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On the integral equation of the boundary value problem for the essentially loaded differential heat operator

In the article the Volterra integral equations of the second kind with the given kernel are investigated. This kind of integral equations arises in the solving of some boundary value problems for essential loaded differential heat operator in an unbounded domain. The theory of boundary value problems for essential loaded differential parabolic equations is very important not only for the modeling of the physical, technical and application processes, but also in the experimental studies. The test problems are also connected with mathematical modeling of thermal processes in the electric arc of the high-current breaking devices. Experimental studies of these phenomena are difficult because of their transience and in some cases only a mathematical model is capable to provide adequate information about their dynamics, so the test material is highly relevant in modern science.

Key words: Volterra integral equations of the second kind, kernel of integral equation, modified Bessel function, gamma function, incomplete gamma function, beta function, the generalized hypergeometric function, symbol of Pochhammer.

The necessity for the study of Volterra integral equations of the second kind in the given form naturally arises in finding of the solutions of some boundary value problems for essentially loaded differential parabolic equation [1]

$$\mu(t) - \lambda \int_0^t K(t,\tau)\mu(\tau)d\tau = F(t), \tag{1}$$

where $\lambda \in C$ is the numeral parameter of this equation; $F(t)$ is the known function, it is defined on the interval $(0;\infty)$. The kernel $K(t,\tau)$ of the integral equation (1) has the following form

$$K(t,\tau) = \frac{\partial^k Q(x,t-\tau)}{\partial x^k} \mid_{x=\bar{x}(t)},$$

where

$$Q(x,t-\tau) = \frac{x^\beta}{2(t-\tau)} \cdot exp\left(-\frac{x^2}{4(t-\tau)}\right) \cdot P(x,t-\tau), \tag{2}$$

$$P(x,t-\tau) = \int_0^\infty \xi^{1-\beta} exp\left(-\frac{\xi^2}{4(t-\tau)}\right) \cdot I_\beta\left(\frac{\xi \cdot x}{2(t-\tau)}\right) \cdot d\xi \tag{3}$$

and $x = \bar{x}(t)$ is the given function with $t \in (0,\infty)$, $I_\beta(z)$ is the modified Bessel function, β is the numeral parameter, $0 < \beta < 1$, $\mu(t)$ is the unknown function.

The function $Q(x,t-\tau)$ defines the kernel of the integral equation (1). We calculate the function $Q(x,t-\tau)$ and present its various interpretations. Taking into account that [2]

$$\int_0^\infty x^{\alpha-1} exp\left(-px^r\right) \cdot I_\nu(cx)dx = \frac{c^\nu}{2^\nu r \cdot p^{\frac{\alpha+\nu}{r}}\Gamma(\nu+1)} \cdot \Sigma_{k=0}^{\infty} \frac{1}{k!(\nu+1)_k} \Gamma\left(\frac{2k+\alpha+\nu}{r}\right) \cdot \left(\frac{c}{2p^{\frac{1}{r}}}\right)^{2k},$$

where Rep, $Re(\alpha+\nu)>0$; $r>1$ and $(\nu+1)_0=1$, $(\nu+1)_k=(\nu+1)(\nu+2)\cdot\ldots\cdot(\nu+k)$, $k=1,2,3...$ are the symbols of Pochhammer, from (3) we get

$$P(x,t-\tau)=\frac{\frac{x^\beta}{2^\beta\cdot(t-\tau)^\beta}}{2^{\beta+1}\cdot\frac{1}{4(t-\tau)}\cdot\Gamma(\beta+1)}\times$$

$$\times\Sigma_{k=0}^{\infty}\frac{1}{k!(\beta+1)_k}\Gamma(k+1)\left(\frac{c}{2p^{\frac{1}{r}}}\right)^{2k}\cdot\left(\frac{x}{2(t-\tau)}\right)^{2k}:\frac{1}{2}\cdot\left(\frac{1}{4(t-\tau)}\right)^{2k}=$$

$$=\frac{4(t-\tau)}{2^{2\beta+1}\cdot\Gamma(\beta+1)}\cdot\frac{x^\beta}{(t-\tau)^\beta}\cdot\Sigma_{k=0}^{\infty}\frac{1}{k!(\beta+1)_k}\Gamma(k+1)\left(\frac{x}{2\sqrt{t-\tau}}\right)^{2k}=$$

