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On multipliers in weighted Sobolev spaces. Part I

Let X, Y be Banach spaces whose elements are functions $y\colon \Omega \to \mathbb{R}$. We say that a function $z\colon \Omega \to \mathbb{R}$ is a pointwise multiplier on the pair (X, Y), if $Tx = zx \in Y$ and the operator $T\colon X \to Y$ is bounded. $M(X \to Y)$ denotes the multiplier space on the pair (X, Y). We introduce the norm $\|z;\, M(X \to Y)\| = \|T;\, X \to Y\|$ in $M(X \to Y)$. Let $1 \le p < \infty$. Let m be an integer. $W^m_{p,\omega_0,\omega_1}$ denotes the weighted Sobolev space with the finite norm $\|u\|_{W^m_{p,\omega_0,\omega_1}} = \|u; W^m_{p,\omega_0,\omega_1}\| = \|\omega_0^{1/p}|\nabla_m u|\|_{L_p} + \|\omega_1^{1/p} u\|_{L_{p,v}}$. The aim of this work is to obtain descriptions of multiplier spaces for the pair of weighted Sobolev spaces $(W^l_{p,\rho,v}, W^m_{q,\omega_0,\omega_1})$.

Key words: weighted Sobolev space, pointwise multiplier.

It is well known that multipliers have their own importance in the theory of function spaces, and in the theory of differential and integral operators acting in these spaces. Applications of the theory of pointwise multipliers in function spaces are relevant both for direct boundary value problems, and for their inverse counterparts. The theory of pointwise multipliers acting on several function spaces like Sobolev, Besov and Triebel-Lizorkin spaces have been developed by several mathematicians. In particular, we mention the contribution of A.Devinatz, I.I.Hirschman, R.Strichartz, J.C.Polking, J.Peetre, V. Maz'ya, T. Shaposhnikova, and more recently the contribution of W.Sickel, T.Runst, J.Frank, and H.Koch. For the latest developments on pointwise multipliers of function spaces we refer to the monographs [1, 2], which is entirely devoted to this topic. Let us point out some specific directions through the works [3–10].

Let Ω be a domain (an open connected set) in the n-dimensional Euclidian space $\mathbb{R}^n$ with the norm $|x| = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{1/2}$. We denote by $L_p(\Omega)$, $1 \le p < \infty$, the space of all real valued measurable functions $f\colon \Omega \to \mathbb{R}$ with the finite norm

$$\|f\|_{L_p(\Omega)} = \|f;\, L_p(\Omega)\| = \left(\int_\Omega |f(x)|^p\, dx\right)^{\frac{1}{p}}.$$

(It is commonly supposed that $f = 0$, if $f(x) = 0$ for a.a. $x \in \Omega$). We denote by $L_{p,loc}(\Omega)$ the space of functions f defined a.e. in Ω such that $f \in L_p(F)$ for any compact $F \subset \Omega$. Here $L^+_{p,loc}(\Omega)$ is the space of all a.e. positive functions of $L_{p,loc}(\Omega)$, $L_{loc}(\Omega) = L_{1,loc}(\Omega)$. By $L^+_{loc}(\Omega)$ is denoted the space of a.e. positive functions of $L_{loc}(\Omega)$. A function v of $L^+_{loc}(\Omega)$ is called weight in Ω. Let α be a measure on Ω. Below $L_{p,\alpha}(\Omega)$ is the space of all real valued functions having in Ω the finite weighted Lebesgue norm

$$\|u\|_{L_{p,\alpha}(\Omega)} = \left(\int\limits_\Omega |u|^p\, d\alpha(x)\right)^{\frac{1}{p}} \quad (1 \le p < \infty).$$

If $d\alpha(x) = v(x)\, dx,\ v \in L^+_{loc}(\Omega)$, we write $L_{p,v}(\Omega)$ instead of $L_{p,\alpha}(\Omega)$. Note that $L_p(\Omega) = L_{p,v}(\Omega)$, if $v \equiv 1$.

By C^∞, C_0^∞ we denote the space of all infinitely differentiable functions in $\mathbb{R}^n$ and the space of functions of C^∞ with compact support $\operatorname{supp} f$ in $\mathbb{R}^n$, respectively. When the domain is not indicated in the notation of a space or a norm then it is assumed to be $\mathbb{R}^n$.

