollary 3.4 imply

Corollary 4.8. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, which is not ω-categorical, A and B be finite sets in $\mathcal{M}$. The following conditions are equivalent:

(1) A and B are separated in $H(\mathcal{M})$;

(2) A and B are separated in $H_{\omega_1}(\mathcal{M})$;

(3) A and B are separated in $H_p(\mathcal{M})$;

(4) A and B are separated in $H_l(\mathcal{M})$;

(5) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

In conditions of Corollary 4.6 (accordingly 4.8) having $\mathrm{acl}(A) \cap B = \varnothing$ ($\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$) we can separate A from B (A and B) constructing countable saturated models step-by-step realizing types over finite sets by elements c with $\mathrm{acl}(\bar{a}c) \cap B = \varnothing$ ($\mathrm{acl}(\bar{a}c) \cap \mathrm{acl}(B) = \varnothing$). Thus Corollaries 4.6 and 4.8 can be extended by the following:

Corollary 4.9. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, which is not ω-categorical, A and B be finite sets in $\mathcal{M}$. The following conditions are equivalent:

(1) A is separated from B in $H_h(\mathcal{M})$;

(2) A is separated from B in $H_s(\mathcal{M})$;

(3) $\mathrm{acl}(A) \cap B = \varnothing$.

Corollary 4.10. Let $\mathcal{M}$ be an ω-saturated model of a small theory T, which is not ω-categorical, A and B be finite sets in $\mathcal{M}$. The following conditions are equivalent:

(1) A and B are separated in $H_h(\mathcal{M})$;

(2) A and B are separated in $H_s(\mathcal{M})$;

(3) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

5 Separability of finite sets by hypergraphs of npl*-models*

Now we consider the separability of elements and finite sets in $\mathcal{H}_{\mathrm{npl}}(\mathcal{M})$, i. e., by npl-*models* — countable models which are neither almost prime nor limit.

Assume that $\mathrm{acl}(\bar{a}) \cap B = \varnothing$, $\varphi(x, \bar{a})$ is a consistent formula, and the tuple $\bar{a}$ belongs to a npl-model $\mathcal{N}$. Constructing a npl-model $\mathcal{N}_0 \preccurlyeq \mathcal{M}$, which separates $\bar{a}$ from B, by Theorem 4.2, we have to find a tuple $\bar{a}'$ extending $\bar{a}$ such that $\mathrm{acl}(\bar{a}') \cap B = \varnothing$ and $\bar{a}'$ can not be extended to a tuple $\bar{b}'$ in a npl-model, where every consistent formula $\varphi(\bar{x}, \bar{b}')$ belongs to an isolated type over $\bar{b}'$. Such $\bar{a}'$ exists since as in Lemma 2.2 we can take $\bar{a}'$ as a realization of a type $t\bar{y}, \bar{a}$ witnessing the required property. Having $\bar{a}'$ we construct step-by-step the model $\mathcal{N}_0$ preserving the separability from B and such that for every $\bar{b}'$ containing $\bar{a}'$, some consistent formula $\varphi(\bar{x}, \bar{b}')$ does not belong to an isolated type over $\bar{b}'$.

Theorem 5.1. Let $\mathcal{M}$ be an ω-saturated model of a theory T, A and B be finite sets in $\mathcal{M}$, and there is a npl-model of T containing the set A. The following conditions are equivalent:

(1) A is separated from B in $H(\mathcal{M})$;

(2) A is separated from B in $H_{\omega_1}(\mathcal{M})$;

(3) A is separated from B in $H_{\mathrm{npl}}(\mathcal{M})$;

(4) $\mathrm{acl}(A) \cap B = \varnothing$.

Similar arguments imply

Theorem 5.2. Let $\mathcal{M}$ be an ω_1-saturated model of a theory T, A and B be finite sets in $\mathcal{M}$ belonging to npl-models. The following conditions are equivalent:

(1) A and B are separated in $H(\mathcal{M})$;

(2) A and B are separated in $H_{\omega_1}(\mathcal{M})$;

(3) A and B are separated in $H_{\mathrm{npl}}(\mathcal{M})$;

(4) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

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С.В. Судоплатов

Теория модельдерінде гиперграфтардағы жиындар мен элементтердің бөлінгіштігі жайында

Мақалада теория модельдері гиперграфтарының топологиялық қасиеттері зерттелген. Алгебралық түйықталу терминінде осы гиперграфтардан элементтердің бөлінгіштігі сипатталған. Осы сияқты гиперграфтардағы жиындар бөлінгіштігінің қажетті және жеткілікті шарттары алынған. Шекті модельдермен, сонымен қатар қарапайым немесе шекті дерлік саналымды модельдермен берілген арнайы түрдегі гиперграфтар үшін ақырлы жиындардың бөлінгіштігі анықталған.

С.В. Судоплатов

Об отделимости элементов и множеств в гиперграфах моделей теории

В статье исследованы топологические свойства гиперграфов моделей теории. В терминах алгебраических замыканий охарактеризована отделимость элементов в этих гиперграфах. Аналогичным образом найдены необходимые и достаточные условия отделимости множеств в гиперграфах. Изучена отделимость конечных множеств для гиперграфов специального вида, задаваемых предельными моделями, а также счетными моделями, не являющимися ни почти простыми, ни предельными.

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