$$=\frac{4}{2^{2\beta+1}\cdot\Gamma(\beta+1)}\cdot\Sigma_{k=0}^{\infty}\frac{1}{2^{2k}(\beta+1)_k}\cdot\frac{x^{2k+\beta}}{(t-\tau)^{k+\beta-1}}=$$

$$=\Sigma_{k=0}^{\infty}\frac{1}{2^{2k+2\beta-1}\cdot(\beta+1)_k\cdot\Gamma(\beta+1)}\cdot\frac{x^{2k+\beta}}{(t-\tau)^{k+\beta-1}}\Longrightarrow$$

$$P(x,t-\tau)=\Sigma_{k=0}^{\infty}\frac{1}{2^{2k+2\beta-1}\cdot(\beta+1)_k\cdot\Gamma(\beta+1)}\cdot\frac{x^{2k+\beta}}{(t-\tau)^{k+\beta-1}}. \tag{4}$$

Substituting (4) in (2), we obtain the following representation of the function $Q(x,t-\tau)$

$$Q(x,t-\tau)=\Sigma_{k=0}^{\infty}\frac{1}{2^{2k+2\beta}\cdot(\beta+1)_k\cdot\Gamma(\beta+1)}\cdot\frac{x^{2k+2\beta}}{(t-\tau)^{k+\beta}}\cdot exp\left(-\frac{x^2}{4(t-\tau)}\right)$$

or

$$Q(x,t-\tau)=\left(\frac{x}{2}\right)^{2\beta}\cdot\frac{1}{\Gamma(\beta+1)(t-\tau)^\beta}\cdot exp\left(-\frac{x^2}{4(t-\tau)}\right)\cdot\Sigma_{k=0}^{\infty}\frac{1}{(\beta+1)_k\cdot(t-\tau)^k}\cdot\left(\frac{x}{2}\right)^{2k}.$$

It is possible to obtain a different ratio for the function $Q(x,t-\tau)$ with using the integral representation of the modified Bessel function [3]

$$I_\beta(z)=\frac{(\frac{z}{2})^\beta}{\Gamma(\beta+\frac{1}{2})\cdot\Gamma(\frac{1}{2})}\cdot\int_{-1}^{1}(1-t^2)^{\beta-\frac{1}{2}}\cdot exp(\pm zt)dt, Re(\beta+\frac{1}{2})>0.$$

From the relation (3) we get

$$P(x,t-\tau)=\int_0^\infty\xi^{1-\beta}exp(-\frac{\xi^2}{4(t-\tau)})\cdot\frac{(\frac{\xi\cdot x}{4(t-\tau)})^\beta}{\Gamma(\beta+\frac{1}{2})\cdot\Gamma(\frac{1}{2})}d\xi\times$$

$$\times\int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp(\pm\frac{\xi\cdot x\cdot\eta}{2(t-\tau)})d\eta=\frac{1}{\Gamma(\beta+\frac{1}{2})\cdot\Gamma(\frac{1}{2})\cdot 4^\beta}\cdot\frac{x^\beta}{(t-\tau)^\beta}\times$$

$$\times\int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}d\eta\int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}d\eta\int_0^\infty\xi\cdot exp(-\frac{1}{4(t-\tau)}(\xi^2\mp 2x\eta\xi))\cdot d\xi. \tag{5}$$

Taking into account that [3]

$$\int_0^\infty x\cdot exp(-\mu x^2-2\nu x)\cdot dx=\frac{1}{2\mu}+\frac{\nu}{2\mu}\cdot\sqrt{\frac{\pi}{\mu}}\cdot exp\left(\frac{\nu^2}{\mu}\right)\cdot\left[1-\Phi\left(\frac{\nu}{\sqrt{\mu}}\right)\right]$$

and $|arg\nu| < \frac{\pi}{2}, Re\mu > 0$, the relation (5) is transformed to the form

$$P(x,t-\tau) = \frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\cdot\int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}\times$$

$$\times\left[2(t-\tau)-\frac{x\eta}{2}\sqrt{4\pi(t-\tau)}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\left(1-\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)\right)\right]d\eta =$$