Let $1 \le p < \infty$. Let m be an integer, $\omega_0,\ \omega_1 \in L^+_{loc}$. We denote by $W^m_{p,\omega_0,\omega_1}$ the completion of the set of $u \in C_0^\infty$ in the finite norm

$$\|u\|_{W^m_{p,\omega_0,\omega_1}} = \|u; W^m_{p,\omega_0,\omega_1}\| = \||\nabla_m u|\|_{L_{p,\omega_0}} + \|u\|_{L_{p,\omega_1}},$$

where $|\nabla_m u| = \left(\sum\limits_{|\alpha|=m} |D^\alpha u|^2\right)^{1/2}$. Here $W^m_{p,\omega} = W^m_{p,\omega_0,\omega_1}$ with $\omega_0 = 1,\ \omega_1 = \omega$, $W^m_p = W^m_{p,\omega_0,\omega_1}$ with $\omega_0 = 1,\ \omega_1 = 1$. Henceforth, χ_e is the characteristic function of a subset e in $\mathbb{R}^n$, $\bar{e}$ is the closure of e. By $W^m_{p,loc}$ we denote the space [1] $\{u\colon \eta u \in W^m_p \quad \text{for all} \quad \eta \in C_0^\infty\}$. Here I^n is the family of all cubes Q in the form

$$Q = Q_h = Q_h(x) = \{y \in \mathbb{R}^n\colon |y_i - x_i| < \frac{h}{2}, i = 1, \dots, n\},\ \lambda Q = Q_{\lambda h}(x).$$

By c we denote constants depending only on the assigned numerical parameters, for example, $c = c(l, p, n)$, etc.

Let $h(\cdot)$ be a positive locally bounded function in $\mathbb{R}^n$. $\mathfrak{B}$ denotes the family (basis) of cubes $Q(x) = Q_{h(x)}(x)$, $x \in \mathbb{R}^n \setminus e$, where e is a set with measure 0. We use the following notation

$$\mathfrak{B} = \{Q(x)\} \quad \text{or} \quad \mathfrak{B} = \{Q(x) = Q_h(x)\}.$$

Definition. Let $\rho \in L^+_{loc.}$ We say that a weight ρ satisfies the slow variation condition with respect to the basis of cubes $\mathfrak{B} = \{Q(x)\}$, if there exist $b > 1$ such that for a.a. x

$$b^{-1}\rho(x) < \rho(y) < b\,\rho(x), \quad \text{for a.a. } y \in Q(x). \tag{1}$$

Examples.

Example 1. Let $-\infty < \mu < \infty$, $h(x) = (1 + |x|)^{-s},\ s > 0$. It is easily seen that

$$|y - x| < \frac{\sqrt{n}h(x)}{2} \text{ for all } y \in Q(x) = Q_{h(x)}(x). \tag{2}$$

Thus

$$\frac{1 + |y|}{1 + |x|} \le \frac{1 + |x| + |y - x|}{1 + |x|} \le 1 + \frac{\sqrt{n}}{2} \text{ for all } y \in Q(x). \tag{3}$$

If $|x| \le \sqrt{n}$, then

$$\frac{1 + |y|}{1 + |x|} \ge \frac{1}{1 + \sqrt{n}} \quad \text{for} \quad y \in Q(x). \tag{4}$$

If $|x| > \sqrt{n},\ y \in Q(x)$, then implies that (2)

$$|x - y| < \frac{\sqrt{n}}{2} < \frac{|x|}{2},\ |y| \ge 2^{-1}|x|$$

and we have

$$\frac{1 + |y|}{1 + |x|} > \frac{1}{2},\ y \in Q(x). \tag{5}$$

Inequalities (3)–(5) imply that

$$b^{-1} \le \left(\frac{\rho(y)}{\rho(x)}\right)^\mu \le b, \quad \text{if} \quad y \in Q(x),$$

where $b = (1 + \sqrt{n})^{|\mu|}$.

Example 2. Let Γ be a compact set in $\mathbb{R}^n$, which has no interior points. We set $|x-y|_\infty = \max_{1\le j\le n} |x_j - y_j|$ $(x,\ y \in \mathbb{R}^n)$. Let us show that the function $\sigma(x) = \inf_{y\in\Gamma} |x-y|_\infty$ satisfies the slow variation condition with respect to the basis $\{(1-\tau)Q_{\sigma(x)}(x)\}$, $0<\tau<1$. Since σ is a continuous function, then $\sigma(x) = \min_{y\in\Gamma} |x-y|_\infty = 0$ only for $x\in\Gamma$. Since $\mathrm{mes}_n(\Gamma)=0$, then $\sigma \in L^+_{loc}$. Let $h=\sigma(x)>0$, $0<\varepsilon<1$. Let us prove that $\Gamma \bigcap \varepsilon Q(x) = \emptyset$. If $y \in \Gamma \bigcap \varepsilon Q(x) \ne \emptyset$, then $0 < \sigma(x) < \varepsilon\sigma(x)/2$, which is impossible. Thus, $Q(x) = \bigcup_{0<\varepsilon<1} \varepsilon Q(x) \subset \mathbb{R}^n \setminus \Gamma$. Hence, $|z-x|_\infty \ge \sigma(x)/2$, $|z-y|_\infty \ge \tau\sigma(x)/2$ for any $z\in\Gamma$, $y \in (1-\tau)Q(x)$. Therefore $\sigma(y) = \inf_{z\in\Gamma} |z-y| \ge \tau\sigma(x)/2$. Since $|z-y|_\infty \le |z-x|_\infty + |x-y|_\infty$, then $\sigma(y) \le (2-\tau)\sigma(x)$ for $y\in(1-\tau)Q(x)$. By taking $\tau = 1/2$, we get

$$\frac{1}{2}\sigma(x) \le \sigma(y) \le 2\sigma(x), \quad \text{if} \quad y \in \frac{1}{2}Q(x).$$