$$= \frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\cdot\frac{x^\beta}{4^\beta(t-\tau)^\beta}\times$$

$$\times\left[2(t-\tau)\cdot\int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}d\eta - x\sqrt{\pi(t-\tau)}\cdot\int_{-1}^{1}\eta(1-\eta^2)^{\beta-\frac{1}{2}}\times\right.$$

$$\left.\times exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\left(1-\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)\right)d\eta\right] = \frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\times$$

$$\times\frac{x^\beta}{4^\beta(t-\tau)^\beta}\cdot\left[2(t-\tau)\cdot A_1 - x\sqrt{\pi(t-\tau)}\cdot A_2(x,(t-\tau))\right], \tag{6}$$

where

$$A_1 = \int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}d\eta; \tag{7}$$

$$A_2(x,t-\tau) = \int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\left(1-\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)\right)d\eta.$$

Since we have [3]

$$\int_0^a(a^\mu-x^\mu)^{p-1}dx = \mu^{-1}\cdot a^{\mu(p-1)+1}\cdot B\left(p,\frac{1}{\mu}\right),$$

where $a,\mu,Rep>0$, then

$$A_1 = \int_{-1}^{1}(1-\eta^2)^{\beta-\frac{1}{2}}d\eta = B\left(\beta+\frac{1}{2},\frac{1}{2}\right).$$

Now we calculate $A_2(x,t-\tau)$

$$A_2(x,t-\tau) = \int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\left(1-\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)\right)d\eta =$$

$$= \int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)erfc\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)d\eta =$$

$$= \int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)d\eta -$$

$$-\int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)d\eta.$$

Taking into account that the integrands in the first and the second integrals of the last relation are odd and even respectively we obtain

$$A_2(x,t-\tau)=-2\int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)d\eta.$$

Since we have [2]

$$\int_0^a x^{\alpha-1}(a^2-x^2)^{p-1}exp(c^2x^2)\cdot\left\{\begin{matrix}erf(cx)\\ erfc(cx)\end{matrix}\right\}dx=$$

$$=\pm\frac{a^{\alpha+2p-1}c}{\sqrt{\pi}}B\left(\frac{\alpha+1}{2},p\right)\cdot_2F_2\left(1,\frac{\alpha+1}{2};\frac{3}{2},p+\frac{\alpha+1}{2};a^2c^2\right)+$$

$$+\left\{\begin{matrix}0\\1\end{matrix}\right\}\frac{a^{\alpha+2p-2}c}{\sqrt{\pi}}B\left(\frac{\alpha}{2},p\right)\cdot_1F_1\left(\frac{\alpha}{2};\frac{\alpha}{2}+p;a^2c^2\right),ifa,Rep>0;Re\alpha>-\frac{1\pm1}{2},$$

where

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdot...\cdot(a_p)_k}{(b_1)_k\cdot...\cdot(b_p)_k}\cdot\frac{z^k}{k!}$$

is the generalized hypergeometric function,

$${}_1F_1(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\cdot\frac{z^k}{k!}$$

is the singular hypergeometric function,

$$a_0=1,(a)_k=a(a+1)\cdot...\cdot(a+k-1),k=1,2,3,...$$

are the symbols of Pochhammer,

$$erfc(z)=1-\Phi(z),erf(z)=\Phi(z),$$

then the correspondence for $A_2(x,t-\tau)$ takes the form

$$A_2(x,t-\tau)=-2\int_{-1}^{1}\eta\cdot(1-\eta^2)^{\beta-\frac{1}{2}}\cdot exp\left(\frac{x^2\eta^2}{4(t-\tau)}\right)\cdot\Phi\left(\frac{x\eta}{2\sqrt{t-\tau}}\right)d\eta=$$

$$=\pm\frac{x}{\sqrt{\pi(t-\tau)}}B\left(\frac{3}{2},\beta+\frac{1}{2}\right)\cdot_2F_2\left(1,\frac{3}{2};\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right). \quad (8)$$

Using (7) and (8) we obtain the representation (6) in the form

$$P(x,t-\tau)=\frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\cdot\frac{x^\beta}{4^\beta(t-\tau)^\beta}\times$$

$$\times\left[2(t-\tau)\cdot B\left(\beta+\frac{1}{2},\frac{1}{2}\right)\pm x\sqrt{\pi(t-\tau)}\cdot\frac{x}{\sqrt{\pi(t-\tau)}}\cdot B\left(\frac{3}{2},\beta+\frac{1}{2}\right)\times\right.$$

$$\left.\times{}_2F_2\left(1,\frac{3}{2};\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)\right]=\frac{1}{\Gamma\left(\beta+\frac{1}{2}\cdot\right)\Gamma\left(\frac{1}{2}\right)}\times$$

$$\times\left[B\left(\beta+\frac{1}{2},\frac{1}{2}\right)\cdot\frac{x^\beta(t-\tau)^{1-\beta}}{2^{2\beta-1}}\pm B\left(\frac{3}{2},\beta+\frac{1}{2}\right)\cdot\frac{x^{\beta+2}}{2^{2\beta}(t-\tau)^\beta}\cdot_2F_2\left(1,\frac{3}{2};\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)\right].$$

Considering the properties of the gamma function and the beta function

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi},$$

we can rewrite the last relation for $P(x,t-\tau)$ to the following form

$$P(x,t-\tau)=\frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}\times$$

$$\times\left[\frac{\Gamma\left(\beta+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\beta+1\right)}\cdot\frac{x^{\beta}(t-\tau)^{1-\beta}}{2^{2\beta-1}}\pm\frac{x^{\beta+2}}{2^{2\beta}(t-\tau)^{\beta}}\cdot{}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma\left(\beta+2\right)}\right]=$$

$$=\frac{x^{\beta}(t-\tau)^{1-\beta}}{2^{2\beta-1}\Gamma(\beta+1)}\pm\frac{x^{\beta+2}}{2^{2\beta+1}\Gamma(\beta+2)(t-\tau)^{\beta}}\cdot{}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right),$$

that is

$$P(x,t-\tau)=\frac{x^{\beta}(t-\tau)^{1-\beta}}{2^{2\beta-1}\Gamma(\beta+1)}\pm\frac{x^{\beta+2}}{2^{2\beta+1}\Gamma(\beta+2)(t-\tau)^{\beta}}\cdot{}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right). \tag{9}$$

Substituting (9) in (2), we obtain the following representation for the function $Q(x,t-\tau)$

$$Q(x,t-\tau)=exp\left(-\frac{x^2}{4(t-\tau)}\right)\times$$

$$\times\left[\frac{x^{2\beta}}{2^{2\beta}\Gamma(\beta+1)}\cdot\frac{1}{(t-\tau)^{\beta}}\pm\frac{x^{2\beta+2}}{2^{2\beta+2}\Gamma(\beta+2)(t-\tau)^{\beta+1}}\cdot{}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)\right], \tag{10}$$

where

$${}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)=\sum_{k=0}^{\infty}\frac{(1)_k\left(\frac{3}{2}\right)_k}{\left(\frac{3}{2}\right)_k(\beta+2)_k}\cdot\left(\frac{x^2}{4(t-\tau)}\right)^k\cdot\frac{1}{k!}.$$

Since we have $(1)_k=1\cdot 2\cdot ...\cdot(1+k-1)=k!$,

$${}_2F_2\left(1,\frac{3}{2},\frac{3}{2},\beta+2;\frac{x^2}{4(t-\tau)}\right)=\sum_{k=0}^{\infty}\frac{1}{(\beta+2)_k}\cdot\left(\frac{x^2}{4(t-\tau)}\right)^k. \tag{11}$$

The equation (10) with (11) takes the form

$$Q(x,t-\tau)=exp\left(-\frac{x^2}{4(t-\tau)}\right)\times$$

$$\times\left[\frac{x^{2\beta}}{2^{2\beta}\Gamma(\beta+1)}\cdot\frac{1}{(t-\tau)^{\beta}}\pm\frac{x^{2\beta+2}}{2^{2\beta+2}\Gamma(\beta+2)(t-\tau)^{\beta+1}}\cdot\sum_{k=0}^{\infty}\frac{1}{(\beta+2)_k}\cdot\left(\frac{x^2}{4(t-\tau)}\right)^k\right].$$