Let ρ satisfy the slow variation condition (1). We denote by $W^m_p(\rho^\mu, \rho^\nu)$ the space $W^m_{p,\omega_0,\omega_1}$ with $\omega_0 = \rho^{\mu p}$, $\omega_1 = \rho^{\nu p}$. Throughout the paper we assume that $0<m<l$ are integers.

Let us assume that

$$K_{(\tau),p,q}(x|\rho,\gamma) = |Q(x)|^{l/n-1/p} \left(\int_{\tau Q(x)} |\nabla_m \gamma|^q \right)^{1/q} + |Q(x)|^{(l-m)/n-1/p} \left(\int_{\tau Q(x)} |\gamma|^q \right)^{1/q}$$

for $0<\tau\le 1$, $Q(x) = Q_h(x)$, $h=\rho(x)^s$.

Theorem 1. Let $1<p\le q<\infty$, $lp>n$, $-\infty<\mu, s<\infty$. Let ρ, ω satisfy the slow variation condition with respect to the basis $\mathfrak{B} = \{Q(x) = Q_{h(x)}(x)\}$, where $h(x) = \rho(x)^s$. Assume that $\gamma \in W^m_{q,loc}$ and

$$\mathbf{C} = \operatorname*{ess\,sup}_x \omega(x)^{1/q}\rho(x)^{-\mu} K_{(1),p,q}(x|\rho,\gamma) < \infty,$$

then

$$\gamma \in M(W^l_p(\rho^\mu, \rho^{\mu-sl}) \to W^m_{q,\omega,\omega\rho^{-smq}}).$$

The norm satisfies the following estimates

$$c_0\, \mathbf{C}_{1/2} \le \|\gamma; M(W^l_p(\rho^\mu, \rho^{\mu-sl}) \to W^m_{q,\omega,\omega\rho^{-smq}})\| \le c_1\, \mathbf{C},$$

where $\mathbf{C}_{1/2} = \operatorname*{ess\,sup}_x \omega(x)^{1/q}\rho(x)^{-\mu} K_{(1/2),p,q}(x|\rho,\gamma)$.

We begin the proof of Theorem 1 by formulating some propositions which will be useful below.

Theorem A [2]. Let $0<\delta,\gamma<1$. Let $h(\cdot)$ be a positive function, which is bounded in the each compact $F\subset\mathbb{R}^n$ such that for all $x\in\mathbb{R}^n$

$$h(y) \ge \delta h(x), \quad \text{if} \quad y \in Q(x) = Q_{h(x)}(x).$$

Then the family of cubes $\{(1-\gamma)\overline{Q}(x),\ x\in F\}$ has a finite or countable subfamily $\{(1-\gamma)\overline{Q}^j,\ j\in J\}$, $Q^j = Q(x^j)$ such that:

1) $F \subset \bigcup_{j\in J} (1-\gamma)\overline{Q}^j$;
2) $\sum_{j\in J} \chi_{(1-\gamma)\overline{Q}^j}(x) \le \varkappa_1$ for any $x\in\mathbb{R}^n$;
3) $\sum_{j\in J} \chi_{Q^j}(x) \le \widetilde{\varkappa}_1$ for any $x\in\mathbb{R}^n$;
4) the family $\{(1-\gamma)\overline{Q}^j,\ j\in J\}$ splits into no more than $\varkappa_2$ subfamilies of disjoint cubes;

5) the family $\{Q^j,\ j\in J\}$ splits into no more than $\widetilde{\varkappa}_2$ subfamilies of disjoint cubes,

where $\varkappa_1$, $\varkappa_2$, $\widetilde{\varkappa}_1$, $\widetilde{\varkappa}_2$ depend only on n, δ, γ.

Remark 1. Coverings $\{Q^j,\ j\in J\}$, $\{(1-\gamma)\overline{Q}^j,\ j\in J\}$, considered in a pair, will be called Besicovitch double coverings. In the sequel to simplify the notation we use the following notation

$$\{\hat{Q}^j, Q^j\}_{j\in J},\ \hat{Q}^j=(1-\gamma)\overline{Q}^j.$$

Theorem B [2]. Let $h(\cdot)$ be a positive function satisfying the conditions of Theorem D. Let $\{\hat{Q}^j, Q^j\}_{j\in J}$ be a Besicovitch double covering of the compact $F\subset\mathbb{R}^n$. Then there exist a family of functions $\Psi=\{\psi_j\}_{j\in J}$ of $C_0^\infty(\mathbb{R}^n)$ such that

1) $\operatorname{supp}\psi_j\subset Q^j$;
2) $0\le\psi_j\le 1,\ \psi_j=1$ in $\hat{Q}^j$;
3) $\sum\limits_{j\in J}\psi_j=1$;
4) $\max\limits_{Q^j}|D^\alpha\psi_j|\le c(h_j)^{-|\alpha|}$ for $|\alpha|\le l$, where c depends only on α,n,δ,γ.

Remark 2. The equality is true

$$D^\alpha u(x)=D^\alpha\left(u(x)\sum_{j\in J}\psi_j(x)\right)=\sum_{j\in J}D^\alpha\left(u\psi_j\right)(x)$$

for all functions $u\in C_0^\infty$, $x\in F=\operatorname{supp}u$.

For mappings $f\colon X\to\mathbb{R}$, $g\colon X\to\mathbb{R}$, the notation $f(x)\ll g(x)$ means that there exists a constant $c>0$ such that $f(x)\le c\,g(x)$ for all $x\in X$. The notation $f(x)\sim g(x)$ means that $f(x)\ll g(x)\ll f(x)$. For values $A>0$, $B>0$, the notation $A\sim B$ means that there exist constants $0<c_1<c_2$ such that $c_1B<A<c_2B$.

Theorem C [1]. Let $1<p\le q<\infty$, $pl>n$. Then

$$\|\gamma;M(W_p^l\to W_q^m)\|\sim\sup_x\|\gamma;W_q^m(B_1(x))\|.$$

The extension theorem of E. Stein [3] implies the following proposition:

Proposition 1. Let $1\le p<\infty$, $Q_1=Q_1(0)$. There exists a bounded extension operator $S\colon W_p^m(Q_1)\to W_p^m$ such that $U=Su\in C_0^\infty(Q_{\frac{3}{2}}(0))$ for all $u\in C^\infty\cap W_p^m(\overline{Q}_1)$.

Proposition 2. Let $1<p<\infty$. Let $0\le k<m$ be integers. Then the following inequalities hold

$$\int_{Q_h}|\nabla_k u|^p\le c\,h^{p(m-k)}\int_{Q_h}(|\nabla_m u|^p+h^{-mp}|u|^p); \tag{6}$$

$$\sup_{Q_h}|u|\le c\,h^{mp-n}\int_{Q_h}(|\nabla_m u|^p+h^{-mp}|u|^p)\quad (mp>n) \tag{7}$$

for all $u\in W_p^m(Q_h)$.

Inequalities (6), (7) follow from embedding theorems for the class $W_p^m(Q_1)$ [4].

Lemma 1. Let $1<p\le q<\infty$, $-\infty<\mu,\nu<\infty$. Let $\gamma\in W_q^m$, $\gamma(y)=0$, if $y\notin Q_h(x)$, $\tilde{\gamma}(\xi)=\gamma(x+h\xi)$ $(\xi\in\mathbb{R}^n)$. Assume, that weighted functions ρ, ω_i $(i=0,1)$ satisfy the following conditions:

1) There exist $0<c_0<1<c_1$ such that

$$0<c_0\,\rho(x)\le\rho(y)\le c_1\,\rho(x)\quad\text{for a.a.}\quad y\in Q_h(x);$$

2) $\omega_0(y) \le c_2\omega_0(x)$ for a.a. $y \in Q_h(x)$;
3) $\omega_1(y) \le c_3\omega_0(x)h^{-mq}$ for a.a. $y \in Q_h(x)$;
4) $\rho^{\mu-\nu}(y) \le c_4 h^l$ for a.a. $y \in Q_h(x)$.

Then the following inequality holds

$$\rho^{\mu q}(x)\int_{Q_h(x)} (|\nabla_m(\gamma u)|^q\omega_0(y) + |u\gamma|^q\omega_1(y))\,dy \le$$
$$\le c\,\omega_0(x)h^{(l-n/p)q}\left(\int_{Q_h(x)}\left(|\nabla_m\gamma|^q + h^{-mq}|\gamma|^q\right)dy\right)\times\left(\int_{Q_h(x)}(|\rho^\mu\nabla_l u|^p + |\rho^\nu u|^p)\,dy\right)^{q/p}$$

for all $u \in C_0^\infty$ and $\tilde{\gamma}(\xi) = \gamma(x+h\xi)$ $(\xi \in \mathbb{R}^n)$.