If for calculation of the function $P(x,t-\tau)$ from (3)

$$P(x,t-\tau)=\int_0^{\infty}\xi^{1-\beta}exp\left(-\frac{\xi^2}{4(t-\tau)}\right)\cdot I_{\beta}\left(\frac{\xi\cdot x}{2(t-\tau)}\right)\cdot d\xi$$

we use the relation [2]

$$\int_0^\infty x^{\alpha-1} exp\left(-px^2\right) \cdot I_\nu(cx) \cdot dx = A_\nu^\alpha, if Rep, Re(\alpha+\nu) > 0; |argc| < \pi,$$

where

$$A_\nu^\alpha = 2^{-\nu-1} c^\nu p^{\frac{\alpha+\nu}{2}} \cdot \frac{\Gamma\left(\frac{\alpha+\nu}{2}\right)}{\Gamma(\nu+1)} \cdot {}_1F_1\left(\frac{\alpha+\nu}{2}, \nu+1; \frac{c^2}{4p}\right),$$

$$A_\nu^{2-\nu} = \frac{(2p)^{\nu-1}}{c^\nu \Gamma(\nu)} \cdot exp\left(\frac{c^2}{4p}\right) \cdot \gamma\left(\nu, \frac{c^2}{4p}\right),$$

$$\gamma(\nu, x) = \Gamma(\nu) - \Gamma(\nu, x) = \int_0^x t^{\nu-1} \cdot e^{-t} dt$$

is the incomplete gamma function,

$$\Gamma(\nu, x) = \int_x^\infty t^{\nu-1} \cdot e^{-t} dt$$

is the additional incomplete gamma function, then from the representation (3) we get

$$P(x, t-\tau) = A_\beta^{2-\beta} = \left(\frac{1}{2(t-\tau)}\right)^{\beta-1} : \left(\frac{x}{2(t-\tau)}\right)^\beta \cdot \frac{1}{\Gamma(\beta)} \cdot exp\left(\frac{x^2}{4(t-\tau)} \cdot (t-\tau)\right) \times$$

$$\times \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right) = \frac{2(t-\tau)}{x^\beta \Gamma(\beta)} \cdot exp\left(\frac{x^2}{4(t-\tau)}\right) \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right) \times$$

$$\times P(x, t-\tau) = \frac{2(t-\tau)}{x^\beta \Gamma(\beta)} \cdot exp\left(\frac{x^2}{4(t-\tau)}\right) \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right). \quad (12)$$

Substituting (12) in (2), we calculate the function $Q(x, t-\tau)$

$$Q(x, t-\tau) = \frac{x^\beta}{2(t-\tau)} \cdot exp\left(-\frac{x^2}{4(t-\tau)}\right) \cdot P(x, t-\tau) =$$

$$= \frac{1}{\Gamma(\beta)} \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right) = \frac{1}{\Gamma(\beta)} \cdot \int_0^{\frac{x^2}{4(t-\tau)}} e^{-\xi} \cdot \xi^{\beta-1} d\xi;$$

$$Q(x, t-\tau) == \frac{1}{\Gamma(\beta)} \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right). \quad (13)$$

The function $Q(x, t-\tau)$ can also be written as

$$Q(x, t-\tau) = \frac{1}{\Gamma(\beta)} \cdot \gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right) = \frac{1}{\Gamma(\beta)} \cdot \left[\Gamma(\beta) - \Gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right)\right];$$

$$Q(x, t-\tau) = 1 - \frac{1}{\Gamma(\beta)} \cdot \Gamma\left(\beta, \frac{x^2}{4(t-\tau)}\right).$$

Taking into account the relation [2]

$$\gamma(\nu, x) = \frac{z^\nu}{\nu} \cdot {}_1F_1(\nu, \nu+1; -z),$$

the representation (13) is transformed into the form

$$Q(x,t-\tau)=\frac{1}{\beta\cdot\Gamma(\beta)}\cdot\frac{x^2}{2^{2\beta}(t-\tau)^{\beta}}\cdot{}_1F_1\left(\beta,\beta+1;-\frac{x^2}{4(t-\tau)}\right), \qquad (14)$$

where

$${}_1F_1\left(\beta;\beta+1;-\frac{x^2}{4(t-\tau)}\right)=\sum_{k=0}^{\infty}\frac{(\beta)_k}{(\beta+1)_k}\cdot\frac{1}{k!}\cdot\left(-\frac{x^2}{4(t-\tau)}\right)^k. \qquad (15)$$