Proof. Let us do the following calculations:

$$\int_{Q_h(x)} (|\nabla_m(\gamma u)|^q\omega_0(y) + |u\gamma|^q\omega_1(y))\,dy \ll$$
$$\ll \omega_0(x)h^{-mq+n}\int_{Q_1(0)}(|\nabla_m(\tilde{u}\tilde{\gamma})|^q + |\tilde{u}\tilde{\gamma}|^q)\,d\xi. \quad (8)$$

Let $U \in C_0^\infty$ be the continuation of $\tilde{u}(\xi) = u(x+h\xi)$ from $Q_1 = Q_1(0)$ to the cube $\frac{3}{2}Q_1$ such that $\operatorname{supp} U \subset \frac{3}{2}Q_1$ and

$$\|U; W_p^l\| \le c_5\|\tilde{u}; W_p^l(Q_1)\|. \quad (9)$$

Inequality (9) implies that

$$\int_{Q_1}(|\nabla_m(\tilde{u}\tilde{\gamma})|^q + |\tilde{u}\tilde{\gamma}|^q)d\xi \ll \int(|\nabla_m(U\tilde{\gamma})|^q + |U\tilde{\gamma}|^q)d\xi \ll$$
$$\ll \|\tilde{\gamma}; M(W_p^l \to W_q^m)\|^q\|\tilde{u}; W_p^l(Q_1)\|^{q/p}. \quad (10)$$

By virtue of Theorem C, we have

$$\|\tilde{\gamma}; M(W_p^l \to W_q^m)\| \ll \sup_x\|\tilde{\gamma}; W_q^m(B_1(x))\| \ll$$
$$\ll \sup_x\left(\int_{Q_1(0)\bigcap B_1(x)}(|\nabla_m\tilde{\gamma}|^q + |\tilde{\gamma}|^q)\,d\xi\right)^{1/q} \le \left(\int_{Q_1(0)}(|\nabla_m\tilde{\gamma}|^q + |\tilde{\gamma}|^q)\,d\xi\right)^{1/q} = \quad (11)$$
$$= h^{m-n/q}\left(\int_{Q_h(x)}\left(|\nabla_m\gamma|^q + h^{-mq}|\gamma|^q\right)dy\right)^{1/q}.$$

Next, by using conditions 2)-4), we obtain

$$\|\tilde{u}; W_p^l(Q_1)\|^{q/p} \ll$$
$$\ll h^{(l-n/p)q}\left(\int_{Q_h(x)}\rho(x)^{-\mu p}|\rho^\mu\nabla_l u|^p\,dy + h^{-lp}\int_{Q_h(x)}\rho(x)^{-\nu p}|\rho^\nu u|^p\,dy\right)^{q/p} \ll \quad (12)$$
$$\ll \rho(x)^{-\mu q}h^{(l-n/p)q}\left(\int_{Q_h(x)}|\rho^\mu\nabla_l u|^p\,dy + \int_{Q_h(x)}|\rho^\nu u|^p\,dy\right)^{q/p}.$$

Then by (8) and (10)–(12) it follows the statement of the lemma. The proof of Lemma 1 is complete.

Let us begin to prove Theorem 1.

Proof. Let $u \in C_0^\infty$, $F = \operatorname{supp} u$. Let $\{(Q^j, \hat{Q}^j), j \in J\}$ be a Besicovitch double covering extracted from the family $\{Q(x), x \in F\}$ ($Q^j = Q(x^j)$, $\hat{Q}^j = \frac{1}{2}Q^j$) (see Theorem A). In Lemma 1 we denote by $\omega_0 = \omega$, $\omega_1 = \omega\rho^{-smq}$. We have

$$\int (|\nabla_m(\gamma u)|^q\omega_0 + |\gamma u|^q\omega_1)\, dy = \int_F (|\nabla_m(\gamma u)|^q\omega_0 + |\gamma u|^q\omega_1)\, dy. \tag{13}$$

By Theorem B γ can be introduced in the form $\gamma = \sum\limits_j \gamma\psi_j$. We denote by $\gamma_j = \gamma\psi_j$. Let us consider each summand on the right hand side of (13) separately.

$$\int_F |\nabla_m(\gamma u)|^q\omega_0(y)\, dy = \int_F \left(\sum_{|\alpha=m|} |D^\alpha(\gamma u)|^2\right)^{q/2} \omega_0(y)\, dy \ll$$

$$\ll \int_F \left(\sum_{|\alpha=m|}\left|\sum_j D^\alpha(\gamma_j u)(y)\right|\right)^q \omega_0(y)\, dy \ll \widetilde{\varkappa}_1^q\widetilde{\varkappa}_2 \max_{1\le i\le\widetilde{\varkappa}_2} \sum_{k\in J_i}\int_{Q^k}\sum_{j\in J}|\nabla_m(\gamma_j u)|^q\omega_0(y)\, dy \ll \tag{14}$$

$$\ll \widetilde{\varkappa}_1^q\widetilde{\varkappa}_2 \max_{1\le i\le\widetilde{\varkappa}_2} \sum_{j\in J}\sum_{k\in J_i}\int_{Q^k\bigcap Q^j}|\nabla_m(\gamma_j u)|^q\omega_0(y)\, dy \ll \widetilde{\varkappa}_1^q\widetilde{\varkappa}_2\sum_{j\in J}\int_{Q^j}|\nabla_m(\gamma_j u)|^q\omega_0(y)\, dy.$$

In the same way we show that

$$\int_F |\gamma u|^q\omega_1(y)\, dy \ll \widetilde{\varkappa}_1^q\widetilde{\varkappa}_2\sum_{j\in J}\int_{Q^j}|\gamma_j u|^q\omega_1(y)\, dy. \tag{15}$$

Then (14), (15) imply that

$$\int_F (|\nabla_m(\gamma u)|^q\omega_0 + |\gamma u|^q\omega_1)\, dy \ll \widetilde{\varkappa}_1^q\widetilde{\varkappa}_2\sum_{j\in J}\int_{Q^j}(|\nabla_m(\gamma_j u)|^q\omega_0(y) + |\gamma_j u|^q\omega_1(y))\, dy. \tag{16}$$

By Lemma 1, we have

$$\int_{Q^j}(|\nabla_m(\gamma_j u)|^q\omega_0 + |\gamma_j u|^q\omega_1)\, dy \ll$$

$$\ll \omega(x)\rho(x)^{-\mu q+sq(l-n/p)}\left(\int_{Q^j}|\nabla_m\gamma_j|^q + \rho(x^j)^{-smq}\int_{Q^j}|\gamma_j|^q\right)\left[\int_{Q^j}(|\rho^\mu\nabla_l u|^p + |\rho^\nu u|^p)\right]^{q/p}.$$

By virtue of Theorem B, we have

$$|\nabla_m\gamma_j|(y) \ll \sum_{k=0}^m h^{k-m}|\nabla_k\gamma|(y)$$

for each $y \in Q^j$. Thus, inequality (16) implies that

$$\int_{Q^j}|\nabla_m\gamma_j|^q\, dy \ll \sum_{k=0}^m h^{q(k-m)}\int_{Q^j}|\nabla_k\gamma|^q\, dy \ll$$

$$\ll \int_{Q^j}\left(|\nabla_m\gamma|^q + h_j^{-mq}|\gamma|^q\right) dy = \int_{Q^j}\left(|\nabla_m\gamma|^q + \rho(x^j)^{-smq}|\gamma|^q\right) dy. \tag{17}$$

Then (17) implies that

$$\int_{Q^j} (|\nabla_m(\gamma_j u)|^q \omega_0 + |\gamma_j u|^q \omega_1)\, dy \ll$$

$$\ll \omega(x)\rho(x)^{-\mu q+sq(l-n/p)} \left(\int_{Q^j} |\nabla_m \gamma|^q + \rho(x^j)^{-smq} \int_{Q^j} |\gamma|^q \right) \left[\int_{Q^j} (|\rho^\mu \nabla_l u|^p + |\rho^\nu u|^p) \right]^{q/p}. \qquad (18)$$

We return to the estimate(16). Relation (18) implies that

$$\int_F (|\nabla_m(\gamma u)|^q \omega_0 + |\gamma u|^q \omega_1)\, dy \ll$$

$$\ll \operatorname*{ess\,sup}_x \frac{\omega(x)}{\rho(x)^{\mu q}} \left[\rho^{s(l-n/p)}(x) \left(\int_{Q(x)} |\nabla_m \gamma|^q \right)^{1/q} + \rho^{s(l-m-n/p)}(x) \left(\int_{Q(x)} |\gamma|^q \right)^{1/q} \right]^q \times$$

$$\times \sum_{j\in J} \left[\int_{Q^j} (|\rho^\mu \nabla_l u|^p + |\rho^\nu u|^p)\, dy \right]^{q/p}.$$

Hence it follows the upper estimate of $\|\gamma; M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W^m_{q,\omega,\omega\rho^{-smq}})\|$.

We take $\eta \in C_0^\infty(Q_1)$, $0 \le \eta \le 1$, $\eta = 1$ in $\frac{1}{2}Q_1$. Let $u_0(y) = \eta\left(h^{-1}(y-x)\right)$. Then

$$\|\gamma; M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W^m_{q,\omega,\omega\rho^{-smq}})\| \gg \frac{\int_{\frac{1}{2}Q(x)} (|\nabla_m(\gamma u_0)|^q \omega_0 + |\gamma u_0|^q \omega_1)\, dy}{\left[\int_{Q(x)} (|\rho^\mu \nabla_l u_0|^p + |\rho^{\mu-sl} u_0|^p)\, dy\right]^{1/p}} \gg$$

$$\gg \frac{\omega(x)^{1/q} \left[\left(\int_{\frac{1}{2}Q(x)} |\nabla_m \gamma|^q \right)^{1/q} + \rho^{-smq}(x) \left(\int_{\frac{1}{2}Q(x)} |\gamma|^q \right)^{1/q} \right]}{\rho^\mu(x)\rho^{-s(l-n/p)} \left[\int_{Q_1} (|\nabla_l \eta|^p + |\eta|^p)\, d\xi \right]^{1/p}}.$$

Hence it follows the lower estimate $\|\gamma; M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W^m_{q,\omega,\omega\rho^{-smq}})\|$. The proof of Theorem 1 is complete.