Since we have

$$\frac{(\beta)_k}{(\beta+1)_k}=\frac{\beta(\beta+1)(\beta+2)\cdot\ldots\cdot(\beta+k-1)}{(\beta+1)(\beta+2)\cdot\ldots\cdot(\beta+k)}=\frac{\beta}{\beta+k},$$

then from (15) we receive

$${}_1F_1\left(\beta;\beta+1;-\frac{x^2}{4(t-\tau)}\right)=\sum_{k=0}^{\infty}\frac{\beta}{(\beta+k)\cdot k!}\cdot\left(-\frac{x^2}{4(t-\tau)}\right)^k=$$

$$=\beta\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{(\beta+k)\cdot k!}\cdot\left(\frac{x^2}{4(t-\tau)}\right)^k. \qquad (16)$$

From (14) with (16) we get the following representation of the function $Q(x,t-\tau)$

$$Q(x,t-\tau)=\frac{1}{\Gamma(\beta)}\cdot\frac{x^2}{2^{2\beta}(t-\tau)^{\beta}}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{(\beta+k)\cdot k!}\cdot\left(\frac{x^2}{4(t-\tau)}\right)^k.$$

Taking into account that [2]

$$\gamma(\nu+1,x)=\nu\cdot\gamma(\nu,z)-z^{\nu}e^{-z},$$

we obtain the representation for the function $Q(x,t-\tau)$ in the form

$$Q(x,t-\tau)=\frac{1}{\Gamma(\beta)}\cdot\left[(\beta-1)\cdot\gamma\left(\beta-1,\frac{x^2}{4(t-\tau)}\right)-\left(\frac{x}{2}\right)^{2\beta-2}(t-\tau)^{1-\beta}exp\left(-\frac{x^2}{4(t-\tau)}\right)\right].$$

Using different representations of the original function of the integral equation kernel we will explore questions of solvability of the integral equation (1).

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Елеулі жүктелген дифференциалдық жылуөткізгіштік операторы үшін шекті есептің интегралды теңдеуі туралы

Мақалада берілген ядромен екінші ретті Вольтерраның интегралды теңдеуі зерттелген. Осындай интегралды теңдеулер интегралды шешімнің нүктесіндегі жүктелген дифференциалдық жылуөткізгіштік операторы үшін маңызды жүктелген есепті қарастырғанда пайда болады. Интегралды шешімнің нүктесіндегі жүктелген дифференциалдық жылуөткізгіштік параболалық теңдеулер үшін теория физикалық, техникалық және қолданбалы процесс үшін ғана емес, сондай-ақ эксперименталды зерттеулер үшін де өзекті болып табылады. Зерттелген есептер математикалық модельдеумен жылуфизикалық процестерді электр доғасында күшті нүктелі ажыратушы аппараттармен байланысты. Бұл құбылыстарды эксперименттік түрде зерттеу олардың ағыны және динамикасы туралы бірқатар жағдайларда ғана емес, математикалық моделі теңбе-тең ақпарат беруге қиындық туғызады, сондықтан бҮл тақырып қазіргі заманғы жаратылыстануда өте өзекті болып есептеледі.

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Об интегральном уравнении граничной задачи для существенно нагруженного дифференциального оператора теплопроводности

В статье исследовано интегральное уравнение Вольтерра второго рода с заданным ядром. Такого рода интегральные уравнения возникают при решении некоторых граничных задач для существенно нагруженного дифференциального оператора теплопроводности в неограниченной области. Теория граничных задач для существенно нагруженных дифференциальных параболических уравнений весьма актуальна не только для моделирования физических, технических и прикладных процессов, но и в экспериментальных исследованиях. Исследуемые задачи связаны также с математическим моделированием теплофизических процессов в электрической дуге сильноточных отключающих аппаратов. Экспериментальные исследования этих явлений затруднены вследствие их быстротечности, и в ряде случаев лишь математическая модель способна дать адекватную информацию об их динамике, поэтому исследуемый материал весьма актуален в современном естествознании.

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