Theorem 2. Let $1 < p \le q < \infty$, $lp > n$, $-\infty < \mu, s < \infty$. Let $\gamma \in W^m_{q,loc}$. Let ρ satisfy the slow variation condition in the basis $\mathfrak{B} = \{Q(x) = Q_{h(x)}(x)\}$, where $h(x) = \rho(x)^s$. If

$$K_\rho = \operatorname*{ess\,sup}_x K_{(1),p,q}(x|\rho,\gamma) < \infty,$$

then

$$\gamma \in M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W_q^m(\rho^\mu, \rho^{\mu-sm})).$$

Here the norm satisfies the following inequalities

$$c_0\, K_{\rho,(1/2)} \le \|\gamma; M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W_q^m(\rho^\mu, \rho^{\mu-sm}))\| \le c_1\, K_\rho,$$

where $K_{\rho,(1/2)} = \operatorname*{ess\,sup}_x K_{(1/2),p,q}(x|\rho,\gamma)$.

The statement of Theorem 2 is a simple consequence of Theorem 1. We formulate it as theorem because here there is the scale of spaces $M(W_p^l(\rho^\mu, \rho^{\mu-sl}) \to W_q^m(\rho^\mu, \rho^{\mu-sm}))$.

Remark 3. We can set $\rho(x) = 1$. In this case, Theorem 2 leads to the well-known description of the space $M(W_p^l \to W_q^m)$ [1].

Corollary 1. Let $1 < p \le q < \infty,\ lp > n,\ \mu > 0,\ s > 0,\ \mathfrak{B} = \{Q(x) = Q_{h(x)}(x)\}$, where $h(x) = (1+|x|)^{-s}$. If $\gamma \in W^m_{q,loc}$ and

$$K = \sup_{Q(x)\in\mathfrak{B}} \left\{ (1+|x|)^{-s(l-n/p)} \left(\int_{\frac{1}{2}Q(x)} |\nabla_m \gamma|^q \right)^{1/q} + \right.$$
$$\left. +(1+|x|)^{-s(l-m-n/p)} \left(\int_{\frac{1}{2}Q(x)} |\gamma|^q \right)^{1/q} \right\} < \infty,$$

then
$$\gamma \in M(W^l_p((1+|x|)^\mu, (1+|x|)^{\mu-sl}) \to W^m_q((1+|x|)^\mu, (1+|x|)^{\mu-sm})).$$
Here the norm satisfies the following relation

$$\|\gamma; M(W^l_p((1+|x|)^\mu, (1+|x|)^{\mu-sl}) \to W^m_q((1+|x|)^\mu, (1+|x|)^{\mu-sm}))\| \sim K.$$

Theorem 3. Let $1 < p \le q < \infty,\ lp > n,\ -\infty < \mu, \nu < \infty$. Let $\rho(\cdot)$ satisfy the slow variation condition with respect to the basis of cubes $\mathfrak{B} = \{Q(x) = Q_{h(x)}(x)\}$ and a.e.

$$\rho(x)^\mu \ll \rho(x)^\nu h(x)^l. \tag{19}$$

Assume that $\gamma \in L^{loc}_{q,\alpha}$. If

$$\mathbf{C} = \operatorname*{ess\,sup}_x h(x)^{l-n/p} \rho(x)^{-\mu} \left(\int_{Q(x)} |\gamma|^q \, d\alpha \right)^{1/q} < \infty,$$

then $\gamma \in M(W^l_p(\rho^\mu, \rho^\nu) \to L_{q,\alpha})$.

Theorem 3 is proved by arguments similar to those that were carried out in the proof of Theorem 1. But instead of Lemma 1 we use the local estimate

$$\int_{Q^j} |\gamma u|^q \, d\alpha \le c\, h_j^{(l-n/p)q} \rho(x^j)^{-\mu q} \int_{Q^j} |\gamma|^q \, d\alpha \times \left(\int_{Q^j} (|\rho^\mu \nabla_l u|^p + |\rho^\nu u|^p) \right)^{q/p},$$

which follows from condition (19) and inequality (7).

Let Γ be a compact manifold in $\mathbb{R}^n$ with dimension $\le n-1$. Let α be a measure on Γ, $\alpha(\Gamma) < \infty$. Let $L_{q,\alpha}(\Gamma)$ be the space of all continuous functions in Γ with the norm

$$\|u; L_{q,\alpha}(\Gamma)\| = \left(\int_\Gamma |u|^q \, d\alpha \right)^{1/q} < \infty.$$

Below we consider the manifold Γ satisfying the condition: there exist $B > 0,\ 0 < \beta < n$ such that

$$\sup_{x\in\Gamma} |\{y \in \mathbb{R}^n\colon |y-x| < R,\ \sigma(y) < \varepsilon\}| \le B\, \varepsilon^\beta R^{n-\beta} \tag{20}$$

for all $\varepsilon > 0,\ R > 0$.

Surfaces with Lipschitz condition satisfy the condition (20). Thus, the following estimate takes place
$$\alpha_\beta(Q \cap \Gamma) = \mathcal{H}_{n-\beta}(Q \cap \Gamma) \le c\, B\, |Q|^{1-\beta/n},\ Q \in I^n,$$
for the Hausdorff measure $\alpha_\beta = \mathcal{H}_{n-\beta}$ ([2], §1.10).

Corollary 2. Let $1 < p \leq q < \infty$, $lp > n$, $mq > n$, $0 < \beta < n$. Let Γ satisfy the condition (20), $\alpha_\beta(\Gamma) < \infty$. Assume that $\gamma \in W_q^m$, and

$$\mathbf{C} = \sup_x \widetilde{\sigma}(x)^{m-n/p-\beta/q} \left(\int_{\widetilde{Q}(x)} \left(|\nabla_m \gamma|^q + |\widetilde{\sigma}^{-m}\gamma|^q \right) dy \right)^{1/q} < \infty.$$

Then $\gamma \in M(W_p^l(\widetilde{\sigma}^l, 1) \to L_{q,\alpha_\beta}(\Gamma))$. Here $\widetilde{\sigma}(\cdot) = \frac{1}{2}\min(1, \sigma(\cdot))$, $\widetilde{Q}(x) = Q_h(x)$, $h = \widetilde{\sigma}(x)$.

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Салмақты Соболев кеңістіктеріндегі мультипликаторлар жайлы. I-бөлім

X, Y — $y \colon \Omega \to \mathbb{R}$ функцияларынан тұратын банах кеңістіктері болсын. Егер $Tx = zx \in Y$ және $T \colon X \to Y$ операторы шенелген болса, онда $z \colon \Omega \to \mathbb{R}$ функциясы (X, Y) жұбындағы нүктелік мультипликатор деп аталады. $M(X \to Y)$ арқылы (X, Y) жұбындағы мультипликаторлар кеңістігін белгілейміз. $M(X \to Y)$ мультипликаторлар кеңістігінде норманы келесідей анықтаймыз: $\|z;\ M(X \to Y)\| = \|T;\ X \to Y\|$. $1 \leq p < \infty$, m – бүтін сан болсын. $W^m_{p,\omega_0,\omega_1}$ арқылы салмақты Соболев кеңістігін белгілеп, норманы келесідей анықтаймыз: $\|u\|_{W^m_{p,\omega_0,\omega_1}} = \|u; W^m_{p,\omega_0,\omega_1}\| = \|\omega_0^{1/p}|\nabla_m u|\|_{L_p} + \|\omega_1^{1/p} u\|_{L_{p,v}}$. Аталмыш жұмыстың мақсаты — салмақты Соболев кеңістіктерінің $(W^l_{p,\rho,v}, W^m_{q,\omega_0,\omega_1})$ жұбы үшін мультипликаторлар кеңістіктерін сипаттау.

Л.Кусаинова, А.Мырзагалиева

О мультипликаторах в весовых пространствах Соболева. Часть I

Пусть X, Y — банаховы пространства функций $y\colon \Omega \to \mathbb{R}$. Функция $z\colon \Omega \to \mathbb{R}$ называется точечным мультипликатором в паре (X, Y), если $Tx = zx \in Y$ и оператор $T\colon X \to Y$ ограничен. Через $M(X \to Y)$ обозначается пространство мультипликаторов в паре (X, Y). В $M(X \to Y)$ вводится норма $\|z; M(X \to Y)\| = \|T; X \to Y\|$. Пусть $1 \le p < \infty$. Пусть m – целое. Через $W^m_{p,\omega_0,\omega_1}$ обозначается весовое пространство Соболева с конечной нормой вида $\|u\|_{W^m_{p,\omega_0,\omega_1}} = \|u; W^m_{p,\omega_0,\omega_1}\| = \|\omega_0^{1/p}|\nabla_m u|\|_{L_p} + \|\omega_1^{1/p} u\|_{L_{p,v}}$. Цель данной работы заключается в описании пространств мультипликаторов для пары весовых пространств Соболева $(W^l_{p,\rho,v}, W^m_{q,\omega_0,\omega_1})$.

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